SEQUENTIAL CONVERGENCES IN A VECTOR LATTICE

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Abstract. In the present paper we deal with sequential convergences on a vector lattice $L$ which are compatible with the structure of $L$.

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In this paper we will investigate the system $\text{Conv} \ L$ of all sequential convergences in a vector lattice $L$. The analogously defined notions of sequential convergences in a lattice ordered group or in a Boolean algebra were studied in [3]–[12].

The following results will be established.

The set $\text{Conv} \ L$ is nonempty if and only if $L$ is archimedean. Let $L$ be archimedean. Then $\text{Conv} \ L$ has the least element (it need not have, in general, a greatest element). Each interval of $\text{Conv} \ L$ is a Brouwerian lattice. If $L$ is $(\aleph_0, 2)$-distributive, then $\text{Conv} \ L$ is a complete lattice. There is a convex vector sublattice $L_1$ of $L$ such that (i) $\text{Conv} \ L_1$ is a complete lattice; (ii) if $L_2$ is a convex vector sublattice of $L$ such that $\text{Conv} \ L_2$ is a complete lattice, then $L_2 \subseteq L_1$. Let $X_i$ ($i = 1, 2$) be archimedean vector lattices; if $X_1$ and $X_2$ are isomorphic as lattices and if $\text{Conv} \ X_1$ is a complete lattice, then $\text{Conv} \ X_2$ is a complete lattice as well. If $L$ is a direct sum of linearly ordered vector lattices, then $\text{Conv} \ L$ is a complete lattice and has no atom. Some further results (concerning orthogonal sequences and strong units) are also proved.
1. Preliminaries

The notion of a vector lattice is applied here in the same sense as in [1], Chap. XV. (In [16], the term “Riesz space” is used; in [13] vector lattices are called K-lineals.)

Let $L$ be a vector lattice and let $\mathbb{N}$ be the set of all positive integers. The direct product $\prod_{n \in \mathbb{N}} L_n$, where $L_n = L$ for each $n \in \mathbb{N}$, will be denoted by $L^N$. The elements of $L^N$ are denoted, e.g., as $(x_n)_{n \in \mathbb{N}}$, or simply $(x_n)$; instead of $n$, sometimes other indices will be applied. $(x_n)$ is said to be a sequence in $L$. If $x \in L$ and $x_n = x$ for each $n \in \mathbb{N}$, then we denote $(x_n) = \text{const } x$. The notion of a subsequence has the usual meaning.

If $a \subseteq L^N \times L$, then instead of $((x_n), x) \in a$ we also write $x_n \rightarrow^a x$.

If the partial order (as defined in $L$) is not taken into account, then we obtain a linear space which will be denoted by $\ell(L)$; similarly, if we disregard the multiplication of elements of $L$ by reals, then we get a lattice ordered group; we denote it by $G(L)$.

The set of all reals will be denoted by $\mathbb{R}$. The symbol $0$ denotes both the real number zero and the neutral element of $L$; the meaning of this symbol will be clear from the context. For $(a_n) \in \mathbb{R}^N$ and $a \in \mathbb{R}$ the symbol $a_n \rightarrow a$ has the usual meaning.

1.1. Definition. (Cf., e.g., [15].) A nonempty subset $a$ of $L^N \times L$ will be said to be a convergence in $\ell(L)$ if it satisfies the following conditions:

(i) If $x_n \rightarrow^a x$ and if $(y_n)$ is a subsequence of $(x_n)$, then $y_n \rightarrow^a x$.
(ii) If $x_n \rightarrow^a x$ and $x_n \rightarrow^a y$, then $x = y$.
(iii) If $x_n \rightarrow^a x$ and $y_n \rightarrow^a y$, then $x_n + y_n \rightarrow^a x + y$.
(iv) If $x_n \rightarrow^a x$ and $a \in \mathbb{R}$, then $ax_n \rightarrow^a ax$.
(v) If $x \in L$, $(a_n) \in \mathbb{R}^N$, $a \in \mathbb{R}$ and $a_n \rightarrow a$, then $a_n x \rightarrow^a ax$.

The system of all convergences in $\ell(L)$ will be denoted by $\text{Conv } \ell(L)$.

1.2. Definition. (Cf. [3].) A nonempty subset $a$ of $L^N \times L$ will be said to be a convergence in $G(L)$ if it satisfies the conditions (i), (ii), (iii) from 1.1, and if also the following conditions are fulfilled:

(i) If $((x_n), x) \in L^N \times L$ and if each subsequence $(y_n)$ of $(x_n)$ has a subsequence $(z_n)$ such that $z_n \rightarrow^a x$, then $x_n \rightarrow^a x$.
(ii) If $x \in L$ and $(x_n) = \text{const } x$, then $x_n \rightarrow^a x$.
(iii) If $x_n \rightarrow^a x$, then $-x_n \rightarrow^a -x$.
(iv) If $x_n \rightarrow^a x$ and $y_n \rightarrow^a y$, then $x_n \wedge y_n \rightarrow^a x \wedge y$ and $x_n \vee y_n \rightarrow^a x \vee y$.
(v) If $x_n \rightarrow^a x$, $y_n \rightarrow^a x$, $(z_n) \in L^N$ and $x_n \leq z_n \leq y_n$ for each $n \in \mathbb{N}$, then $z_n \rightarrow^a x$. 

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The system of all convergences in $G(L)$ will be denoted by $\text{Conv}_g L$.

Let us remark that in the paper [14] the Urysohn property $(i_1)$ (which will be systematically applied below) was not assumed to be valid when investigating a sequential convergence in a lattice ordered group.

