SPECTRA OF AUTOMETRIZED LATTICE ALGEBRAS

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(Received October 2, 1996)

Abstract. Autometrized algebras are a common generalization e.g. of commutative lattice ordered groups and Brouwerian algebras. In the paper, spectra of normal autometrized lattice ordered algebras (i.e. topologies of sets (and subsets) of their proper prime ideals) are studied. Especially, the representable dually residuated lattice ordered semigroups are examined.

Keywords: autometrized algebra, dually residuated lattice ordered semigroup, prime ideal, spectrum

MSC 1991: 06F05, 06F20, 06D20

1. Introduction

K.L.N. Swamy introduced in [7] the notion of an autometrized algebra which is a common generalization, for example, of commutative $\ell$-groups and Brouwerian algebras. (A Brouwerian algebra is a lattice $A$ with the greatest element in which for each $a, b \in A$ there exists a smallest $x \in A$ such that $b \lor x \geq a$.) Ideals in autometrized algebras were introduced and studied by K.L.N. Swamy and N.P. Rao in [9]. Their work has been continued by J.Rachůnek in [4], [5], [6], M.E.Hansen in [1], [2], and T.Kovár in [3]. The notion of a prime ideal in an autometrized algebra was defined in [4] and minimal prime ideals were studied in [1].

In this paper, spectra of autometrized lattice ordered algebras, i.e. topological spaces of some sets of their proper prime ideals, are studied.

A system $A = (A, +, 0, \leq, *)$ is called an autometrized algebra if

1. $(A, +, 0)$ is a commutative monoid;
2. $(A, \leq)$ is an ordered set, and

\[ \forall a, b, c \in A ; a \leq b \implies a + c \leq b + c \]
(3) \( * : A \to A \) is an autometric on \( A \), i.e.

\[
\forall a, b \in A; \ a * b \geq 0, \\
\forall a, b \in A; \ a * b = 0 \iff a = b, \\
\forall a, b \in A; \ a * b = b * a, \\
\forall a, b, c \in A; \ a * (b * c) \leq (a * b) + (b * c).
\]

An autometized algebra is called normal if

\[
\forall a \in A; \ a \leq a * 0, \\
\forall a, b, c, d \in A; \ (a + c) * (b + d) \leq (a * b) + (c * d), \\
\forall a, b, c, d \in A; \ (a * c) * (b * d) \leq (a * b) + (c * d), \\
\forall a, b \in A; \ (a \leq b \implies \exists x \geq 0; a + x = b).
\]

If \( (A, \leq) \) is a lattice and

\[
\forall a, b, c \in A; \ a + (b \lor c) = (a + b) \lor (a + c), \\
\forall a, b, c \in A; \ a + (b \land c) = (a + b) \land (a + c),
\]

then \( A = (A, +, 0, \leq, *) \) is called an autometized lattice algebra (an autometized \( \ell \)-algebra).

For instance, every commutative \( \ell \)-group and every Brouwerian algebra is a normal autometized \( \ell \)-algebra.

If \( A \) is an autometized algebra, then \( \emptyset \neq I \subseteq A \) is called an ideal in \( A \) if and only if

\[
\forall a, b \in I; \ a + b \in I, \\
\forall a \in I, \ x \in A; \ x * 0 \leq a * 0 \implies x \in I.
\]

In [9] it is proved that the set \( I(A) \) of all ideals in a normal autometized algebra \( A \) is a complete algebraic lattice with respect to the order by set inclusion. If \( \emptyset \neq B \subseteq A \), then the ideal generated by \( B \) is

\[
I(B) = \{ x \in A; \ x * 0 \leq m_1(a_1 * 0) + \ldots + m_k(a_k * 0), \text{ where } m_1, \ldots, m_k \in \mathbb{N} \text{ and } a_1, \ldots, a_k \in B \}.
\]

For the principal ideal \( I(a) \) generated by an element \( a \in A \) we have

\[
I(a) = \{ x \in A; \ x * 0 \leq m(a * 0) \text{ for some } m \geq 0 \}.
\]
An ideal \( I \) of an autometrized algebra \( A \) is called \emph{prime} if
\[
\forall J, K \in \mathcal{I}(A); \ J \cap K = I \implies J = I \text{ or } K = I,
\]
and it is called \emph{regular} if
\[
I = \bigcap_{\alpha \in \Gamma} J_\alpha,
\]
where \( J_\alpha \in \mathcal{I}(A) \) for all \( \alpha \in \Gamma \) implies the existence of \( \beta \in \Gamma \) such that \( I = J_\beta \).

