ON EXISTENCE OF KNESER SOLUTIONS OF A CERTAIN CLASS
OF n-TH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

The aim of our paper is to give some conditions for existence of Kneser solutions of the differential equation

\[(L) \quad L(y) = 0,\]

where

\[L(y) = L_n y + \sum_{k=1}^{n-1} P_k(t) L_k y + P_0(t) f(y),\]

\[L_0 y(t) = y(t),\]

\[L_1 y(t) = p_1(t)(L_0 y(t))' = p_1(t) \frac{dy(t)}{dt},\]

\[L_k y(t) = p_k(t)(L_{k-1} y(t))' \quad \text{for} \quad k = 2, 3, \ldots, n - 1,\]

\[L_n y(t) = (L_{n-1} y(t))'.\]
n is an arbitrary positive integer, \( n \geq 2 \), \( P_h(t) \), \( k = 0, 1, \ldots, n - 1 \), \( p_i(t) \), \( i = 1, 2, \ldots, n - 1 \) are real-valued continuous functions on the interval \( I_a = [a, \infty) \), \(-\infty < a < \infty\); \( f(t) \) is a real-valued function continuous on \( E_1 = (-\infty, \infty) \).

If \( n = 1 \), then \( L(y) = L_1y + P_0(t)f(y) = y' + P_0(t)f(y) \), \( P_0(t) \) and \( f(t) \) are real-valued continuous functions on \( I_a \) and on \( E_1 \), respectively.

It is assumed throughout that

(A) \( P_h(t) \leq 0 \), \( p_i(t) > 0 \) for all \( t \in I_a \), \( k = 0, 1, \ldots, n - 1 \), \( i = 1, 2, \ldots, n - 1 \); \( f(0) \neq 0 \), \( f(t) \geq 0 \) for all \( t \in E_1 \); \( P_0(t) \) is not identically zero in any subinterval of \( I_a \); \( n \) is an arbitrary positive integer, \( n \geq 2 \). If \( n = 1 \), then \( P_0(t) \leq 0 \) and \( f(t) \geq 0 \) for all \( t \in I_a \) and \( E_1 \), respectively.

The problems of existence of monotone or Kneser solutions for third order ordinary differential equations with quasi-derivatives were studied in several papers ([5], [7], [8], [10]). The equation (L), where \( p_i(t) = 1 \), \( i = 1, 2, 3 \) \( (n = 4) \) was studied, for example, in ([6], [9], [12]). Equations of the fourth order with quasi-derivatives were also studied, for instance, in ([11], [3], [13]).

Existence of monotone solutions for \( n \)-th order equations with quasi-derivatives was studied in [4].

In our paper, Theorem 1 and Theorem 2 give sufficient conditions for existence of a Kneser solution of (L) on \( [a, \infty) \) for an even number or for an odd one, respectively.

Now we explain the concept of a Kneser solution, and other useful ones:

**Definition 1.** A nontrivial solution \( y(t) \) of a differential equation of the \( n \)-th order is called a Kneser solution on \( I_a = [a, \infty) \) if \( (y(t) > 0, (-1)^kL_ky(t) \geq 0) \) or \( (y(t) < 0, (-1)^kL_ky(t) \leq 0) \) for all \( t \in I_a \), \( k = 1, 2, \ldots, n - 1 \).

**Definition 2.** Let \( J \) be an arbitrary type of an interval with endpoints \( t_1, t_2 \), where \(-\infty \leq t_1 < t_2 \leq \infty \). The interval \( J \) is called the maximum interval of existence of \( u: J \to E^n \), where \( u(t) \) is a solution of the differential system \( u' = F(t, u) \) if \( u(t) \) can be continued neither to the right nor to the left of \( J \).

**Definition 3.** Let \( y' = U(t, y) \) be a scalar differential equation. Then \( y_0(t) \) is called the maximum solution of the Cauchy problem

\[
(*) \quad y' = U(t, y), \quad y(t_0) = y_0
\]

if \( y_0(t) \) is a solution of \( (*) \) on the maximum interval of existence and if \( y(t) \) is another solution of \( (*) \), then \( y(t) \leq y_0(t) \) for all \( t \) belonging to the common interval of existence of \( y(t) \) and \( y_0(t) \).

We give some preliminary results.
Lemma 1. Let $A(t, s)$ be a nonpositive and continuous function for $a \leq t \leq s \leq t_0$. If $g(t)$, $\psi(t)$ are continuous functions in the interval $[a, t_0]$ and

$$
\psi(t) \geq g(t) + \int_{t_0}^{t} A(t, s)\psi(s) \, ds \quad \text{for } t \in [a, t_0],
$$

then every solution $y(t)$ of the integral equation

$$
y(t) = g(t) + \int_{t_0}^{t} A(t, s)y(s) \, ds
$$

satisfies the inequality $y(t) \leq \psi(t)$ in $[a, t_0]$.

Proof. See [6], page 331. □

Lemma 2. (Wintner) Let $U(t, u)$ be a continuous function on a domain $t_0 \leq t \leq t_0 + \alpha$, $\alpha > 0$, $u \geq 0$, let $u(t)$ be a maximum solution of the Cauchy problem

$$
u' = U(t, u), \quad u(t_0) = u_0 \geq 0 \quad (u' = U(t, u) \text{ is a scalar differential equation}) \text{ existing on } [t_0, t_0 + \alpha]; \text{ for example, let } U(t, u) = \psi(u), \text{ where } \psi(u) \text{ is a continuous and positive function for } u \geq 0 \text{ such that}
$$

$$
\int_{t_0}^{\infty} \frac{du}{\psi(u)} = \infty.
$$

Let us assume $f(t, y)$ to be continuous on $t_0 \leq t \leq t_0 + \alpha$, $y \in E^n_1$, $y$ arbitrary, and to satisfy the condition

$$
|f(t, y)| \leq U(t, |y|).
$$

Then the maximum interval of existence of a solution of the Cauchy problem

$$
y' = f(t, y), \quad y(t_0) = y_0,
$$

where $|y_0| \leq u_0$, is $[t_0, t_0 + \alpha]$.

Proof. See [2], Theorem III.5.1. □

Lemma 3. Let (A) hold, and let there exist real nonnegative constants $a_1$, $a_2$ such that $f(t) \leq a_1 |t| + a_2$ for all $t \in E^n_1$. Let initial values $L_k y(a) = b_k$ be given for $k = 0, 1, \ldots, n - 1$. Then there exists a solution $y(t)$ of (L) on $[a, \infty)$, which fulfills these initial conditions.