We denote by $d$ the system of all elements $((x_n), x) \in L^N \times L$ having the property that there is $m \in \mathbb{N}$ such that $x_n = x$ for each $n \geq m$. It is easy to verify that $d$ belongs to $\text{Conv}_g L$, hence $\text{Conv}_g L$ is nonempty. The system $\text{Conv}_g L$ will be considered to be partially ordered by inclusion. It is obvious that $d$ is the least element of $\text{Conv}_g L$.

Let us remark that the conditions $(i)$, $(ii)$, $(iii)$, $(i_1)$, $(ii_1)$ and $(iii_1)$ define a convergence group in the sense of [18] or a FLUSH convergence on the corresponding group (cf. [17]).

1.3. **Definition.** A nonempty subset $\alpha$ of $L^N \times L$ will be said to be a convergence in $L$ if $\alpha \in \text{Conv}_L \cap \text{Conv}_g L$. The system of all convergences in $L$ will be denoted by $\text{Conv} L$. If $\text{Conv} L \neq \emptyset$, then the set $\text{Conv} L$ will be partially ordered by inclusion.

The vector lattice $L$ is said to be archimedean if, whenever $x, y \in L$ and $0 \leq nx \leq y$ for each $n \in \mathbb{N}$, then $x = 0$.

1.4. **Lemma.** Let $L$ be non-archimedean. Then $\text{Conv} L = \emptyset$.

**Proof.** There exist $x, y \in L$ such that $0 < nx \leq y$ for each $n \in \mathbb{N}$. By way of contradiction, assume that $\alpha \in \text{Conv} L$. Because $\frac{1}{n} \rightarrow 0$ in $R$, in view of 1.1, (v) we infer that $\frac{1}{n} y \rightarrow_\alpha 0$. Since $0 < x \leq \frac{1}{n} y$ for each $n \in \mathbb{N}$, according to $(ii_1)$ and $(v_1)$ of 1.2 the relation $x_n \rightarrow_\alpha x$ is valid, where $(x_n) = \text{const } x$. Thus in view of $(ii_1)$ and (ii) we have arrived at a contradiction. \[ \square \]

1.5. **Lemma.** Let $\alpha \in \text{Conv}_g L$. Then $\alpha$ satisfies the condition $(iv)$ from 1.1.

**Proof.** Let $x_n \rightarrow_\alpha x$ and let $a \in R$. There is $m \in \mathbb{N}$ with $|a| \leq m$. We have

$$x_n \rightarrow_\alpha x \Rightarrow |x_n - x| \rightarrow_\alpha 0,$$

whence in view of (iii) and by induction we get $m|x_n - x| \rightarrow_\alpha 0$. Since

$$0 \leq |ax_n - ax| = |a| |x_n - x| \leq m|x_n - x|,$$

according to $(v_1)$ we obtain $|ax_n - ax| \rightarrow_\alpha 0$, thus $ax_n \rightarrow_\alpha ax$. \[ \square \]

1.6. **Corollary.** Let $\alpha \in \text{Conv}_g L$. Then $\alpha \in \text{Conv} L$ if and only if $\alpha$ satisfies the condition $(v)$ from 1.1.

If $L \neq \{0\}$, then the element $d$ of $\text{Conv}_g L$ does not satisfy the condition $(v)$ of 1.1. Hence if $L \neq \{0\}$, then $\text{Conv}_g L$ fails to be a subset of $\text{Conv} L$.

The positive cone $\{x \in L : x \geq 0\}$ of $L$ will be denoted by $L^+$. Under the inherited partial order and the operation $+$, $L^+$ is a lattice ordered semigroup.
1.7. Definition. A convex subsemigroup $\beta$ of $(L^+)^N$ will be said to be a 0-convergence in $G(L)$ if the following conditions are satisfied:

(I) If $(g_n) \in \beta$, then each subsequence of $(g_n)$ belongs to $\beta$.
(II) If $(g_n) \in (L^+)^N$ and if each subsequence of $(g_n)$ has a subsequence belonging to $(\beta)$, then $(g_n)$ belongs to $\beta$.
(III) Let $x \in L^+$. Then const $x$ belongs to $\beta$ if and only if $x = 0$.

The system of all 0-convergences in $G(L)$ will be denoted by 0-Conv$_g$L. Let $d_0$ be the set of all $(x_n) \in (L^+)^N$ such that $((x_n), 0) \in d$. Then $d_0 \subseteq 0$-Conv$_g$L. Hence 0-Conv$_g$L is partially ordered by inclusion.

Let $\alpha \in \text{Conv}_g L$. Put

$$\varphi_1(\alpha) = \{ (|x_n - x|): x_n \to_x x \}.$$  

Conversely, let $\beta \in 0$-Conv$_g$L. Denote

$$\varphi_2(\beta) = \{ ((x_n), x): (|x_n - x|) \in \beta \}.$$

1.8. Lemma. (Cf. [4], Lemma 1.4 and Theorem 1.6.) $\varphi_1$ and $\varphi_2$ are inverse isomorphisms of Conv$_g$L onto 0-Conv$_g$L, or of 0-Conv$_g$L onto Conv$_g$L, respectively.

1.9. Definition. A nonempty subset $\beta$ of $(L^+)^N$ will be said to be a 0-convergence in $L$ if $\beta \subseteq \text{Conv}_g L$ and if, moreover, the following condition is satisfied:

(IV) If $x \in L$ and $a_n \to 0$ in $R$, then $(a_n, x) \in \beta$.

Let 0-Conv$L$ be the set of all 0-convergences in $L$. If this set is nonempty, then it will be considered to be partially ordered by inclusion.

Now let $\alpha$ and $\beta$ run over the set Conv$L$ or 0-Conv$L$, respectively, and let $\varphi_1$ and $\varphi_2$ be defined as in (1) and (2). Then by a routine proof and by using 1.5 we obtain the following result which is analogous to 1.8:

1.10. Lemma. (i) Conv$L = 0$ $\Rightarrow$ 0-Conv$L = 0$. (ii) If Conv$L \neq 0$, then $\varphi_1$ and $\varphi_2$ are inverse isomorphisms of Conv$L$ onto 0-Conv$L$, or of 0-Conv$L$ onto Conv$L$, respectively.

