DOMINATING FUNCTIONS OF GRAPHS WITH TWO VALUES

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Abstract. The Y-domination number of a graph for a given number set Y was introduced by D. W. Bange, A. E. Barkauskas, L. H. Host and P. J. Slater as a generalization of the domination number of a graph. It is defined using the concept of a Y-dominating function. In this paper the particular case where \( Y = \{0, 1/k\} \) for a positive integer \( k \) is studied.

Keywords: Y-dominating function of a graph, Y-domination number of a graph

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This paper will concern a certain generalization of the domination number of a graph. All graphs considered will be finite undirected graphs without loops and multiple edges.

A subset \( D \) of the vertex set \( V(G) \) of a graph \( G \) is called dominating in \( G \), if for each vertex \( x \in V(G) - D \) there exists a vertex \( y \in D \) adjacent to \( x \). The minimum number of vertices of a dominating set in \( G \) is called the domination number of \( G \) and denoted by \( \gamma(G) \).

This well-known concept can be defined in another way, using domination functions. We will speak about functions \( f \) which map \( V(G) \) into some set of numbers. If \( S \subseteq V(G) \), then we denote \( f(S) = \sum_{x \in S} f(x) \). If \( x \in V(G) \), then by \( N(x) \) we denote the closed neighbourhood of \( x \) in \( G \), i.e. the set consisting of \( x \) and all vertices which are adjacent to \( x \) in \( G \). Besides, we will also consider the open neighbourhood \( N(x) = N[x] - \{ x \} \). Now we can formulate the alternative definition of the domination number.

An function \( f : V(G) \to \{0, 1\} \) is called a dominating function of \( G \), if \( f(N[x]) \geq 1 \) for each \( x \in V(G) \). The minimum sum \( \sum_{x \in V(G)} f(x) \) taken over all dominating functions \( f \) of \( G \) is called the domination number of \( G \) and denoted by \( \gamma(G) \).
It is evident that these two definitions are equivalent. Namely, if \( D \) is a dominating set in \( G \), then the function \( f \) defined so that \( f(x) = 1 \) for \( x \in D \) and \( f(x) = 0 \) for \( x \in V(G) - D \) is a dominating function of \( G \). Conversely, if \( f \) is a dominating function of \( G \), then the set \( D = \{ x \in V(G) ; f(x) = 1 \} \) is a dominating set in \( D \).

The concept of a dominating function and obviously also the related concept of the domination number were generalized by some authors in such a way that the set of values \( \{0, 1\} \) was replaced by another number set. In [1] the signed dominating function and the signed domination number were defined by replacing the set \( \{0, 1\} \) by \( \{-1, 1\} \) and in [2] the minus dominating function and the minus domination number were defined by using the set \( \{-1, 0, 1\} \). The fractional dominating function and the fractional domination number were introduced in [3] by using the set of real numbers. The most general case is the \( Y \)-dominating function and the \( Y \)-domination number, where a quite arbitrary set \( Y \) of values of \( f \) is used [4].

Therefore, following [4], a function \( f : V(G) \to Y \), where \( Y \) is a given set of numbers, is called a \( Y \)-dominating function of \( G \), if \( f(N[x]) \geq 1 \) for each \( x \in V(G) \). The minimum of \( f(V(G)) \) taken over all \( Y \)-dominating functions \( f \) of \( G \) is called the \( Y \)-dominating number of \( G \) and is denoted by \( \gamma_Y(G) \).

We will not treat the domination is such a general way. We restrict our considerations to natural generalizations of the set \( \{0, 1\} \), namely to two-element number sets \( \{0, t\} \), where \( t \) is a positive real number.

The following proposition is easy to prove.

**Proposition 1.** Let \( Y = \{0, t\} \), where \( t \) is a positive real number. Let \( G \) be a graph. The \( Y \)-domination number \( \gamma_Y(G) \) of \( G \) is defined and at least one \( Y \)-dominating function of \( G \) exists if and only if \( \delta(G) \geq 1/t - 1 \), where \( \delta(G) \) denotes the minimum degree of a vertex of \( G \).