Proof. See [4], Lemma 3. □
2. Results

**Lemma 4.** Let us assume \( g(t, z) \) to be continuous on \( t_0 - \alpha \leq t \leq t_0, \alpha \) a positive constant, \( z \in E^n_1 \), \( z \) is arbitrary and satisfies a condition

\[
|g(t, z)| \leq \psi(|z|),
\]

where \( \psi(t) \) is a continuous and positive function for \( t \geq 0 \) such that

\[
\int_0^\infty \frac{dt}{\psi(t)} = \infty.
\]

Then the maximum interval of existence of a solution of the Cauchy problem

\[
z' = g(t, z), \quad z(t_0) = z_0,
\]

is \([t_0 - \alpha, t_0]\).

**Proof.** Let us consider the Cauchy problem

\[
(u) \quad u' = \psi(u), \quad u(-t_0) = u_0 = |z_0|,
\]

According to the assumptions, the problem \((u)\) admits a single solution \( u_0(t) \) on \([-t_0, \infty)\), where

\[
u_0(t) = R_{-1}(t + t_0)
\]

and \( R: [u_0, \infty) \to [0, \infty), \quad R(u) = \int_{u_0}^u \frac{1}{\psi(t)} dt, \quad R_{-1}(R(u)) = u, \quad u \in [u_0, \infty) \). Let us consider the Cauchy problems

\[
(U) \quad u' = U(t, u) = \psi(u), \quad u(-t_0) = u_0 = |z_0|, \quad (t, u) \in [-t_0, -t_0 + \alpha] \times [0, \infty),
\]

\[
y'(t) = g(-t, -y), \quad y(-t_0) = -z_0, \quad (t, y) \in [-t_0, -t_0 + \alpha] \times E^n_1,
\]

\[
z'(t) = g(t, z), \quad z(t_0) = z_0, \quad (t, z) \in [t_0 - \alpha, t_0] \times E^n_1.
\]

Then \( u_0(t) = R_{-1}(t + t_0) \) is the maximum solution of \((U)\) on the maximum interval of existence \([-t_0, -t_0 + \alpha]\). According to Lemma 2 there exists a solution \( y_0(t) \) of \((y)\) on \([-t_0, -t_0 + \alpha]\). Then the Cauchy problem \((z)\) admits the solution \( z_0(t) = -y_0(-t) \) on \([t_0 - \alpha, t_0]\) because of

\[
z_0'(t) = y_0'(-t) = g(t, -y_0(-t)) = g(t, z_0(t))
\]

on \([t_0 - \alpha, t_0]\). So the maximum interval of existence of \((z)\) is \([t_0 - \alpha, t_0]\). \( \Box \)
Lemma 5. Let (A) hold, and let there exist nonnegative real constants \( a_1, a_2 \) such that \( f(t) \leq a_1 |t| + a_2 \) for all \( t \in E_1 \). Let initial values \( y(t_0) = b_k \) be given for \( k = 0, 1, \ldots, n - 1 \), to \( t > a \). Then there exists a solution \( y(t) \) of (L) on \( [a, \infty) \), which fulfills these initial conditions.

Proof. According to Lemma 3 there exists a solution of (L) on \( [t_0, \infty) \) such that the initial conditions hold. To prove our lemma we need to prove existence of a solution \( y(t) \) of (L) on \( [a, t_0] \) satisfying the given initial conditions. Consider now the following system (S), which corresponds to the equation (L):

\[
\begin{aligned}
    u_k'(t) &= \frac{u_{k+1}(t)}{p_k(t)}, \quad k = 1, 2, \ldots, n - 1, \\
    u_n'(t) &= -\sum_{k=1}^{n-1} P_k(t) u_{k+1}(t) - P_0(t) f(u_1(t)),
\end{aligned}
\]

(S)

where \( u_k(t) = L_{k-1} y(t) \), \( k = 1, 2, \ldots, n \), \( f_k = u_{k+1}/p_k \), \( k = 1, \ldots, n - 1 \), \( f_n = -\sum P_k u_{k+1} - P_0 f(u_1) \), \( F = (f_1, f_2, \ldots, f_n) \), \( u = (u_1, u_2, \ldots, u_n) \), \( u' = (u_1', u_2', \ldots, u_n') \), \( u_n' = |u| = \sum_{k=1}^{n} |u_k| \), \( |F| = \sum_{k=1}^{n} |f_k| \), \((t, u) \in [a, t_0] \times E_1^n \). Then

\[
\begin{aligned}
    |F(t, u)| &= \sum_{k=1}^{n-1} \frac{|u_{k+1}|}{p_k} - \sum_{k=1}^{n-1} P_k u_{k+1} - P_0 f(u_1) \\
    &\leq \sum_{k=1}^{n-1} (-P_k + \frac{1}{p_k}) |u_{k+1}| - P_0 (a_1 |u_1| + a_2) \leq K_1 |u| + K_2 = \psi(|u|),
\end{aligned}
\]

where \( K_1, K_2 \) are appropriate positive real constants. It is obvious that

\[
\int_{E_1}^\infty \frac{ds}{\psi(s)} = \infty
\]

for \( s \in E_1, s > 0 \). Lemma 4 yields existence of a solution of (S) on \( [a, t_0] \). This fact implies existence of a solution \( y(t) \) of the equation (L) on \( [a, t_0] \) which satisfies the given initial conditions. The lemma is proved.