As we remarked in the introduction, we are interested in studying the partially ordered system Conv$L$. Now, in view of 1.10, it suffices to investigate the system 0-Conv$L$. Next, according to 1.4, it suffices to consider the case when $L$ is archimedean.
2. Regular sets

In what follows we assume that $L$ is an archimedean vector lattice. Let $\emptyset \neq A \subseteq (L^+)^\mathbb{N}$. The set $A$ will be said to be regular with respect to $G(L)$ (or $L$, respectively) if there is $\alpha \in 0\text{-Conv}_g L$ (or $\alpha \in 0\text{-Conv} L$) such that $A \subseteq \alpha$.

2.1. Lemma. Let $\emptyset \neq A \subseteq (L^+)^\mathbb{N}$. Then the following conditions are equivalent:

(i) $A$ fails to be regular with respect to $G(L)$.

(ii) There exist $0 < z \in L$, positive integers $m, k$, elements $(y_1^k), \ldots, (y_m^k)$ of $A$ and subsequences $(x_1^k), \ldots, (x_m^k)$ of $(y_n^k)$ such that

$$z \leq m(x_1^k \lor x_2^k \lor \ldots \lor x_m^k) \quad \text{for each } n \in \mathbb{N}.$$

Proof. The implication (ii)$\Rightarrow$(i) is obvious. Let (i) be valid. In view of the results of [4] (cf. also [10], Proposition 2.1) there exist $0 < z \in L$, positive integers $m, k$, elements $(y_1^k), \ldots, (y_m^k)$ of $A$ and subsequences $(x_1^k), \ldots, (x_m^k)$ of $(y_n^k)$ such that

$$z \leq m_1(x_1^k + x_2^k + \ldots + x_m^k) \quad \text{for each } n \in \mathbb{N}.$$

Hence according to Lemma 2.4, [10] there is $m \in \mathbb{N}$ with

$$z \leq m(x_1^k \lor x_2^k \lor \ldots \lor x_m^k) \quad \text{for each } n \in \mathbb{N}.$$

$\Box$

Let $A_0$ be the set of all sequences $(x_n)$ in $L$ having the property that there are $0 \leq x \in L$ and $(a_n) \in (\mathbb{R}^+)^\mathbb{N}$ such that $a_n \to 0$ in $\mathbb{R}$ and $x_n = a_n x$ for each $n \in \mathbb{N}$.

2.2. Lemma. The set $A_0$ is regular with respect to $G(L)$ and also with respect to $L$.

Proof. By way of contradiction, assume that $A_0$ fails to be regular with respect to $G(L)$. Then the condition (ii) from 2.1. holds for $A_0$.

For each $i \in \{1, 2, \ldots, k\}$ there are $0 < x_i^k \in L$ and $(a_n^i) \in (\mathbb{R}^+)^\mathbb{N}$ such that $a_n^i \to 0$ in $\mathbb{R}$ and

$$x_n^i = a_n^i x^i \quad \text{for each } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ we put $a_n = \max\{a_n^1, a_n^2, \ldots, a_n^k\}$. Then $a_n \to 0$ in $\mathbb{R}$ and

$$0 < z \leq m(x_1 \lor x_2 \lor \ldots \lor x_n) = m(a_n^1 x^1 \lor \ldots \lor a_n^k x^k) \leq ma_n(x^1 \lor \ldots \lor x^k) \quad \text{for each } n \in \mathbb{N}.$$
Next, for each \( n \in \mathbb{N} \) there is \( n(1) \in \mathbb{N} \) such that \( ma_{n(1)} < \frac{1}{n} \), hence
\[
0 < z < \frac{1}{n}(x^1 \lor \ldots \lor x^k) \quad \text{for each } n \in \mathbb{N}.
\]
Thus \( nz < x^1 \lor \ldots \lor x^k \) for each \( n \in \mathbb{N} \), which is impossible, because \( L \) is archimedean. Thus there is \( \alpha \in 0\text{-Conv}_{0} L \) with \( A_0 \subseteq \alpha \). Then \( \alpha \) fulfills the condition (IV), hence \( \alpha \in 0\text{-Conv}_{0} L \).

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\begin{proof}
\end{proof}
\]

2.3. **Theorem.** Let \( L \) be an archimedean vector lattice. Then \( \text{Conv } L \neq \emptyset \).

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\begin{proof}
\end{proof}
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2.4. **Lemma.** Let \( \alpha \in 0\text{-Conv}_{0} L \). Then \( A_0 \subseteq \alpha \).

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\begin{proof}
\end{proof}
\]

2.5. **Corollary.** Let \( I \) be a nonempty set and for each \( i \in I \) let \( \alpha_i \in 0\text{-Conv}_{0} L \) . Then \( \emptyset \neq \bigcap_{i \in I} \alpha_i \in 0\text{-Conv}_{0} L \).

Let us denote by \( d^0 \) the intersection of all \( \alpha_i \in 0\text{-Conv}_{0} L \) with \( A_0 \subseteq \alpha_i \) (such \( \alpha_i \) do exist in view of 2.2). According to 2.4 and 2.5 we obtain:

2.6. **Corollary.** \( d^0 \) is the least element of \( 0\text{-Conv}_{0} L \). If \( \alpha \in 0\text{-Conv}_{0} L \), then the interval \( [d^0, \alpha] \) of the partially ordered set \( 0\text{-Conv}_{0} L \) is a complete lattice.