Let \( f \) be a function which maps \( V(G) \) into the set of real numbers and let \( x \in V(G) \). The vertex set \( x \) will be called a zero vertex of \( f \), if \( f(x) = 0 \).

The following theorem enables us to restrict our consideration to numbers \( t \) which are inverses of positive integers.

**Theorem 1.** Let \( t \) be a positive real number, let \( G \) be a graph with \( \delta(G) \geq 1/t - 1 \). Let \( k = \lfloor 1/t \rfloor \) and \( Y_1 = \{0, t\} \), \( Y_2 = \{0, 1/k\} \). Then \( \gamma_{Y_1}(G) = k \gamma_{Y_2}(G) \) and there exists a one-to-one correspondence between \( Y_1 \)-dominating functions of \( G \) and \( Y_2 \)-dominating functions of \( G \) such that the corresponding functions have the same set of zero vertices.

**Proof.** Let \( f : V(G) \to Y_1 \), \( g : V(G) \to Y_2 \) and suppose that \( f, g \) have the same set of zero vertices. Then \( f(x) = k g(x) \) and also \( f(N[x]) = k g(N[x]) \) for
each \( x \in V(G) \). Suppose that \( g \) is a \( Y_2 \)-dominating function of \( G \): then \( g(N[x]) \geq 1 \) for each \( x \in V(G) \). Evidently \( kt \geq 1 \) and thus \( f(N[x]) \geq g(N[x]) \geq 1 \) for each \( x \in V(G) \) and \( f \) is a \( Y_1 \)-dominating function of \( G \). Now suppose that \( g \) is not a \( Y_2 \)-dominating function of \( G \). There exists \( x \in V(G) \) such that \( g(N[x]) < 1 \). If \( k = 1 \), then \( g(N[x]) \) must be a non-negative integer and therefore \( g(N[x]) = 0 \). This is possible only if \( f(y) = 0 \) for each \( y \in N[x] \). But then also \( f(y) = 0 \) for each \( y \in N[x] \) and \( f(N[x]) = 0 \); the function \( f \) is not a \( Y_1 \)-dominating function of \( G \). If \( k \geq 2 \), then the number of vertices of \( N[x] \) which are not zero vertices of \( g \) is at most \( k - 1 \). But these vertices are exactly those vertices which are not zero vertices of \( f \). We have \( f(N[x]) \leq (k - 1)t \). Evidently \( 1/t > k - 1 \) and thus \( f(N[x]) = (k - 1)t < 1 \); the function \( f \) is not a \( Y_1 \)-dominating function of \( G \). If \( g_0 \) is a minimal (i.e. with the minimum sum on \( V(G) \)) \( Y_2 \)-dominating function, then the corresponding function \( f_0 \) is a minimal \( Y_1 \)-dominating function. We have 
\[
\gamma_{Y_1}(G) = \sum_{x \in V(G)} f_0(x) = \sum_{x \in V(G)} kt g_0(x) = kt \sum_{x \in V(G)} g_0(x) = kt \gamma_{Y_2}(G).
\]

\[\square\]

For each positive integer \( k \) we denote \( Y(k) = \{0, 1/k\} \) and \( \gamma(k, G) = \gamma_Y(k)G \). From Proposition 1 we have the following corollary.

**Corollary 1.** Let \( k \) be a positive integer, let \( G \) be a graph. The \( Y(k) \)-domination number \( \gamma(k, G) \) is defined and at least one \( Y(k) \)-dominating function of \( G \) exists if and only if \( \delta(G) \geq k - 1 \).

Note that \( \gamma(1, G) = \gamma(G) \), the usual domination number of \( G \).