Lemma 6. Let (A) hold, and let \( y(t) \) be a solution of (L) on \( [t_1, \infty) \), where \( t_1 \geq a \). Let (B) hold, where \( (s_0 = s) \)

\[
\sum_{k=1}^{n-1} (-1)^{k-1} M_k(t, s) \leq 0, \quad N_n(t) \leq 0, \quad n \geq 2
\]

(B)
and

\[ M_k(t, s) = \int_0^s \frac{ds_1}{p_{n-k}(s_1)} \int_0^{s_1} \frac{ds_2}{p_{n-k-2}(s_2)} \cdots \int_0^{s_{k-2}} \frac{ds_k}{p_{n-k-1}(s_k)} \cdot \frac{-P_{n-k}(s_{k-1})}{p_{n-1}(s)} \, ds_k, \]

\[ M_1(t, s) = -P_{n-1}(s), \quad N_n = \int_{t_2}^{t_1} \sum_{k=1}^{n-1} \left( -P_{n-k}(s)G_k(s) \right) \, ds, \]

\[ G_k(s) = L_{n-k}y(t_2) + (-1)^kL_{n-k+1}y(t_2) \int_s^{t_2} \frac{ds_1}{p_{n-k-1}(s_1)} + (-1)^2L_{n-k+2}y(t_2) \]

\[ \times \int_s^{t_2} \frac{ds_1}{p_{n-k-1}(s_1)} \int_s^{t_2} \frac{ds_2}{p_{n-k-2}(s_2)} + \cdots + (-1)^{k-2}L_{n-k+2}y(t_2) \]

\[ \times \int_s^{t_2} \frac{ds_1}{p_{n-k-1}(s_1)} \int_s^{t_2} \frac{ds_2}{p_{n-k-2}(s_2)} \cdots \int_s^{t_2} \frac{ds_{k-2}}{p_{n-2}(s_{k-2})} \]

for \( k = 2, 3, \ldots, n-1, G_1(s) = 0. \)

a) Let \( n \) be an even number and \( t_2 \in (t_1, \infty) \) such that \((-1)^kL_{k}y(t_2) \geq 0\) for \( k = 0, 1, \ldots, n-1. \) Then \((-1)^kL_{k}y(t) \geq 0\) for \( t \in [t_1, t_2], k = 0, 1, \ldots, n-1. \)

b) Let \( n \) be an odd number and \( t_2 \in (t_1, \infty) \) such that \((-1)^kL_{k}y(t_2) \leq 0\) for \( k = 0, 1, \ldots, n-1. \) Then \((-1)^kL_{k}y(t) \leq 0\) for \( t \in [t_1, t_2], k = 0, 1, \ldots, n-1. \)

**Proof.** Let \( n \geq 2. \) Integration of the identity \( L_{n-1}y = (L_{n-1}y)' \) over \([t_2, t]\), where \( t_1 \leq t \leq t_2 \) \((n \) can be an even number as well as an odd one) yields

\[ L_{n-1}y(t) = L_{n-1}y(t_2) - \int_{t_2}^{t} \sum_{k=1}^{n-1} P_k(s)L_{k}y(s) \, ds - \int_{t_2}^{t} P_0(s)f(y(s)) \, ds \]

\[ = L_{n-1}y(t_2) + \int_{t_2}^{t} (-P_0(s)f(y(s))) \, ds + \int_{t_2}^{t} \sum_{k=1}^{n-1} (-P_{n-k}(s)L_{n-k}y(s)) \, ds. \]

Let us denote the expression \( L_{n-1}y(t_2) + \int_{t_2}^{t} (-P_0(s)f(y(s))) \, ds \) by \( K_n(t) \). It is obvious that \( K_n(t) \leq 0 \) for all \( t \in [t_1, t_2]. \) We have

\[ L_{n-1}y(t) = K_n(t) + \int_{t_2}^{t} \sum_{k=1}^{n-1} (-P_{n-k}(s)L_{n-k}y(s)) \, ds. \]
It can be proved that

\[
L_{n-k}y(s) = L_{n-k}y(t_2) + \left( -1 \right)^1 L_{n-k+1}y(t_2) \int_{t_2}^{s} \frac{ds_1}{p_{n-k+1}(s_1)} + 
+ \left( -1 \right)^2 L_{n-k+2}y(t_2) \int_{t_2}^{s} \frac{ds_1}{p_{n-k+1}(s_1)} \int_{t_2}^{s_1} \frac{ds_2}{p_{n-k+2}(s_2)} + \ldots + 
+ \left( -1 \right)^{k-2} L_{n-k}y(t_2) \int_{t_2}^{s} \frac{ds_1}{p_{n-k+1}(s_1)} \int_{t_2}^{s_2} \frac{ds_2}{p_{n-k+2}(s_2)} \ldots \int_{t_2}^{s_{k-2}} \frac{ds_{k-1}}{p_{n-k+1}(s_{k-1})} 
\]

for \( k = 2, 3, \ldots, n-1 \). By interchanging the upper and lower bounds in the previous integrals, we have

\[
L_{n-k}y(s) = L_{n-k}y(t_2) + \left( -1 \right)^1 L_{n-k+1}y(t_2) \int_{s}^{t_2} \frac{ds_1}{p_{n-k+1}(s_1)} + 
+ \left( -1 \right)^2 L_{n-k+2}y(t_2) \int_{s}^{t_2} \frac{ds_1}{p_{n-k+1}(s_1)} \int_{s_1}^{t_2} \frac{ds_2}{p_{n-k+2}(s_2)} + \ldots + 
+ \left( -1 \right)^{k-2} L_{n-k}y(t_2) \int_{s}^{t_2} \frac{ds_1}{p_{n-k+1}(s_1)} \int_{s_2}^{t_2} \frac{ds_2}{p_{n-k+2}(s_2)} \ldots \int_{s_{k-2}}^{t_2} \frac{ds_{k-1}}{p_{n-k+1}(s_{k-1})} \int_{t_2}^{s_{k-2}} \frac{ds_{k-2}}{p_{n-k+1}(s_{k-1})} 
\]

Denoting the last \((k - 1)\)-dimensional integral by \( I_k(s) \), the previous sum by \( G_k(s) \),
\( I_1(s) = L_{n-1}y(s) \), \( G_1(s) = 0 \) for \( k = 1, 2, \ldots, n-1 \) \((s_0 = s)\) we obtain

\[
L_{n-k}y(s) = G_k(s) + \left( -1 \right)^{k-1} I_k(s).
\]