\textbf{Note.} An autometrized algebra \( A \) is called
\begin{enumerate}[a)]
\item \emph{semiregular} if
\[
\forall a \in A; \ a \geq 0 \implies a \ast 0 = a;
\]
\item \emph{interpolation} if
\[
\forall a, b, c \in A; \ (0 \leq a, b, c, \ a \leq b + c \implies (\exists 0 \leq b_1 \leq b, 0 \leq c_1 \leq c; \ a = b_1 + c_1)).
\]
\end{enumerate}

Clearly, commutative \( \ell \)-groups and Brouwerian algebras are both semiregular and interpolation.

Many of properties of prime ideals were proved in [4], [5] and [6] for interpolation semiregular \( \ell \)-algebras. But using [1], Lemma 1.2, one can easily prove that the assumption “\( A \) is interpolation” is unnecessary. Further (as shown in [3]), Lemma 5 in [9] (i.e. if \( A \) is an autometrized algebra and \( a, b \in A \) then \( (a \ast b) \ast 0 = a \ast b \)) makes it often possible to omit also the requirement of semiregularity.

\section{2. Normal Autometrized \( \ell \)-Algebras}

Let \( A \) be an autometrized algebra. Let us denote by \( \text{Spec} A \) the set of proper prime ideals in \( A \). If \( M \subseteq A \), we put
\[
\begin{align*}
S(M) &= \{ P \in \text{Spec} A; \ M \not\subseteq P \}, \\
H(M) &= \{ P \in \text{Spec} A; \ M \subseteq P \}.
\end{align*}
\]

Especially, for \( M = \{ a \} \) where \( a \in A \), we will write
\[
S(\{ a \}) = S(a) \text{ and } H(\{ a \}) = H(a).
\]

It is obvious that for any \( M \subseteq A \) we have \( S(M) = S(I(M)) \) and \( H(M) = H(I(M)) \), hence we will consider only \( S(I) \) and \( H(I) \) for all \( I \in \mathcal{I}(A) \) and \( S(a) \) and \( H(a) \) for each \( a \in A \).
**Lemma 1.** If $A$ is a normal autometrized $\ell$-algebra then:

1. $S(0) = 0, S(A) = \text{Spec } A$.
2. $\forall I, J \in \mathcal{I}(A); S(I \cap J) = S(I) \cap S(J)$.
3. $\forall I_\gamma \in \mathcal{I}(A), \gamma \in \Gamma; S(\bigvee_{\gamma \in \Gamma} I_\gamma) = \bigcup_{\gamma \in \Gamma} S(I_\gamma)$.
4. $\forall a, b \in A; S((a \cdot 0) \vee (b \cdot 0)) = S(a) \cup S(b)$.
5. $\forall a, b \in A; S((a \cdot 0) \wedge (b \cdot 0)) = S(a) \cap S(b)$.

**Proof.** 1. Obvious.

2. Let $I, J \in \mathcal{I}(A)$ and $P \in \text{Spec } A$. Then by [4], Theorem 4, and [3], Theorem 9, $I \cap J \not\subseteq P$ if and only if $I \not\subseteq P$ and $J \not\subseteq P$, therefore $S(I \cap J) = S(I) \cap S(J)$.

3. Let $I_\gamma \in \mathcal{I}(A), \gamma \in \Gamma,$ and $P \in \text{Spec } A$. Then for $\bigvee_{\gamma \in \Gamma} I_\gamma$, the join of $I_\gamma$ in $\mathcal{I}(A)$, we have $\bigvee_{\gamma \in \Gamma} I_\gamma \not\subseteq P$ if and only if there exists $\gamma_0 \in \Gamma$ such that $I_{\gamma_0} \not\subseteq P$, and hence $S(\bigvee_{\gamma \in \Gamma} I_\gamma) = \bigcup_{\gamma \in \Gamma} S(I_\gamma)$.