2.7. **Proposition.** \( d^0 = A_0 \).

\[
\begin{proof}
\end{proof}
\]

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For each $X \subseteq (L^+)^N$ let us denote by $X^*$ the set of all $(x_n) \in (L^+)^N$ such that each subsequence of $(x_n)$ has a subsequence which belongs to $X$.

Let $A_1$ be the set of all $(x_n) \in (L^+)^N$ which have the following property: there exist $0 \leq x \in L$ and $m \in \mathbb{N}$ such that $x_n \leq \frac{1}{m}x$ for each $n \geq m$.

Another constructive characterization of $d^0$ is given by the following lemma.

2.8. Lemma. $d^0 = A_1^*$.

Proof. Since $A_1 \subseteq A_0$, we clearly have $A_1^* \subseteq d^0$. Let $(x_n) \in d^0$. In view of 2.7 there are $x \in L^+$ and $(a_n) \in (R^+)^N$ such that $x_n = a_n x$ for each $n \in \mathbb{N}$. Let $(y_n)$ be a subsequence of $(x_n)$ and let $(b_n)$ be the corresponding subsequence of $(a_n)$; hence $y_n = b_n x$ for each $n \in \mathbb{N}$. There exists a subsequence $(c_n)$ of $(b_n)$ such that $c_n \leq \frac{1}{m}$ for each $n \in \mathbb{N}$. Put $z_n = c_n x$ for each $n \in \mathbb{N}$. Then $(c_n x)$ is a subsequence of $(y_n)$ and $(c_n x) \in A_1$. Hence $(x_n) \in A_1^*$ and thus $d^0 \subseteq A_1^*$. \hfill \Box

2.9. Proposition. There exists an archimedean vector lattice $L$ such that $0\text{-Conv} L$ has no greatest element.

Proof. It suffices to apply an analogous example as in [3], Section 5 (with the distinction that the real functions under consideration in the example are not assumed to be integer valued). \hfill \Box

2.10. Theorem. Let $L$ be an archimedean vector lattice. Suppose that $L$ is $(\aleph_0, 2)$-distributive. Then $0\text{-Conv} L$ possesses a greatest element.

Proof. This is a consequence of 2.6 and of the fact that $0\text{-Conv}_g L$ has a greatest element (cf. [12]). \hfill \Box

Lemma 1.10 and Lemma 2.6 yield that each interval of the partially ordered set $0\text{-Conv} L$ is, at the same time, an interval of $0\text{-Conv}_g L$. Hence in view of [5], Theorem 2.5 we obtain:

2.11. Proposition. Each interval of $0\text{-Conv} L$ is a Brouwerian lattice.

3. The sets of the form $\alpha \cup A_0$

Let $\emptyset \neq \alpha \subseteq (L^+)^N$ be such that $\alpha$ is regular with respect to $G(L)$. We shall investigate the problem whether the set $\alpha \cup A_0$ is regular with respect to $L$.

First we shall deal with the case when $L$ is a projectable vector lattice. (Projectable lattice ordered groups and vector lattices were studied by several authors; cf. e.g., [2] and [16].)

For the sake of completeness we recall the following notions.

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Let $L$ be a vector lattice and $X \subseteq L$. We put

$$X^d = \{ y \in L : |y| \wedge |x| = 0 \quad \text{for each } x \in X \}.$$ 

Then $X^d$ is said to be a polar of $L$. The vector lattice $L$ is called projectable if for each $x \in L$, the set $\{x^d\}$ is a direct factor of $L$.

An element $e \in L$ is called a strong unit of $L$ if for each $x \in L$ there is $n \in \mathbb{N}$ such that $x \leqne$.

Since each strong unit of an archimedean vector lattice $L_1$ is, at the same time, a strong unit of the Dedekind completion of $L_1$, we have

**3.1. Proposition.** (Cf., e.g., [19], Theorem V.3.1) Let $L_1$ be an archimedean vector lattice having a strong unit. Then there is a set $I$ such that there exists an isomorphism of $L_1$ into the vector lattice $\prod_{i \in I} R_i$, where $R_i = \mathbb{R}$ for each $i \in I$.

**3.2. Lemma.** Let $\alpha \in \text{Conv}_2 L$. Then the following conditions are equivalent:

(i) The set $\alpha \cup A_0$ fails to be regular with respect to $G(L)$.

(ii) There are $t, z \in L$ and $(z_n) \in \alpha$ such that $0 < z \leq t$ and

$$z = z_n \vee \left( z \wedge \frac{1}{n} \right) \quad \text{for each } n \in \mathbb{N}.$$

**Proof.** According to 2.1, (ii)$\Rightarrow$(i). Suppose that (i) is valid. Thus in view of 2.7 and 2.8, the set $\alpha \cup A_1$ fails to be regular with respect to $G(L)$. Hence the condition (ii) from 2.1 holds, where $A = \alpha \cup A_1$.

If $(x_n^1), \ldots, (x_n^k) \in \alpha$, then $\alpha$ would not be regular with respect to $G(L)$, which is a contradiction. If $(x_n^1), \ldots, (x_n^k) \in A_1$, then we obtain a contradiction with respect to 2.2. Hence without loss of generality we can suppose that there is $k(1) \in \mathbb{N}$ with

$$1 < k(1) < k$$

such that

$$(x_n^1), \ldots, (x_n^{k(1)}) \in \alpha \quad \text{and} \quad (x_n^{k(1)+1}), \ldots, (x_n^k) \in A_1.$$ 

Put $z_n = m(x_n^1 \vee \ldots \vee x_n^{k(1)})$ for each $n \in \mathbb{N}$. Then $(z_n) \in \alpha$.