Hence

\[
L_{n-1}y(t) = K_n(t) + \int_{t_2}^{t} \sum_{k=1}^{n-1} \left( -P_{n-k}(s) [G_k(s) + \left( -1 \right)^{k-1} I_k(s)] \right) ds,
\]

\[
K_n(t) + \int_{t_2}^{t} \sum_{k=1}^{n-1} \left( -P_{n-k}(s) G_k(s) \right) ds + \int_{t_2}^{t} \sum_{k=1}^{n-1} \left( -P_{n-k}(s) \left( -1 \right)^{k-1} I_k(s) \right) ds.
\]
Denoting $K_n(t) + \int_{t}^{t_1} \sum_{k=1}^{n-1} (-P_{n-k}(s) G_k(s)) \, ds$ by $g_n(t)$ and denoting $\int_{t}^{t_1} (-P_{n-k}(s) \times (-1)^{k-1} I_k(s)) \, ds$ by $(-1)^{k-1} J_k(t)$ we have

$$L_{n-1} y(t) = g_n(t) + \sum_{k=1}^{n-1} (-1)^{k-1} J_k(t),$$

where $J_k(t)$ is the $k$-dimensional integral

$$J_k(t) = - \int_{t}^{t_1} (-P_{n-k}(s)) \, ds \int_{t}^{s} \frac{ds_1}{p_{n-k+1}(s_1)} \int_{s_1}^{t_2} \frac{ds_2}{p_{n-k+2}(s_2)} \cdots \int_{s_{k-2}}^{t_2} \frac{ds_{k-1}}{p_{n-k+1}(s_{k-1})} L_{n-1} y(s_{k-1}) \, ds_{k-1},$$

for $k = 2, 3, \ldots, n-1$ and $J_1(t) = - \int_{t}^{t_2} (-P_{n-1}(s) L_{n-1} y(s)) \, ds$.

By changing the notation of the variables we have

$$J_k(t) = - \int_{t}^{t_1} (-P_{n-k}(s_{k-1})) \, ds_{k-1} \int_{s_{k-1}}^{t_2} \frac{ds_{k-2}}{p_{n-k+1}(s_{k-2})} \int_{s_{k-2}}^{t_2} \frac{ds_{k-3}}{p_{n-k+2}(s_{k-3})} \cdots \int_{s_{2}}^{t_2} \frac{ds_{1}}{p_{n-1}(s_{1})} L_{n-1} y(s_{1}) \, ds_{1}.$$

$J_k(t)$ is a $k$-dimensional integral on a $k$-dimensional domain. This domain can be described as an elementary domain in the following way:

$$t \leq s_{k-1} \leq t_2$$

$$s_{k-1} \leq s_{k-2} \leq t_2$$

$$s_{k-2} \leq s_{k-3} \leq t_2$$

$$\vdots$$

$$s_2 \leq s_1 \leq t_2$$

$$s_1 \leq s \leq t_2,$$

as well as like

$$t \leq s \leq t_2$$

$$t \leq s_1 \leq s$$

$$t \leq s_2 \leq s_1$$

$$\vdots$$

$$t \leq s_{k-2} \leq s_{k-3}$$

$$t \leq s_{k-1} \leq s_{k-2}$$

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for \( k = 2, 3, \ldots, n - 1 \). Hence

\[
J_k(t) = - \int_t^{t_2} L_{n-1}y(s) \, ds \int_t^{s_1} \frac{ds_1}{p_{n-2}(s_1)} \int_t^{s_2} \frac{ds_2}{p_{n-3}(s_2)} \cdots \int_t^{s_{k-2}} \frac{ds_{k-2}}{p_{n-k}(s_{k-1})} \frac{P_{n-k}(s_{k-1})}{p_{n-1}(s)} \, ds_{k-1}.
\]

The last integral can be rewritten into the form

\[
J_k(t) = - \int_t^{t_2} M_k(t, s) L_{n-1}y(s) \, ds = \int_t^{t_2} M_k(t, s) L_{n-1}y(s) \, ds,
\]

where

\[
M_k(t, s) = \int_t^{s_1} \frac{ds_1}{p_{n-2}(s_1)} \int_t^{s_2} \frac{ds_2}{p_{n-3}(s_2)} \cdots \int_t^{s_{k-2}} \frac{ds_{k-2}}{p_{n-k}(s_{k-1})} \frac{P_{n-k}(s_{k-1})}{p_{n-1}(s)} \, ds_{k-1}
\]

for \( k = 2, 3, \ldots, n - 1 \), \( M_1(t, s) = -P_{n-1}(s) \). Hence

\[
L_{n-1}y(t) = g_n(t) + \sum_{k=1}^{n-1} (-1)^{k-1} J_k(t) = g_n(t) + \sum_{k=1}^{n-1} (-1)^{k-1} \int_t^{t_2} M_k(t, s) L_{n-1}y(s) \, ds
\]

\[
= g_n(t) + \int_t^{t_2} \left[ \sum_{k=1}^{n-1} (-1)^{k-1} M_k(t, s) \right] L_{n-1}y(s) \, ds = g_n(t) + \int_t^{t_2} A_n(t, s) L_{n-1}y(s) \, ds,
\]

where \( A_n(t, s) = \sum_{k=1}^{n-1} (-1)^{k-1} M_k(t, s) \). We note that \( s \leq t_2 \), \( s_i \leq t_2 \), \( t \leq s_i \) for \( i = 1, 2, \ldots, n - 3 \). According to the assumptions of the lemma, we have \( g_n(t) = K_n(t) + N_n(t) \) and \( g_n(t) \leq 0 \), \( A_n(t, s) \leq 0 \). According to Lemma 1 we have \( L_{n-1}y(t) \leq 0 \) for all \( t \in [t_1, t_2] \). By virtue of

\[
L_{n-2}y(t) = L_{n-2}y(t_2) + \int_{t_2}^{t} \frac{L_{n-1}y(s)}{p_{n-1}(s)} \, ds \geq 0
\]

we have \( L_{n-2}y(t) \geq 0 \) on \([t_1, t_2]\). By using of a similar procedure (\( n \) can be an even number or an odd one), we get for \( n \geq 2 \):

a) \( (-1)^k L_ky(t) \geq 0 \) on \([t_1, t_2]\) for \( k = 0, 1, \ldots, n - 1 \), for \( n \) an even number,

b) \( (-1)^k L_ky(t) \leq 0 \) on \([t_1, t_2]\) for \( k = 0, 1, \ldots, n - 1 \), for \( n \) an odd number.

If \( n = 1 \), then the assertion of the lemma is obvious.
Lemma 7. Consider a solution \( y(t) \) of (L) on \([t_1, \infty)\), \( t_1 \geq \alpha \) such that (A) holds, let \( n \) be an even number and \( t_2 \in (t_1, \infty) \) such that \((-1)^k L_k y(t_2) \geq 0\) for \( k = 0, 1, \ldots, n - 1 \). Let \( p_k(t) \equiv 0 \) on \([t_1, t_2]\) for all even integers \( k \in [1, n] \). Then (B) holds.