If we speak about a function \( f : V(G) \to Y(k) \), we will use the notation \( V^0 = \{x \in V(G); f(x) = 0\} \), \( V^+ = \{x \in V(G); f(x) = 1/k\} \).

**Theorem 2.** Let \( G \) be a regular graph of degree \( k - 1 \) with \( n \) vertices. Then \( \gamma(k, G) = n/k \).

**Proof.** The neighbourhood \( N[x] \) for each \( x \in V(G) \) has exactly \( k \) vertices. If \( f \) is a \( Y(k) \)-dominating function, then \( f \) must assign the value \( 1/k \) to all vertices of \( N[x] \). As \( x \) was chosen arbitrarily, it assigns \( 1/k \) to all vertices of \( G \), which implies the assertion. \[\square\]

By \( G^2 \) we denote the square of the graph \( G \), i.e. the graph such that \( V(G^2) = V(G) \) and two vertices are adjacent in \( G^2 \) if and only if their distance in \( G \) is at most \( 2 \). The symbol \( \alpha_0(G) \) denotes the independence number of \( G \), i.e. the maximum number of pairwise non-adjacent vertices in \( G \).

**Theorem 3.** Let \( G \) be a regular graph of degree \( k \) with \( n \) vertices. Then \( \gamma(k, G) = (n - \alpha_0(G^2))/k \).

265
Proof. For each vertex \( x \) of \( G \) the set \( N[x] \) has \( k + 1 \) vertices. If \( f \) is a \( Y(k) \)-dominating function of \( G \), then \( N[x] \) contains at most one zero vertex of \( f \). The distance between two zero vertices of \( f \) cannot be 1; then the closed neighbourhood of either of them would contain them both. This distance cannot be 2; then there would exist a vertex adjacent to both of them and its closed neighbourhood would contain them both. Therefore the distance between two zero vertices of \( f \) in \( G \) is at least 3 and in \( G^2 \) at least 2; they form an independent set in \( G^2 \). Therefore there are at most \( a_0(G^2) \) zero vertices of \( f \) and at least \( n - a_0(G^2) \) vertices \( x \) such that \( f(x) = 1/k \). This implies the assertion. \( \square \)

Corollary 2. Let \( C_n \) be the circuit of length \( n \). Then \( \gamma(3, C_n) = n/3 \) and \( \gamma(2, C_n) = n/3 \) for \( n \equiv 0(\text{mod } 3) \), \( \gamma(2, C_n) = n/3 - 1/6 \) for \( n \equiv 1(\text{mod } 3) \), \( \gamma(2, C_n) = n/3 + 1/3 \) for \( n \equiv 2(\text{mod } 3) \).

A path is a similar case. If \( f \) is a \( Y(2) \)-dominating function of a path \( P_n \) of length \( n \), then again the distance between any two zero vertices of \( f \) is at least 3 and moreover neither the vertices of degree 1, not the vertices adjacent to them may be zero vertices of \( f \). This yields the result.

Proposition 2. Let \( P_n \) be a path of length \( n \). Then \( \gamma(2, P_n) = n/3 + 1 \) for \( n \equiv 0(\text{mod } 3) \), \( \gamma(2, P_n) = n/3 + 2/3 \) for \( n \equiv 1(\text{mod } 3) \), \( \gamma(2, P_n) = n/3 + 5/6 \) for \( n \equiv 2(\text{mod } 3) \).

Now we turn to complete graphs and complete bipartite graphs.

Theorem 4. Let \( k, n \) be positive integers, \( k \leq n \). Then \( \gamma(k, K_n) = 1 \).