For each $j \in \{k(1)+1, \ldots, k\}$ there are $0 < y^j \in L$ and $(a_n^j) \in (\mathbb{R}^+)^k$ such that $a_n^j \to 0$ in $\mathbb{R}$ and $y_n^j = a_n^j y^j$ for each $n \in \mathbb{N}$. Denote

$$a_n = \max\{a_n^{k(1)+1}, \ldots, a_n^k\}, \quad t = y^{k(1)+1} \vee \ldots \vee y^k.$$ 

There is a subsequence $(n(1))$ of the sequence $(n)$ such that

$$ma_{n(1)} < \frac{1}{n} \quad \text{for each } n \in \mathbb{N}.$$
Hence we have
\[ m\left(x_n^{(k+1)} \vee \ldots \vee x_n^{(k)}\right) \leq \frac{1}{m} \] for each \( n \in \mathbb{N} \).

Therefore
\[ 0 < z \leq z_n^{(k)} \vee \frac{1}{n} \] for each \( n \in \mathbb{N} \).

Because \((z_n^{(k)}) \in \alpha\), it suffices to write \( z_n \) instead of \( z_n^{(k)} \). Thus

\[ (1) \quad z = z \land (z_n \lor \frac{1}{n}) = (z \land z_n) \lor (z \land \frac{1}{n}) \] for each \( n \in \mathbb{N} \).

If \( z \land t = 0 \), then \( z \land \frac{1}{n} = 0 \) for each \( n \in \mathbb{N} \), whence \( z \leq z_n \) for each \( n \in \mathbb{N} \) and thus \( \alpha \) fails to be regular, which is a contradiction. Therefore \( z \land t > 0 \) and then, without loss of generality, we can take \( z \land t \) instead of \( z \); hence we have \( z \leq t \). Next, \((z \land z_n) \in \alpha\), thus without loss of generality we can take \((z \land z_n)\) instead of \((z_n)\).

Hence in view of (1) we infer that (ii) is valid. \( \square \)

3.3. Proposition. Assume that \( L \) is projectable. Let \( \alpha \in 0-\text{Conv}_g L \). Then \( \alpha \cup A_0 \) is regular with respect to \( L \).

Proof. In view of 2.7 it suffices to verify that \( \alpha \cup A_0 \) is regular with respect to \( G(L) \).

By way of contradiction, suppose that \( \alpha \cup A_0 \) fails to be regular with respect to \( G(L) \). Then the condition (ii) from 3.2 is valid. There exists \( m \in \mathbb{N} \) such that \( z \notin \frac{1}{m} \). Thus

\[ (1') \quad z^0 = (z - \frac{1}{m})^+ > 0. \]

Let us denote by \( P \) the polar of \( L \) generated by \( z^0 \); i.e., \( P = \{z^0\}^{ud} \). Since \( L \) is projectable, \( P \) is a direct factor in \( L \). For each \( g \in L \) let \( g(P) \) be the component of \( g \) in \( P \). In view of the condition (ii) of 3.2 we have

\[ (2) \quad z(P) = z_n(P) \lor (z(P) \land \frac{1}{n}(P)) \] for each \( n \in \mathbb{N} \).

If \( z(P) = 0 \), then \( z^0 = z^0(P) = 0 \), which is a contradiction. Thus \( z(P) > 0 \). Next, from \( z \leq t \) we infer that \( z(P) \leq t(P) \).

Let \( L_1 \) be the convex \( t \)-subgroup of \( G(P) \) generated by the element \( t(P) \). Then \( t(P) \) is a strong unit of \( L_1 \) and \( L_1 \) is a linear subspace of \( L \). Let \( I \) and \( \varphi \) be as in 3.2. For each \( i \in I \) we have \( \varphi(z(P))(i) \geq 0 \). According to the definition of \( P \) we obtain

\[ (z - \frac{1}{m})^- \in P^d \]
whence \((z - \frac{1}{m}t)(P) = z_0(P)\). In view of (1'),
\[(3) \quad 0 < z^0 = z^0(P) = z(P) - \frac{1}{m}t(P),\]
hence the set \(I_1 = \{i \in I : \varphi(z(P))(i) > 0\}\) is nonempty.

Let \(i \in I_1\) and \(n > m\). According to (3),
\[(4) \quad \varphi(z(P))(i) \geq \frac{1}{m}\varphi(t(P))(i),\]
Also, in view of (2),
\[
\varphi(z(P))(i) = \varphi(z_n(P))(i) \lor (\varphi(z(P))(i) \land \frac{1}{m}\varphi(t(P))(i)) \\
= \max\{\varphi(z_n(P))(i), \min\{\varphi(z(P))(i), \frac{1}{m}\varphi(t(P))(i)\}\}.
\]
Thus according to (4),
\[
\varphi(z(P))(i) = \max\{\varphi(z_n(P))(i), \frac{1}{m}\varphi(t(P))(i)\}.
\]
By applying (4) again we get
\[
\varphi(z(P))(i) = \varphi(z_n(P))(i).
\]
Therefore \(\varphi(z(P))(i) = \varphi(z_n(P))(i)\) for each \(i \in I\). Hence
\[(5) \quad 0 < z(P) = z_n(P) \quad \text{for each } n > m.\]

Since \(z_n(P) \leq z_n\) for each \(n \in \mathbb{N}\) and since \((z_n)\) is regular with respect to \(L\), we infer that \((z_n(P))\) is regular with respect to \(L\). Thus in view of (5) we have arrived at a contradiction. \(\Box\)

Now let us drop the assumption that \(L\) is projectable. We denote by \(L'\) the Dedekind completion of \(L\). It is well-known that \(L'\) is projectable.

**3.4. Lemma.** Let \(\emptyset \neq \alpha \subseteq (L^+)^\mathbb{N}\). Assume that \(\alpha\) is regular with respect to \(G(L)\). Then \(\alpha\) is regular with respect to \(G(L')\).