Proof. We have \( G_k(s) \geq 0 \) for all even numbers \( k \in [1, n] \), and \( G_k(s) \leq 0 \) for all odd ones. If \( k \) is an odd number, then \( n - k \) is an odd number too, and \( p_{n-k}(t) \equiv 0 \) on \([t_1, t_2]\). Therefore \( N_n(t) = \int_{t_1}^{t_2} (-1)^k G_k(s) ds \leq 0 \). Similarly, \( M_k(t, s) = 0 \) for all odd \( k \leq n \). So \( A_n(t, s) = \sum_{k=1}^{n} (-1)^{k-1} M_k(t, s) \leq 0 \) because \( M_k(t, s) \geq 0 \) for all \( k = 1, 2, \ldots, n - 1 \).

Lemma 8. Consider a solution \( y(t) \) of (L) on \([t_1, \infty)\), \( t_1 \geq \alpha \) such that (A) holds, let \( n > 1 \) be an odd number and \( t_2 \in (t_1, \infty) \) such that \((-1)^k L_k y(t_2) \leq 0\) for \( k = 0, 1, \ldots, n - 1 \). Let \( p_k(t) \equiv 0 \) on \([t_1, t_2]\) for all even integers \( k \in [1, n] \). Then (B) holds.

Proof. The proof is similar to the proof of the previous lemma, so it is omitted.

Lemma 9. Let \( \{y_m(t)\}_{m=n_0}^{\infty} \) be a sequence of solutions of (L) on \([t_0, \infty)\), where \( a < t_0 < n_0 \), \( n \) is an even number, and \( L_k y_m(m) = (-1)^k \) for all \( m \geq n_0 \), \( k = 0, 1, \ldots, n - 1 \). Let (A) hold, and let \( p_k(t) \equiv 0 \) on \([a, \infty)\) for all odd integer numbers \( k \in [1, n] \). Let \(-\infty < \int_{t_0}^{s} p_k(s) ds = P < 0\), \( \int_{t_0}^{s} p_k(s) ds \geq -\frac{1}{2} \) for \( k = 1, 2, \ldots, n - 1 \), let \( p_k \) be nondecreasing functions for \( k = 0, 1, \ldots, n - 1 \), \( \int_{t_0}^{\infty} 1/p_r(s) ds \leq \frac{1}{2} \) for \( r = 1, 2, \ldots, n - 1 \), and let \( K \) be a real positive constant such that \( 0 \leq f(t) \leq K \) for \( t \in (-\infty, \infty) \). Then there exists a subsequence of \( \{y_m(t)\}_{m=n_0}^{\infty} \) which converges to \( \varphi_0(t) \). This function \( \varphi_0(t) \) is a solution of (L) on \([t_0, \infty)\), and \((-1)^k L_k \varphi_0(t) \geq 0 \) on \([t_0, \infty)\) for \( k = 0, 1, \ldots, n - 1 \).

Proof. Because \( L_n y_m(t) \geq 0 \) on \([t_0, m]\) for \( m = n_0, n_0 + 1, \ldots \) (this follows from Lemma 7 and Lemma 6, part a)), we have that \( L_{n-1} y_m(t) \) is nondecreasing and negative on \([t_0, n_0]\) for \( m > n_0 \). If we prove that \( L_{n-1} y_m(t_0) \) is bounded from below, it means \( L_{n-1} y_m(t) \) is uniformly bounded on \([t_0, n_0]\). Using the expression (C) several times, where

\[
(C) \quad L_k y_m(s) = L_k y_m(m) + \int_m^s \left( L_{k+1} \frac{y_m(s)}{p_{k+1}(s)} \right) ds \text{ for } k = 0, 1, \ldots, n - 2,
\]
we obtain for \( n > 3, \ 2 \leq k < n - 1 \) \((s_0 = s)\):

\[
L_{k+1}(s) = L_k(m) + L_{k+1}(m) \int_m^s \frac{ds_1}{p_{k+1}(s_1)} \\
+ L_{k+2}(m) \int_m^s \frac{ds_1}{p_{k+1}(s_1)} \int_{s_1}^{s_2} \frac{ds_2}{p_{k+2}(s_2)} + \ldots
\]

(D)

\[
+ L_{n-2}(m) \int_m^s \frac{ds_1}{p_{k+1}(s_1)} \int_{s_1}^{s_2} \frac{ds_2}{p_{k+2}(s_2)} \ldots \int_{s_{n-3}}^{s_{n-2}} \frac{ds_{n-2}}{p_{n-2}(s_{n-2})} \\
+ \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_k(s) \int_{m}^{s} \frac{ds_1}{p_{2k+1}(s_1)} \int_{s_1}^{s_2} \frac{ds_2}{p_{2k+2}(s_2)} \ldots \int_{s_{n-2k-2}}^{s_{n-2k-1}} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} ds.
\]

Integration of (L) over \([t_0, m]\) yields

\[
L_{n-1}(m) = L_{n-1}(m) + \int_{t_0}^m P_k(s)f(y_m(s)) \int_{m}^{s} \frac{ds_1}{p_{2k+1}(s_1)} \int_{s_1}^{s_2} \frac{ds_2}{p_{2k+2}(s_2)} \ldots \int_{s_{n-2k-2}}^{s_{n-2k-1}} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} ds,
\]

where \( C_k(s) \) is the last integral in (D) and \( B_k(s) \) is the rest of the right-hand side of (D). Let us denote the expression \( L_{n-1}(m) + \int_{t_0}^m P_k(s)f(y_m(s)) \int_{m}^{s} \frac{ds_1}{p_{2k+1}(s_1)} \int_{s_1}^{s_2} \frac{ds_2}{p_{2k+2}(s_2)} \ldots \int_{s_{n-2k-2}}^{s_{n-2k-1}} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} ds \) by \( F_m \). Then

\[
L_{n-1}(m)
\]

\[
\geq F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_k(s)B_k(s) \int_{m}^{s} \frac{ds_1}{p_{2k+1}(s_1)} \int_{s_1}^{s_2} \frac{ds_2}{p_{2k+2}(s_2)} \ldots \int_{s_{n-2k-2}}^{s_{n-2k-1}} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} ds,
\]

\[
\geq F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^m P_k(s)B_k(s) \int_{m}^{s} \frac{ds_1}{p_{2k+1}(s_1)} \int_{s_1}^{s_2} \frac{ds_2}{p_{2k+2}(s_2)} \ldots \int_{s_{n-2k-2}}^{s_{n-2k-1}} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} ds,
\]

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(We have used the fact that the last integral has the dimension $n - 2k$, which is an even number, and $t_0 \leq s_i \leq m < \infty$ for $i = 1, 2, \ldots, n - 2k - 2, t_0 \leq s \leq m < \infty$).