Proof. In the complete graph \( K_n \) we have \( N[x] = V(K_n) \) for each vertex \( x \). If \( f \) is a \( Y(k) \)-dominating function, then \( f(V(K_n)) = f(N[x]) \geq 1 \). Moreover, there exists a function \( f \) which assigns the value \( 1/k \) to \( k \) vertices and the value \( 0 \) to the remaining \( n - k \) vertices: then \( f(V(K_n)) = 1 \). \( \square \)

Theorem 5. Let \( k, m, n \) be positive integers, \( k - 1 \leq m \leq n \). If \( k < m \), then \( \gamma(k, K_{m,n}) = 2 \). If \( m = k - 1 \), then \( \gamma(k, K_{m,n}) = (m + n)/k = (k + n - 1)/k \). If \( m = k \), then \( \gamma(k, K_{m,n}) = 2 - 1/k \).

Proof. Let \( k < m \). Let \( A, B \) be the bipartition classes of \( K \), \( |A| = m \), \( |B| = n \). For each vertex \( x \in A \), its open neighbourhood satisfies \( N(x) \subseteq B \). As \( N[x] = \{x\} \cup N(x) \) and \( f(N[x]) \geq 1 \) for a \( Y(k) \)-dominating function \( f \), there are at least \( k - 1 \) vertices \( y \in N(x) \subseteq A \) which are in \( V^+ \). If moreover \( f(x) = 0 \), then there are at least \( k \) such vertices. Therefore either \( f(x) = 1/k \) for all \( x \in A \) and
$f(y) = 1/k$ for at least $k-1$ vertices of $B$, or $f(y) = 1/k$ for at least $k$ vertices of $B$. In the former case $f(V(K_m)) \geq (m + k - 1)/k \geq 2$. In the latter case analogously either $f(x) = 1/k$ for all $x \in B$ and $f(y) = 1/k$ for at least $k-1$ vertices of $A$, or $f(y) = 1/k$ for at least $k$ vertices of $A$. In both these cases again $f(V(K_m)) \geq 2$. A function $f$ which assigns $1/k$ to exactly $k$ vertices of $A$ and to exactly $k$ vertices of $B$ has $f(V(K_m)) = 2$.

Now suppose $m = k - 1$. Then $|A| = k - 1$. Let $x \in B$ and again let $f$ be a $Y(k)$-dominating function of $K_m$. The set $N[x]$ has exactly $k$ vertices and thus $f(x) = 1/k$ for each $y \in N[x]$. This means that $f(y) = 1/k$ for each $y \in A$ and also $f(x) = 1/k$. As $x$ is an arbitrary vertex of $B$, we have $f(x) = 1/k$ for all $x \in V(K_m)$ and $f(V(K_m)) = (k - 1 + n)/k$. Another $Y(k)$-dominating function does not exist and thus $\gamma(k, K_m) = (k - 1 + n)/k$.

Finally, let $k = m$. If $f$ is a $Y(k)$-dominating function, then either $f(x) = 1/k$ for each $x \in A$ and for at least $k - 1$ vertices $x$ of $B$, or $f(x) = 1/k$ for exactly $k - 1$ vertices of $A$ and all vertices $x \in B$. In the former case $f(V(K_m)) \geq (2k - 1)/k = 2 - 1/k$, in the latter case $f(V(K_m)) \geq (k - 1 + n)/k \geq (2k - 1)/k = 2 - 1/k$. If $f$ assigns the value $1/k$ to all vertices of $A$ and to exactly $k - 1$ vertices of $B$, then $f(V(K_m)) = 2 - 1/k$, therefore $\gamma(k, K_m) = 2 - 1/k$.

By the symbol $G \oplus H$ we denote the Zykov sum of graphs $G$ and $H$, i.e. the graph obtained from vertex-disjoint graphs $G$ and $H$ by joining all vertices of $G$ with all vertices of $H$ by edges.

**Theorem 6.** Let $k, q$ be positive integers, let $G, H$ be two graphs such that $\gamma(k, G)$, $\gamma(k, H)$ are defined and $q \leq 1 + \min(\gamma(k, G), \gamma(k, H))$. Then $\gamma(kq, G \oplus H) \leq (\gamma(k, G) + \gamma(k, H))/q$.