4 and 5. By [4], Propositions 2 and 3, and [3], Theorems 6 and 7,

$$I(a) \vee I(b) = I((a \cdot 0) \vee (b \cdot 0)),$$

$$I(a) \wedge I(b) = I((a \cdot 0) \wedge (b \cdot 0)),$$

thus 4 and 5 are special cases of the properties 2 and 3. \qed

**Corollary 2.** The sets $S(I)$, where $I$ is any ideal in $A$, form a topology of $\text{Spec } A$.

**Definition.** If $A$ is a normal autometrized $\ell$-algebra then the topology of Spec $A$ such that its open sets are exactly $S(I)$ for any $I \in \mathcal{I}(A)$ will be called the spectral topology. The topological space Spec $A$ with the spectral topology will be called the spectrum of the algebra $A$.

In this section, $A$ will always denote a normal autometrized $\ell$-algebra.

**Proposition 3.** The mapping $S: I \rightarrow S(I)$ is an isomorphism of the lattice $\mathcal{I}(A)$ onto the lattice of open subsets in $\text{Spec } A$.

**Proof.** By Lemma 1, $S$ is a surjective homomorphism. By [6] (Theorem 3), any ideal is an intersection of regular ideals, and since every regular ideal is prime, we have

$$I = \bigcap\{P; P \in H(I)\}$$

for each $I \in \mathcal{I}(A)$. Hence, if $S(I) = S(J)$, then

$$I = \bigcap\{P; P \in H(I)\} = \bigcap\{Q; Q \in H(J)\} = J,$$

and therefore $S$ is injective. \qed
Theorem 4. The sets $S(a)$, where $a$ is any element in $A$, form a basis of open sets in the spectral topology stable under finite unions and intersections.

Proof. If $I \in \mathcal{I}(A)$, then by Lemma 1 (3),
\[ S(I) = S \left( \bigvee_{a \in I} I(a) \right) = \bigcup_{a \in I} S(a), \]
hence the sets $S(a)$ form a basis of the spectral topology.

The stability of this basis under finite unions and intersections follows from Lemma 1 (4), (5). \hfill \square

Theorem 5. a) $S(a)$ is compact for every $a \in A$.

b) If $B$ is an open compact set of Spec $A$ then $B = S(a)$ for some $a \in A$.

Proof. a) Let $a \in A$, $I_{\gamma} \in \mathcal{I}(A)$, $\gamma \in \Gamma$, and let
\[ S(a) \subseteq \bigcup_{\gamma \in \Gamma} S(I_{\gamma}) = S \left( \bigvee_{\gamma \in \Gamma} I_{\gamma} \right). \]
Then, by Proposition 3, $a \in \bigvee_{\gamma \in \Gamma} I_{\gamma}$, and hence, by [9], Lemma 2,
\[ a * 0 \leq (b_1 * 0) + \ldots + (b_k * 0), \]
where $k \in \mathbb{N}$, $b_i \in I_{\gamma_i}$, $i = 1, \ldots, k$. But this means that $a \in I_{\gamma_1} \vee \ldots \vee I_{\gamma_k}$, and so
\[ S(a) \subseteq S \left( \bigvee_{i=1}^{k} I_{\gamma_i} \right) = \bigcup_{i=1}^{k} S(I_{\gamma_i}). \]

b) Let $B$ be an open compact set. Then there exist $a_1, \ldots, a_n \in A$ such that $B = \bigcup_{i=1}^{n} S(a_i)$. Hence by Lemma 1 (4),
\[ B = S \left( \bigvee_{i=1}^{n} (a_i * 0) \right). \]

\hfill \square

Corollary 6. The spectrum of a normal autometrized \( \ell \)-algebra $A$ is compact if and only if $A$ contains an element $a$ such that $I(a) = A$. 

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This means, if \( A \) is a commutative \( \ell \)-group then \( \text{Spec} \ A \) is compact if and only if \( A \) has a strong unit, and \( \text{Spec} \ A \) is compact for each Brouwerian algebra \( A \).