An easy arrangement yields

$$L_{n-1}y_m(t_0) \left[ 1 + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{\infty} P_{2k}(s) \, ds \int_{t_0}^{\infty} \frac{ds_1}{p_{2k+1}(s_1)} \int_{t_0}^{\infty} \frac{ds_2}{p_{2k+2}(s_2)} \cdots \right]$$

$$\cdots \int_{t_0}^{\infty} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} \right] \geq F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s)B_{2k}(s) \, ds.$$  

According to the assumptions, the expression in the parentheses above is a positive number because of

$$\sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{\infty} [-P_{2k}(s)] \, ds \cdots \int_{t_0}^{\infty} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})} \leq \sum_{k=1}^{\frac{n}{2}-1} \left( \frac{1}{2} \right)^{n-2k} < 1. \text{ Therefore}$$

$$L_{n-1}y_m(t_0) \geq \frac{F_m + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s)B_{2k}(s) \, ds}{1 + \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s) \, ds \int_{t_0}^{m} \frac{ds_1}{p_{2k+1}(s_1)} \cdots \int_{t_0}^{m} \frac{ds_{n-2k-1}}{p_{n-1}(s_{n-2k-1})}}.$$  

We have

$$F_m = L_{n-1}y_m(m) + \int_{t_0}^{m} P_0(s)f(y_m(s)) \, ds \geq -1 + \int_{t_0}^{\infty} P_0(s)f(y_m(s)) \, ds$$

$$\geq -1 + K \int_{t_0}^{\infty} P_0(s) \, ds = -1 + KP,$$

$$B_{2k}(s) = L_{2k}y_m(m) + L_{2k+1}y_m(m) \int_{t_0}^{m} \frac{ds_1}{p_{2k+1}(s_1)} + \cdots + L_{n-2}y_m(m) \int_{t_0}^{m} \frac{ds_1}{p_{2k+1}(s_1)} \cdots$$

$$\cdots \int_{t_0}^{m} \frac{ds_{n-2k-2}}{p_{n-2k-2}(s_{n-2k-2})} = 1 + 1 \int_{s}^{m} \frac{ds_1}{p_{2k+1}(s_1)} + \cdots + 1 \int_{s}^{m} \frac{ds_1}{p_{2k+1}(s_1)}$$

because of $s \leq m, s_i \leq m$ for $i = 1, 2, \ldots, n - 2k - 3$. So we have

$$\sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s)B_{2k}(s) \, ds \geq n \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{m} P_{2k}(s) \, ds$$

$$\geq n \sum_{k=1}^{\frac{n}{2}-1} \int_{t_0}^{\infty} P_{2k}(s) \, ds \geq -n \frac{n - 1}{2}. 60
Hence
\begin{align*}
L_{n-1} y_m(t_0) & \geq \frac{-1 + KP - \frac{2}{d} \left( \frac{d}{2} - 1 \right)}{1 + \sum_{k=1}^{n-1} \int_{t_0}^{s_0} \int_{t_0}^{s} \frac{ds}{p_k(s)} \int_{t_0}^{s} \frac{ds}{p_{k+1}(s)} \cdots \int_{t_0}^{s} \frac{ds}{p_{n-1}(s)_{m-n-1}}} \\
& = S_{n-1} \in (-\infty, 0)
\end{align*}
for \( n > 3 \). If \( n = 2 \), then \( L_{n-1} y_m(t_0) = F_m \geq -1 + KP \in (-\infty, 0) \). It implies that \( \{L_{n-1} y_m(t_0)\}_{m=n_0}^\infty \) is bounded from below for any fixed even number \( n \geq 2 \). So we have

\begin{align*}
0 & \leq L_{n-2} y_m(t_0) = L_{n-2} y_m(m) + \int_{t_0}^{s_0} \frac{L_{n-2} y_m(s)}{p_{n-1}(s)} ds \leq 1 - L_{n-1} y_m(t_0) \int_{t_0}^{s_0} \frac{ds}{p_{n-1}(s)} \\
& \leq 1 - S_{n-1} \int_{t_0}^{s_0} \frac{ds}{p_{n-1}(s)} = S_{n-2} \in (0, \infty),
\end{align*}

\begin{align*}
0 & \geq L_{n-3} y_m(t_0) = L_{n-3} y_m(m) + \int_{t_0}^{s_0} \frac{L_{n-3} y_m(s)}{p_{n-2}(s)} ds \geq -1 - L_{n-2} y_m(t_0) \int_{t_0}^{s_0} \frac{ds}{p_{n-2}(s)} \\
& \geq -1 - S_{n-2} \int_{t_0}^{s_0} \frac{ds}{p_{n-2}(s)} = S_{n-3} \in (-\infty, 0).
\end{align*}

Similarly, it can be proved that \( \{L_k y_m(t_0)\}_{m=n_0}^\infty \) is bounded for \( k = 0, 1, \ldots, n-1 \). However,

\begin{align*}
0 & \leq L_n y_m(t) = - \sum_{k=1}^{n-1} \int_{t_0}^{s_0} \frac{P_{2k}(t) L_{2k} y_m(t) - P_0(t) f(y_m(t))}{p_{n-1}(s)} ds \\
& \leq - \sum_{k=1}^{n-1} \int_{t_0}^{s_0} \frac{P_{2k}(t_0) L_{2k} y_m(t_0) - P_0(t_0) K}{p_{n-1}(s)} ds \\
& \leq - \sum_{k=1}^{n-1} \int_{t_0}^{s_0} \frac{P_{2k}(t_0) S_{2k} - P_0(t_0) K}{p_{n-1}(s)} ds = S_n \in (0, \infty),
\end{align*}