**Proof.** Let $g$ and $h$ be minimal $Y(k)$-dominating functions of $G$ and $H$, respectively. Let $f : V(G) \cup V(H) \to Y(kq)$ be defined so that $f(x) = g(x)/q$ for $x \in V(G)$ and $f(x) = h(x)/q$ for $x \in V(H)$. Consider $x \in V(G)$. The closed neighbourhood of $x$ in $G \oplus H$ is the disjoint union of the closed neighbourhood of $x$ in $G$ and of $V(H)$. The sum of values of $f$ over the closed neighbourhood of $x$ in $G$ is at least $1/q$, its sum over $V(H)$ is at least $\gamma(k, H)/q$. It follows from the assumption that $1/q + \gamma(k, H)/q \geq 1$. For $x \in V(H)$ this may be proved quite analogously. Therefore $f$ is a $Y(kq)$-dominating function of $G \oplus H$. This implies the assertions.

For the particular case $k = 1$ we have a corollary.

**Corollary 3.** Let $q$ be a positive integer, let $G, H$ be two graphs such that $q \leq 1 + \min(\gamma(G), \gamma(H))$. Then $\gamma(q, G \oplus H) \leq (\gamma(G) + \gamma(H))/q$.

267
A similar assertion holds for $G \oplus K_1$, i.e. the graph which is obtained from $G$ by adding a new vertex and joining it with all vertices of $G$ by edges.

**Theorem 7.** Let $k$ be a positive integer, let $G$ be a graph for which $\gamma(k, G)$ is defined. Then

$$\gamma(k + 1, G \oplus K_1) = \gamma(k, G) \cdot \frac{k}{k+1} + \frac{1}{k+1}.$$ 

**Proof.** Let $f$ be a minimal $Y(k)$-dominating function of $G$. Let $w$ be the added vertex. Let $g : V(G) \cup \{w\} \to Y(k + 1)$ be defined so that $g(x) = kf(x)/(k + 1)$ for $x \in V(G)$ and $g(w) = 1/(k + 1)$. Then the sum of $g(x)$ over the closed neighbourhood of $x$ in $G \oplus K_1$ is equal to the sum of $g$ over the closed neighbourhood of $x$ in $G$ plus $g(w)$. The sum of $g$ over the closed neighbourhood of $x$ in $G$ is at least $k/(k + 1)$ and $g(w) = 1/(k + 1)$, therefore the sum of $g$ over the closed neighbourhood of $x$ in $G \oplus K_1$ is at least $1$. The closed neighbourhood of $w$ in $G \oplus K_1$ is $V(G) \cup \{w\}$ and the sum of $g$ over it is greater than or equal to this sum over the closed neighbourhood of any other vertex, therefore it is also at least 1 and

$$\sum_{x \in V(G) \cup \{w\}} g(x) = \frac{k}{k+1} \sum_{x \in V(G)} f(x) + g(w) = \frac{k}{k+1} \gamma(k, G) + \frac{1}{k+1}.$$ 

Hence $\gamma(k + 1, G \oplus K_1) \leq \frac{k}{k+1} \gamma(k, G) + \frac{1}{k+1}$. On the other hand, let $g_0$ be a minimal $Y(k + 1)$-dominating function of $G \oplus K_1$ and let $f_0 : V(G) \to Y(k)$ be defined so that $f_0(x) = (k + 1)g_0(x)/k$ for each $x \in V(G)$. The sum of values of $g$ over the closed neighbourhood of any vertex $x \in V(G)$ in $G$ is at least $1 - 1/(k + 1)$ and thus such a sum of $f_0$ is at least 1. We have

$$\sum_{x \in V(G)} f_0(x) = \sum_{x \in V(G)} (k + 1)g_0(x)/k = \frac{k+1}{k} \sum_{x \in V(G)} g_0(x) = \frac{k+1}{k} \gamma(k + 1, G \oplus K_1) - \frac{1}{k}$$ 

and thus

$$\gamma(k, G) \leq \frac{k+1}{k} \gamma(k + 1, G \oplus K_1) - \frac{1}{k},$$

which yields $\gamma(k + 1, G \oplus K_1) \geq \frac{k}{k+1} \gamma(k, G) + \frac{1}{k+1}$. Hence we have the equality $\gamma(k + 1, G \oplus K_1) = \frac{k}{k+1} \gamma(k, G) + \frac{1}{k+1}$. \qed