**Proof.** By way of contradiction, assume that \(\alpha\) fails to be regular with respect to \(G(L')\). Then the condition (ii) from 2.1 holds (with the distinction that \(z \in L'\) and \(A\) is replaced by \(\alpha\)). There exists \(0 < z_1 \in L\) with \(z_1 \leq z\). But by applying 2.1 again we infer that \(\alpha\) fails to be regular with respect to \(L\), which is a contradiction. \(\Box\)

**3.5. Lemma.** Let \(\emptyset \neq \alpha \subseteq (L^+)^\mathbb{N}\). Assume that \(\alpha\) is regular with respect to \(G(L)\). Then \(\alpha\) is regular with respect to \(G(L)\).

**Proof.** This is an immediate consequence of 2.1. \(\Box\)
3.6. **Theorem.** Let \( \emptyset \neq \alpha \subseteq (\mathbb{L}^+)^n \). Assume that \( \alpha \) is regular with respect to \( G(\mathbb{L}) \). Then \( \alpha \cup A_0 \) is regular with respect to \( G(\mathbb{L}) \) and with respect to \( L \).

**Proof.** In view of 3.4, \( \alpha \) is regular with respect to \( G(\mathbb{L}') \). Because \( G(\mathbb{L}') \) is projectable, according to 3.3 we obtain that \( \alpha \cup A_0 \) is regular with respect to \( G(\mathbb{L}') \). Thus 3.5 yields that \( \alpha \cup A_0 \) is regular with respect to \( G(\mathbb{L}) \). Now it follows from 2.7 that \( \alpha \cup A_0 \) is regular with respect to \( L \). \( \square \)

3.7. **Corollary.** Let \( \alpha \in 0\text{-Conv}_g L \). Then \( \alpha \lor d^0 \) does exist in \( 0\text{-Conv}_g L \) and in \( 0\text{-Conv} L \).

3.8. **Proposition.** The following conditions are equivalent:

(i) \( 0\text{-Conv} L \) has the greatest element.
(ii) \( 0\text{-Conv}_g L \) has the greatest element.

**Proof.** We obviously have \((i) \Rightarrow (ii)\). Let \((i)\) hold and let \( \beta \) be the greatest element of \( 0\text{-Conv} L \). Let \( \alpha \in 0\text{-Conv}_g L \). According to 3.7, the element \( \alpha \lor d^0 \) does exist in \( 0\text{-Conv} L \). Thus \( \alpha \leq \alpha \lor d^0 \leq \beta \). Hence \( \beta \) is the greatest element of \( 0\text{-Conv}_g L \). \( \square \)

3.9. **Corollary.** Let \( 0\text{-Conv} L \) have the greatest element. Then \( 0\text{-Conv} L \) is a complete lattice and \( 0\text{-Conv}_g L \) is a principal dual ideal of \( 0\text{-Conv}_g L \) generated by the element \( d^0 \).

Let us remark that if \( L_1 \) is a convex \( \ell \)-subgroup of \( G(\mathbb{L}) \), then it is a linear subspace of \( L \).

3.10. **Theorem.** There exists a convex \( \ell \)-subgroup \( L_1 \) of \( G(\mathbb{L}) \) such that the following conditions are satisfied:

(i) \( \text{Conv} L_1 \) is a complete lattice.
(ii) If \( L_2 \) is a convex \( \ell \)-subgroup of \( G(\mathbb{L}) \) such that \( \text{Conv} L_2 \) is a complete lattice, then \( L_2 \leq L_1 \).

**Proof.** This follows from 3.8 and from [10], Theorem 5.5. \( \square \)

Let \( L_1 \) be a vector lattice. If neither the operation + nor the multiplication of elements of \( L_1 \) by reals is taken into account, then we obtain a lattice which will be denoted by \( L_1^0 \).

3.11. **Theorem.** Let \( L_i \) \((i = 1, 2)\) be Archimedean vector lattices. Assume that the lattices \( L_1^0 \) and \( L_2^0 \) are isomorphic and that \( \text{Conv} L_1 \) possesses a greatest element. Then \( \text{Conv} L_2 \) possesses a greatest element as well.
Proof. According to 1.10, 0-Conv $L_1$ possesses a greatest element. Then in view of 3.8, 0-Conv$_g L$ has a greatest element. Since $L_1^0$ is isomorphic to $L_2^0$, by applying [10], Theorem 3.5 we conclude that 0-Conv$_g L_2$ has a greatest element as well. Now according to 3.8 and 1.10, Conv $L_2$ possesses a greatest element. □

4. DISJOINT SEQUENCES

A sequence $(x_n)$ in $L$ will be said to be disjoint (or orthogonal) if $x_n \wedge x_m = 0$ whenever $n$ and $m$ are distinct positive integers.

The following assertion follows from the results proved in [4].

(A) Assume that $L$ possesses a disjoint sequence all members of which are strictly positive. Then there exist infinitely many elements $\alpha_i$ of 0-Conv$_g L$ such that each $\alpha_i$ is generated by a disjoint sequence.

4.1. Lemma. (Cf. [4],) Let $(x_n)$ be a disjoint sequence in $L$. Then the set $(x_n)$ is regular with respect to $G(L)$.

4.2. Lemma. Let $(x_n)$ be a disjoint sequence in $L$. Then the set $\{(x_n)\} \cup A_0$ is regular with respect to $G(L)$ and with respect to $L$.

Proof. This is a consequence of 4.1 and 3.6. □

If $(x_n) \in (L^+)^N$ and the set $\{(x_n)\}$ is regular in $G(L)$ then the least element $\alpha$ of 0-Conv$_g L$ satisfying the relation $\{(x_n)\} \cup A_0 \subseteq \alpha$ will be denoted by $\alpha(x_n)$.