and this implies that \( \{L_n y_m(t)\}_{m=n_0}^\infty \) is uniformly bounded on \([t_0, n_0]\) for \( m \geq n_0 \) and so \( L_{n-1} y_m(t) \) are uniformly equicontinuous on \([t_0, n_0]\) for \( m \geq n_0 \). According to Arzelà-Ascoli theorem, there exists a subsequence \( \{L_{n-1} y_{k_m}\}_{m=n_0}^\infty \) of \( \{L_{n-1} y_m\}_{m=n_0}^\infty \) such that \( \{L_{n-1} y_{k_m}\}_{m=n_0}^\infty \) converges uniformly on \([t_0, n_0]\) to, for example, a function \( \varphi_{n-1}(t) \).
To ensure uniform convergence of \( \{L_{m-2}y_{m}\}_{m=n_0}^{\infty} \) on \([t_0, n_0]\) to, for instance, a function \( \varphi_{n-2}(t) \), it suffices to show convergence of \( \{L_{m-2}y_{m}\}_{m=n_0}^{\infty} \) at an inner point of \([t_0, n_0]\). This follows from the fact that \( L_{m-2}y_{m}(t_0 + \varepsilon) \leq L_{m-2}y_{m}(t_0) \leq S_{m-2} \) for \( \varepsilon > 0, \varepsilon < n_0 - t_0 \). Then there exists a convergent subsequence \( \{L_{m-2}y_{k_m}^{(t_0 + \varepsilon)}\}_{m=n_0}^{\infty} \) of \( \{L_{m-2}y_{m}(t_0 + \varepsilon)\}_{m=n_0}^{\infty} \) and therefore \( \{L_{m-2}y_{m}\}_{m=n_0}^{\infty} \) converges uniformly to \( \varphi_{n-2}(t) \) on \([t_0, n_0]\). It is obvious that \( L_{m-1}y_{k_m} \Rightarrow \varphi_{n-1} \) on \([t_0, n_0]\), too. In a similar way we can prove uniform convergence of a subsequence \( \{y_{m}\}_{m=n_0}^{\infty} \) of \( \{y_{m}\}_{m=n_0}^{\infty} \) such that \( L_{k}y_{m}(t) \Rightarrow \varphi_{k}(t) \) on \([t_0, n_0]\) for \( k = 0, 1, \ldots, n \). Due to the fact that uniform convergence makes changing of the order of limit processes possible (a quasi-derivative is a certain kind of limit), we have

\[
0 = \lim_{m \to \infty} L(y_{m}(t)) = \lim_{m \to \infty} Lny_{m}(t) + \sum_{k=1}^{n-1} P_{2k}(t) \lim_{m \to \infty} L_{2k}y_{m}(t) + P_{0}(t)f(\lim_{m \to \infty} y_{m}(t))
\]

\[
= \varphi_{n}(t) + \sum_{k=1}^{n-1} P_{2k}(t)\varphi_{k}(t) + P_{0}(t)f(\varphi_{0}(t))
\]

for all \( t \in [t_0, n_0] \).

But \( \varphi_{k}(t) = \lim_{m \to \infty} L_{k}y_{m}(t) = L_{k}(\lim_{m \to \infty} y_{m}(t)) = L_{k}(\lim_{m \to \infty} L_{0}y_{m}(t)) = L_{k}\varphi_{0}(t) \), so \( \varphi_{0}(t) \) fulfills (L) on \([t_0, n_0]\). It is important that we are able to continue \( \varphi_{0}(t) \) on \([t_0, n_0+1]\) in such a way that \( \varphi_{0}(t) \) be a solution of (L) on \([t_0, n_0+1]\). Indeed, it suffices to repeat the whole previous part of the proof with the sequence \( y_{m} \) for \( m \geq n_0 + 1 \) instead of \( y_{m} \) for \( m \geq n_0 \). Now it is obvious that \( \varphi_{0}(t) \) can be continued on \([t_0, n_0+v]\) \((v \in \mathbb{N}) \) and therefore \( \varphi_{0}(t) \) fulfills (L) on \([t_0, \infty)\).

Now let us take an arbitrary point \( t_1 \in [t_0, \infty) \). Then there exists \( m_0 \in \{1, 2, \ldots\} \), \( t_1 < m_0 \) and a subsequence \( \{y_{m}\}_{m=m_0}^{\infty} \) of \( \{y_{m}\}_{m=n_0}^{\infty} \) such that \( L_{k}y_{m} \Rightarrow L_{k}\varphi_{0}(t) \) on \([t_0, m_0]\). But \( (-1)^{k}L_{k}y_{m}(t) \geq 0 \) on \([t_0, m_0]\). Therefore \( (-1)^{k}L_{k}\varphi_{0}(t_{1}) \geq 0 \). It implies that \( (-1)^{k}L_{k}\varphi_{0}(t_{1}) \geq 0 \) for all \( t \geq t_{0}, k = 0, 1, \ldots, n-1 \).

**Lemma 10.** Let \( \{y_{m}(t)\}_{m=n_0}^{\infty} \) be a sequence of solutions of (L) on \([t_0, \infty)\), where \( a < t_0 < n_0, n \) is an odd number, and \( L_{k}y_{m}(m) = (-1)^{k-1} \) for all \( m \geq n_0 \), \( k = 0, 1, \ldots, n-1 \). Let (A) hold, and let \( P_{k}(t) \equiv 0 \) on \([a, \infty)\) for all even integers \( k \in [1, n]\). Let \( -\infty < \int_{t_0}^{\infty} P_{0}(s)ds = P < \infty \), \( \int_{t_0}^{\infty} P_{k}(s)ds \geq -\frac{1}{k} \) for \( k = 1, 2, \ldots, n-1 \), let \( P_{k} \) be nondecreasing functions for \( k = 0, 1, \ldots, n-1 \), \( \int_{t_0}^{\infty} P_{r}(s)ds \leq \frac{1}{r} \) for \( r = 1, 2, \ldots, n-1 \), and let \( K \) be a real positive constant such that \( 0 \leq f(t) \leq K \) for \( t \in (-\infty, \infty) \). Then there exists a subsequence of \( \{y_{m}(t)\}_{m=n_0}^{\infty} \) which converges to
\( \varphi_0(t) \). This function \( \varphi_0(t) \) is a solution of (L) on \([t_0, \infty)\), and \((-1)^k L_k \varphi_0(t) \leq 0\) on \([t_0, \infty)\) for \(k = 0, 1, \ldots, n-1\).