If \( x \subseteq \text{Spec} \ A \), put

\[
\mathcal{D}(x) = \bigcap \{ P; P \in x \}.
\]

**Proposition 7.** a) The closed sets in \( \text{Spec} \ A \) are exactly all \( H(I) \), where \( I \in \mathcal{I}(A) \),
b) If \( x \subseteq \text{Spec} \ A \), then its closure is \( \bar{x} = H(\mathcal{D}(x)) \).

**Proof.** a) \( H(I) = \text{Spec} \ A \setminus S(I) \).
b) \( x \subseteq H(\mathcal{D}(x)) \), hence \( x \subseteq H(\mathcal{D}(x)) \), and so

\[
\mathcal{D}(x) = D H(\mathcal{D}(x)) \subseteq \mathcal{D}(x).
\]

But \( x \subseteq \bar{x} \), therefore \( \mathcal{D}(x) \subseteq \mathcal{D}(x) \). Thus \( \mathcal{D}(x) = \mathcal{D}(x) \), which means

\[
\bar{x} = H(\mathcal{D}(x)) = H(\mathcal{D}(x)).
\]

\( \square \)

**Corollary 8.** If \( x \subseteq \text{Spec} \ A \), then \( x \) is dense if and only if \( \bigcap \{ P; P \in x \} = \{0\} \).

### 3. Representable DRL-semigroups

Let us recall the notion of a dually residuated lattice ordered semigroup (DRL-semigroup) that has been introduced by K.L.N. Swamy in [8].

A system \( A = (A, +, 0, \leq, \cdot) \) is called a DRL-semigroup if

1. \( (A, +, 0, \leq) \) is a commutative lattice ordered monoid;
2. for each \( a, b \in A \) there exists a least element \( x \in A \) such that \( b + x \geq a \) (such \( x \) is denoted by \( a - b \));
3. \( \forall a, b \in A ; (a - b) + b \leq a \lor b \);
4. \( \forall a \in A ; a - a \geq 0 \).

Let us denote \( a * b = (a - b) \lor (b - a) \) for \( a, b \in A \). Then \( (A, +, 0, \leq, *, *) \) is, by [8] and [9], a normal semiregular autometritized \( \ell \)-algebra.

A DRL-semigroup \( A \) is called representable (see [10]) if \( (a - b) \wedge (b - a) \leq 0 \) for each \( a, b \in A \). (For instance, commutative \( \ell \)-groups and Boolean algebras are representable DRL-semigroups.)

**Proposition 9.** Let \( A \) be a representable DRL-semigroup, let \( P, Q \in \text{Spec} \ A \) and let \( P \parallel Q \). Then \( P \) and \( Q \) have in \( \text{Spec} \ A \) disjoint neighborhoods.
Proof. Let \( P, Q \in \text{Spec} A, P \not\subseteq Q \) and \( Q \not\subseteq P \). Then there exist \( 0 < a \in A, 0 < b \in A \) such that \( a \in P \setminus Q \) and \( b \in Q \setminus P \). Denote \( u = a - (a \wedge b) \) and \( v = b - (a \wedge b) \). Let us show that \( u \notin Q \) and \( v \notin P \). Let, for example, \( u \in Q \). By \([4]\), Lemma 6, \( a = (a \wedge b) + u \), and since \( a \wedge b \in Q \), we have \( a \in Q \), a contradiction. Hence \( P \in S(u), Q \in S(v) \) and by \([4]\), Lemma 6, \( u \wedge v = 0 \). Thus \( S(u) \cap S(v) = S(u \wedge v) = \emptyset \).

If \( x \subseteq \text{Spec} A \) then the topology of \( x \) induced by the spectral topology of \( \text{Spec} A \) will be called the spectral topology on \( x \).

Corollary 10. If \( A \) is a representable DRI-semigroup and \( x \subseteq \text{Spec} A \) is a set of pairwise non-comparable prime ideals, then the spectral topology of \( x \) is a \( T_2 \)-topology.

If \( x \subseteq \text{Spec} A \) and \( M \subseteq A \), put \( S_x(M) = S(M) \cap x \).

Denote by \( m(A) \) the set of all minimal and by \( M(A) \) the set of all maximal prime ideals of a representable DRI-semigroup \( A \).