Let $(x_n)$ be a disjoint sequence in $L$ such that $x_n > 0$ for each $n \in \mathbb{N}$. Then $(x_n) \notin A_0$. On the other hand, $(x_n)$ can belong to $d^0$ (cf. Proposition 4.6 below).

4.3. Lemma. Let $(x_n)$ and $(y_n)$ be disjoint sequences in $L$ such that $x_n \wedge y_m = 0$ for each $m,n \in \mathbb{N}$. Let $y_n > 0$ for each $n \in \mathbb{N}$ and $(y_n) \notin d^0$. Then $(y_n) \notin \alpha(x_n)$.

Proof. By way of contradiction, assume that $y_n \in \alpha(x_n)$. Then in view of [10], Lemma 2.3 there are $m,k \in \mathbb{N}$ and $(z_1^m, \ldots, z_k^m) \in (L^+)^N$ such that each $(z_i^m) (i = 1, 2, \ldots, k)$ is a subsequence of a sequence belonging to $\{(x_n)\} \cup A_0$ and

$$0 < y_n \leq m(z_1^m \vee \ldots \vee z_k^m) \text{ for each } n \in \mathbb{N}.$$ 

Since $(y_n) \notin A_0$, without loss of generality we can assume that $(z_1^m), \ldots, (z_{k-1}^m)$ are subsequences of $(x_n)$ and that $(z_k^m)$ is a subsequence of $(\frac{1}{m} x)$ for some $0 < x \in L$. Thus

$$0 < y_n \leq (mz_1^m \vee \ldots \vee mz_{k-1}^m) \vee \frac{1}{m} x' \text{ for each } n \in \mathbb{N},$$

where $x' = mx$. But $y_n \wedge (mz_1^m \vee \ldots \vee mz_{k-1}^m) = 0$, whence $y_n \leq \frac{1}{m} x'$ for each $n \in \mathbb{N}$. Since $(y_n) \notin d^0$, we have arrived at a contradiction. □
4.4. Theorem. Assume that \( L \) possesses an infinite orthogonal subset. Next, suppose that no disjoint sequence \((x_n)\) in \( L \) with \( x_n > 0 \) for each \( n \in \mathbb{N} \) belongs to \( d^0 \). Then \( 0\text{-Conv} \ L \) is infinite.

Proof. In view of the assumption there are disjoint sequences \((x^i_n)\) \((i \in \mathbb{N})\) in \( L \) such that \( x^i_n > 0 \) for each \( n, i \in \mathbb{N} \), and \( x^i_n \land x^j_m = 0 \) whenever \( m, n, i, j \in \mathbb{N} \) and \( i \neq j \). In view of 4.2 we have \( \alpha(x^i_n) \in 0\text{-Conv}_g L \) for each \( i \in \mathbb{N} \). Let \( i, j \) be distinct elements of \( \mathbb{N} \). According to 4.3, \( \alpha(x^i_n) \neq \alpha(x^j_n) \).

For a relevant result concerning convergences in a lattice ordered group cf. [4].

4.5. Theorem. Assume that \( L \) possesses no infinite orthogonal subset. Then \( 0\text{-Conv} \ L \) is a one-element set.

Proof. The case \( L = \{0\} \) is trivial; let \( L \neq \{0\} \). The system \( 0\text{-Conv}_g L \) was described in [4], Section 6. According to [4], if \( \alpha \in 0\text{-Conv}_g L \) and \( \left( \frac{1}{n}x \right) \in \alpha \) for each \( 0 < x \in L \), then \( \alpha \) is the greatest element of \( 0\text{-Conv}_g L \); hence only this greatest element of \( 0\text{-Conv}_g L \) can belong to \( 0\text{-Conv} \ L \).

4.6. Proposition. Assume that \( L \) is orthogonally complete. Then each disjoint sequence in \( L \) belongs to \( d^0 \).

Proof. Let \((x_n)\) be a disjoint sequence in \( L \). Then \((nx_n)\) is disjoint as well. Since \( L \) is orthogonally complete, there exists \( x = \bigvee nx_n \) in \( L \). For each \( n \in \mathbb{N} \) we have \( 0 \leq x_n \leq \frac{1}{n}x \), whence \((x_n) \in d^0 \).

4.7. Corollary. The assertion (A) does not hold in general if \( 0\text{-Conv}_g L \) is replaced by \( 0\text{-Conv} \ L \).

4.8. Proposition. Assume that \( L \neq \{0\} \) has a strong unit and that \((x_n)\) is a disjoint sequence in \( L \) such that \( x_n > 0 \) for each \( n \in \mathbb{N} \). Then there is a sequence \((a_n)\) with \( a_n \in \mathbb{N} \) for each \( n \in \mathbb{N} \) having the property that \( (a_nx_n) \notin d^0 \).

Proof. Let \( e \) be a strong unit in \( L \). Since \( L \) is archimedean, for each \( n \in \mathbb{N} \) there is \( a_n \in \mathbb{N} \) such that

\[
a_n x_n \not\leq e.
\]

By way of contradiction, assume that \((a_nx_n) \in d^0 \). Hence in view of 2.8 there is a subsequence \((b_ny_n)\) of \((a_nx_n)\) such that \((b_ny_n) \in A_1 \). Thus there are \( m \in \mathbb{N} \) and \( 0 < x \in L \) such that \( b_ny_n \leq \frac{1}{n}x \) for each \( n \geq m \). Next, since \( e \) is a strong unit in \( L \), there is \( k \in \mathbb{N} \) with \( x \leq ke \). Thus

\[
b_ny_n \leq \frac{k}{n}e \quad \text{for each } n \geq m.
\]

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Hence for \( n > \max\{m, k\} \) we have \( b_n y_n \leq c \). But in view of (1) the relation \( b_n y_n \not\leq c \) is valid for each \( n \in \mathbb{N} \), which is a contradiction. \( \square \)

4.9. Proposition. Assume that \( L \) has a strong unit. Then (A) is valid with \( \text{Conv}_g L \) replaced by \( \text{Conv} L \).

