JÓNSSON’S LEMMA FOR NORMALLY PRESENTED VARIETIES

Ivan Chajda, Olomouc

(Received May 28, 1996)

Varieties presented by normal identities were treated in [1]. Let us recall the basic concepts. Let $\tau$ be a similarity type and $\{x_1, x_2, \ldots\}$ a set of variables. For an $n$-ary term $p(x_1, \ldots, x_n)$ of type $\tau$ we denote by $\text{var } p = \{x_1, \ldots, x_n\}$ the set of all variables occurring in $p$. For $n$-ary terms $p, q$ of type $\tau$ the identity

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$$

is said to be normal if it is either trivial, i.e. $x_1 = x_1$, or $p \notin \text{var } p$ and $q \notin \text{var } q$, i.e. neither $p$ nor $q$ is a single variable. A variety $\mathcal{V}$ of type $\tau$ is normally presented if $\text{Id } \mathcal{V}$ contains only normal identities.

If $\mathcal{V}$ is a variety of type $\tau$, denote by $N(\mathcal{V})$ the variety satisfying all normal identities of $\mathcal{V}$. Hence, $\mathcal{V}$ is a subvariety of $N(\mathcal{V})$ and if $\mathcal{V} \neq N(\mathcal{V})$ then $N(\mathcal{V})$ covers $\mathcal{V}$ in the lattice of all varieties of type $\tau$, see [3].

Since every congruence identity is characterized by a Maltsev condition (see [4]) and because every Maltsev condition contains an identity which is not normal, we obtain the following

Observation. For every variety $\mathcal{V}$, the variety $N(\mathcal{V})$ satisfies no congruence identity.

In particular, $N(\mathcal{V})$ is never a congruence distributive variety. Despite of this fact, $N(\mathcal{V})$ satisfies the assertion of Jónsson’s Lemma provided $\mathcal{V}$ is congruence distributive:

Theorem. Let $\mathcal{V}$ be a congruence distributive variety of type $\tau$ and let $N(\mathcal{V})$ be generated by a class $\mathcal{K}$ of algebras of type $\tau$. Then $\text{Si}(N(\mathcal{V})) = \text{HSP}_U(\mathcal{K})$ and, therefore, $N(\mathcal{V}) = \text{IP}_S \text{HSP}_U(\mathcal{K})$. 
Proof. Let $\mathcal{V}$ be a congruence distributive variety of type $\tau$. Denote by $\mathscr{B} = (\{0, 1\}, F)$ an algebra of type $\tau$ such that $f(x_1, \ldots, x_n) = 0$ for every $x_1, \ldots, x_n$ of $\{0, 1\}$. $\mathscr{B}$ is the so-called constant algebra in the sense of [1]. As was pointed out in Theorem 3 of [1], $\text{Si}(N(\mathcal{V})) = \text{Si}(\mathcal{V}) \cup \mathscr{B}$. By Jónsson’s Lemma, we have

$$\text{Si}(N(\mathcal{V})) = \text{HSP}_U(\mathcal{X}) \cup \mathscr{B}.$$ 

If $\mathscr{B} \notin \text{HSP}_U(\mathcal{X})$ then $\mathscr{B} \notin \text{HSP}(\mathcal{X})$ and thus, by [1], $\text{HSP}(\mathcal{X})$ is not normally presented, a contradiction with $N(\mathcal{V}) = \text{HSP}(\mathcal{X})$. Hence $\mathscr{B} \in \text{HSP}_U(\mathcal{X})$ and $\text{Si}(N(\mathcal{V})) = \text{HSP}_U(\mathcal{X})$. \hfill $\square$

References


Author’s address: Ivan Chajda, Department of Algebra and Geometry, Palacký University Olomouc, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: chajda@risc.upol.cz.