Theorem 11. If \( A \) is a representable DRI-semigroup then the spectral topology of \( m(A) \) is a \( T_2 \)-topology and the sets \( S_{m(A)}(a) = \{ P \in m(A) \mid a \notin P \} \), \( a \in A \), form a basis of the space \( m(A) \) composed by closed subsets.

Proof. Let \( A \) be a representable DRI-semigroup. Obviously, the sets \( S_{m(A)}(a) \), where \( a \in A \), form a basis of the spectral topology of \( m(A) \). Let \( a \in A \) and let \( P \) be a minimal prime ideal in \( A \). By \([1]\), Proposition 2.4, either \( a \notin P \) or \( a^\perp \not\subseteq P \). Hence \( S_{m(A)}(a) \cap S_{m(A)}(a^\perp) = \emptyset \) and \( S_{m(A)} \cup S_{m(A)}(a^\perp) = m(A) \). Therefore, since \( S_{m(A)}(a^\perp) \) is open, \( S_{m(A)}(a) \) is closed.

Let \( A \) be a representable DRI-semigroup, \( 0 \neq a \in A \). Let us denote by \( \text{val}(a) \) the set of all values of \( a \), i.e. the set of all ideals maximal with respect to the property of not containing \( a \). (For \( a = 0 \), put \( \text{val}(a) = \emptyset \).) Let \( P \in S(a) \). Then, by \([6]\), Theorem 4, the set of ideals in \( A \) containing \( P \) is linearly ordered and by \([6]\), Theorem 2, there are ideals in \( \text{val}(a) \) that contain \( P \). Hence there is exactly one \( M_P \in \text{val}(a) \) such that \( P \subseteq M_P \).

Let us denote by \( \psi_a : S(a) \to \text{val}(a) \) the mapping such that \( \psi : P \mapsto M_P \).

Proposition 12. The mapping \( \psi_a \) is continuous.

Proof. Let \( a \in A, P \in S(a) \) and let \( U \) be a neighborhood of \( M_P \) in \( \text{val}(a) \). We can suppose that \( U = S(b) \cap \text{val}(a) \) for some \( b \in A \). If \( Q \in \text{val}(a) \setminus S(b) \), then we can choose a neighborhood \( U_Q \) of \( Q \) and a neighborhood \( V_Q \) of \( M_P \) such that \( U_Q \cap V_Q = \emptyset \). It is evident that all \( U_Q \), where \( Q \) runs over \( \text{val}(a) \setminus S(b) \), form a
covering of $S(a) \setminus S(b)$. Since $S(a)$ is compact and $S(a) \setminus S(b)$ is closed in $S(a)$, $S(a) \setminus S(b)$ is compact, too. Hence there exist $n \in \mathbb{N}$ and $Q_1, \ldots, Q_n \in S(a) \setminus S(b)$ such that $S(a) \setminus S(b) \subseteq U_{Q_1} \cup \ldots \cup U_{Q_n}$.

Let us denote $C = S(a) \setminus (U_{Q_1} \cup \ldots \cup U_{Q_n})$. We have $V_Q \cap \ldots \cap V_{Q_n} \subseteq C$, therefore $C$ is a neighborhood of $MP$ which is closed in $S(a)$, and $C \cap \text{val}(a) \subseteq U$. Therefore $C \subseteq \psi^{-1}_a(C \cap \text{val}(a)) \subseteq \psi^{-1}_a(U)$. Moreover, $C$, which is a neighborhood of $MP$, is also a neighborhood of $P$.

**Proposition 13.** If $a \in A$, then the set $\text{val}(a)$ is a compact $T_2$-space.

**Proof.** By Corollary 10, $\text{val}(a)$ is a $T_2$-space. Further, $\text{val}(a)$ is the image of the compact set $S(a)$ in the mapping $\psi_a$ which is, by Proposition 12, continuous, hence $\text{val}(a)$ is also compact. \hfill $\square$

The following theorem is now an immediate consequence.

**Theorem 14.** If $A$ is a representable DRf-semigroup then the space $\mathcal{M}(A)$ of all its maximal prime ideals is a $T_2$-space. If there exists $b \in A$ such that $I(b) = A$ then $\mathcal{M}(A)$ is compact.

**References**


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