**Proof.** The proof is similar to the proof of Lemma 9 (instead of Lemma 6, part a), and Lemma 7 we use Lemma 6, part b) and Lemma 8, respectively, so it is omitted. \qed

**Theorem 1.** Let \( n \) be an even number. Let (A) hold, and let \( P_k(t) \equiv 0 \) on \([a, \infty)\) for all odd integers \( k \in [1, n] \). Let \( P_k(t) \) be nondecreasing functions on \([a, \infty)\) such that \( \int_a^\infty P_k(s) \, ds > -\infty \) for \( k = 0, 1, \ldots, n-1 \), \( \int_a^\infty 1/p_r(s) \, ds < \infty \) for \( r = 1, 2, \ldots, n-1 \), and let \( K \) be a real positive constant such that \( 0 \leq f(t) \leq K \) for all \( t \in (-\infty, \infty) \). Then (L) admits a Kneser solution \( y(t) \) on \([a, \infty)\), i.e. \( y(t) > 0, (-1)^k L_k y(t) \geq 0 \) on \([a, \infty)\) for \( k = 1, 2, \ldots, n-1 \).

**Proof.** Let us take \( t_0 \in (a, \infty) \) such that \( \int_{t_0}^\infty P_k(s) \, ds > -\frac{1}{2}, \int_{t_0}^\infty 1/p_r(s) \, ds \leq \frac{1}{2} \) for \( k = 1, 2, \ldots, n-1; r = 1, 2, \ldots, n-1 \). According to Lemma 5, there exists a sequence \( \{y_m(t)\}_{m=n_0}^\infty \) of solutions of (L) on \([t_0, \infty)\) such that \( L_k y_m(m) = (-1)^k \) for all \( m \geq n_0 > t_0, k = 0, 1, \ldots, n-1 \). Lemma 7 ensures validity of (B), and Lemma 6, part a), yields that \( \{y_m(t)\}_{m=n_0}^\infty \) has the required properties from Lemma 9. According to the last-mentioned lemma, there exists a function \( y(t) \) such that \( L(y(t)) = 0 \) on \([t_0, \infty)\), \((-1)^k L_k y(t) \geq 0\) on \([t_0, \infty)\) for \( k = 0, 1, \ldots, n-1 \). This solution \( y(t) \) of (L) on \([t_0, \infty)\) can be continued onto \([a, \infty)\) by Lemma 5. According to Lemma 6, part a), \( y(t) \) is a Kneser solution of (L) on \([a, \infty)\) because \( y(t) > 0 \) on \([a, \infty)\) (this follows from \( f(0) \neq 0 \)). \qed

**Theorem 2.** Let \( n \) be an odd number. Let (A) hold, and let \( P_k(t) \equiv 0 \) on \([a, \infty)\) for all even integers \( k \in [1, n] \). Let \( P_k(t) \) be nondecreasing functions on \([a, \infty)\) such that \( \int_a^\infty P_k(s) \, ds > -\infty \) for \( k = 0, 1, \ldots, n-1 \), \( \int_a^\infty 1/p_r(s) \, ds < \infty \) for \( r = 1, 2, \ldots, n-1 \) and let \( K \) be a real positive constant such that \( 0 \leq f(t) \leq K \) for all \( t \in (-\infty, \infty) \). Then (L) admits a Kneser solution \( y(t) \) on \([a, \infty)\), i.e. \( y(t) < 0, (-1)^k L_k y(t) \leq 0 \) on \([a, \infty)\) for \( k = 1, 2, \ldots, n-1 \).

**Proof.** The proof is similar to that of the previous theorem (instead of Lemma 6, part a) and Lemma 9 we will use Lemma 6, part b) and Lemma 10, respectively) and so it is omitted. \qed
3. Examples

Example 1. The equation

\[(t^4(t^3(3t^2y')')') - \frac{1}{t^7(t^3(3t^2y')') + [(\frac{72}{t^8} - \frac{1296}{t^{14}}) \sqrt{1 + t^{-18}}] \frac{1}{\sqrt{1 + y^2}}} = 0\]

admits a Kneser solution \(y(t) = t^{-9}\) on \([1, \infty)\) according to Theorem 1 because \(\int (1/p_r(t))dt < \infty\) for \(r = 1, 2, 3, P_k(t)\) is nonpositive and nondecreasing on \([1, \infty)\), \(\int P_k(t)dt > -\infty\) for \(k = 0, 1, 2, 3, 0 \leq \frac{1}{\sqrt{1 + y^2}} \leq 1, f(0) \neq 0.\)

Example 2. The equation of the \(n\)-th order \((n\) is an even number) \(L_ny + \sum_{k=1}^{n-1} P_k(t)L_{2k}y + P_0(t)f(y) = 0,\)

where \(P_k(t) = -t^{2k-2}\) for \(k = 0, 1, \ldots, n\) \(\frac{n}{2} - 1, P_r(t) = t^{3r}\) for \(r = 1, 2, \ldots, n - 1, f(t) = e^{-t^2}\) admits a Kneser solution on \([1, \infty)\) according to Theorem 1 because \(\int (1/p_r(t))dt < \infty\) for \(r = 1, 2, \ldots, n - 1, \int P_k(t)dt > -\infty\) for \(k = 0, 1, \ldots, n\) \(\frac{n}{2} - 1, 0 \leq e^{-t^2} \leq 1, f(0) \neq 0.\)

Example 3. The equation \(L_3y - \frac{1}{t^6}L_3y - \frac{1}{t^7}L_1y + (12t^{-13} + 1188t^{-12} - 14256t^{-3}) \frac{1}{\sqrt{1 + t^{-18}}} \frac{1}{\sqrt{1 + y^4}} = 0,\)

where \(p_r(t) = t^{r+1}\) for \(r = 1, 2, 3, 4\) admits a Kneser solution \(y(t) = -t^{-12} < 0\) on \([1, \infty)\) according to Theorem 2 because \(\int (1/p_r(t))dt < \infty\) for \(r = 1, 2, 3, 4, P_0(t)\) is nonpositive and nondecreasing on \([1, \infty)\), \(\int P_k(t)dt > -\infty\) for \(k = 0, 1, 2, 3, 4, 0 \leq \frac{1}{\sqrt{1 + y^4}} \leq 1, f(0) \neq 0.\)
References


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