Proof. This is a consequence of 4.3 and 4.8. \( \square \)

5. Direct sums of linearly ordered vector lattices

Let us denote by \( S \) the class of all archimedean vector lattices which can be expressed as the direct sum of linearly ordered vector lattices. Next, let \( \mathcal{L} \) be the class of all linearly ordered vector lattices.

In this section it will be shown that if \( L \in S \), then \( 0-\text{Conv} L \) is a complete lattice which has no atom.

The case \( L = \{0\} \) being trivial, we assume in the present section that \( L \) is a nonzero archimedean vector lattice which can be represented as

\[
L = \sum_{i \in I} L_i, \quad \text{where } L_i \in \mathcal{L} \text{ for each } i \in I.
\]

Also, without loss of generality we can suppose that \( L_i \neq \{0\} \) for each \( i \in I \).

5.1. Proposition. \( 0-\text{Conv} L \) is a complete lattice.

Proof. From (1) it follows that \( L \) is completely distributive. Hence in view of 2.10, \( 0-\text{Conv} L \) possesses a greatest element. Thus \( 0-\text{Conv} L \) is a complete lattice. \( \square \)

5.2. Lemma. Let \( (x_n) \) be a disjoint sequence in \( L \) such that \( x_n > 0 \) for each \( n \in \mathbb{N} \). Then \( (x_n) \) is not upper-bounded in \( L \).

Proof. This is an immediate consequence of (1). \( \square \)

In view of 5.2 and 2.8 we obtain

5.3. Corollary. Let \( (x_n) \) be as in 5.2. Then \( (x_n) \) does not belong to \( d^0 \).

5.4. Proposition. Let \( I \) be finite. Then \( 0-\text{Conv} L \) is a one-element set.

Proof. From (1) we infer that \( L \) has no infinite orthogonal subset. Hence in view of 4.5, \( 0-\text{Conv} L \) is a one-element set. \( \square \)
5.5. Proposition. Let \( I \) be infinite. Then \( 0\text{-Conv} \, L \) is infinite.

**Proof.** According to (1), \( L \) possesses an infinite orthogonal subset. Then 4.4 and 5.3 yield that \( 0\text{-Conv} \, L \) is infinite.

5.6. Lemma. Let \( \alpha \in 0\text{-Conv} \, L \). Assume that \( (x_n) \in \alpha, \, x_n > 0 \text{ for each } n \in \mathbb{N}, \) and that the sequence \((x_n)\) is disjoint. Then \( \alpha \) fails to be an atom of \( 0\text{-Conv} \, L \).

**Proof.** Consider the sequences \((x_{2n})\) and \((x_{2n+1})\). In view of 5.3, \((x_{2n}) \notin d^n\) and \((x_{2n+1}) \notin d^n\). Hence by applying the notation from Section 4 we have

\[
d^0 < \alpha(x_{2n}) \leq \alpha, \quad d^0 < \alpha(x_{2n+1}) \leq \alpha.
\]

Next, according to 4.3, \( \alpha(x_{2n}) \neq \alpha(x_{2n+1}) \). Hence \( \alpha \) cannot be an atom of \( 0\text{-Conv} \, L \).

For \( x \in L \) and \( i \in I \), let \( x(i) \) be the component of \( x \) in \( L_i \). We put \( \text{Sup} \, x = \{ i \in I : x(i) \neq 0 \} \). If \((x_n)\) is a sequence in \( L \), then we denote

\[
\text{Sup}(x_n) = \bigcup_{n \in \mathbb{N}} \text{Sup} \, x_n.
\]

5.7. Lemma. Let \( (x_n) \in (L^+)^\mathbb{N} \) be such that \( \{(x_n)\} \) is regular and suppose that \( \text{Sup}(x_n) \) is finite. Then \( \alpha(x_n) = d^0 \).

**Proof.** In view of the assumption there is a finite subset \( I(1) \) of \( I \) such that \( x_n \in L(1) = \sum_{i \in I(1)} L_i \) for each \( n \in \mathbb{N} \). Then according to 4.5, \((x_n)\) belongs to the least element of \( 0\text{-Conv} \, L(1) \). Next, in view of 2.8, \((x_n)\) belongs to \( d^0 \). Hence \( \alpha(x_n) = d^0 \).

5.8. Lemma. Let \( (x_n) \in (L^+)^\mathbb{N} \) be such that \( \{(x_n)\} \) is regular and suppose that \( \text{Sup}(x_n) \) is infinite. Then \( \alpha(x_n) \) contains a disjoint sequence with strictly positive elements.

**Proof.** Since \( \text{Sup}(x_n) \) is infinite and (1) holds, there is a subsequence \((y_n)\) of \((x_n)\) such that for each \( n \in \mathbb{N} \), \( \text{Sup} \, y_n \) is not a subset of the set

\[
\text{Sup} \, y_1 \cup \ldots \cup \text{Sup} \, y_{n-1}.
\]

Therefore the sequence \((y_n)\) is disjoint and belongs to \( \alpha(x_n) \).

5.9. Theorem. Let \( L \in S \). Then \( 0\text{-Conv} \, L \) has no atom.

**Proof.** By way of contradiction, assume that \( \alpha \) is an atom of \( 0\text{-Conv} \, L \). Then there is \((x_n) \in (L^+)^\mathbb{N} \) such that \( \alpha = \alpha(x_n) \). If \( \text{Sup} \, x_n \) is finite, then 5.7 yields a contradiction. If \( \text{Sup} \, x_n \) is infinite, then by means of 5.8 and 5.6 we arrive at a contradiction.

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References


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