In the end we will consider the number $\gamma(k, G)$ for different numbers $k$ and for the same graph $G$.

**Theorem 8.** Let $k, q$ be positive integers. Then there exists a graph $G$ such that $\gamma(k + 1, G) - \gamma(k, G) = q$.

**Proof.** Denote $p = kq + q + 1$ and let $G$ be the Zykov sum $K_2 \oplus \overline{K}_p$, where $\overline{K}_p$ denotes the complement of the complete graph $K_p$, i.e. the graph consisting of $p$ isolated vertices. If $f$ is a function such that $f(x) = 0$ for $x \in V(\overline{K}_p)$ and $f(x) = 1/k$ for $x \in V(K_2)$, then $f$ is a $Y(k)$-dominating function of $G$; namely, we have $V(K_2) \subseteq N[x]$ for each $x \in V(G)$ and $f(V(K_2)) = 1$. We have $\gamma(k, G) = 1$. Each vertex of
\(K_p\) has degree \(k\) in \(G\) and therefore for each \(Y(k+1)\)-dominating function \(g\) we have 
\(g(y) = 1/(k+1)\) for each \(y \in V(G)\) and 
\(\gamma(k+1, G) = (p+k)/(k+1) = q + 1. \)

The next theorem is not expressed for \(k\) in general, but only for \(\gamma(1,G)\) and \(\gamma(2,G)\).

**Theorem 9.** Let \(q\) be a positive integer. Then there exists a graph \(G\) such that 
\(\gamma(1,G) - \gamma(2,G) = q.\)

**Proof.** Let \(H\) be a graph obtained from the circuit of length 4 by adding a new vertex \(u\) and joining it to a vertex \(v\) of the circuit by an edge. Take \(2q\) pairwise vertex-disjoint copies \(H_1, \ldots, H_{2q}\) of \(H\). Take a vertex \(w\) and join it by edges with the vertex corresponding to \(u\) in each of the graphs \(H_1, \ldots, H_{2q}\). Finally, take a new vertex \(x\) and join it with \(w\) by an edge. The resulting graph will be \(G\). For \(q = 4\) this graph is shown Fig. 1. The number \(\gamma(1,G)\) is the usual domination number \(\gamma(G)\) of \(G\),

![Diagram](image-url)

i.e. the minimum number of vertices of a dominating set \(D\) in \(G\). Evidently such a dominating set must contain at least one of the vertices \(w, x\) and at least two vertices from each \(H\) for \(i = 1, \ldots, 2q\); hence \(\gamma(G) \geq 4q + 1\). If \(D\) consists of \(w, x\) of the vertices corresponding to \(v\) in \(H\) and of one other vertex of the circuit in \(H\) for \(i = 1, \ldots, 2q\), then \(D\) is dominating in \(G\) and \(|D| = 4q + 1\), which implies \(\gamma(G) = 4q + 1\). Now let \(V^+\) be the set consisting of all vertices of \(D\) and, moreover, of \(x\) and of one more vertex of the circuit in each \(H\) for \(i = 1, \ldots, 2q\). We have \(|V^+| = 6q + 2\). If \(f(x) = \frac{1}{2}\) for \(x \in V^+\) and \(f(x) = 0\) for \(x \in V(G) - V^+\), then \(f\) is a \(V(2)\)-dominating function of \(G\) and is evidently minimal. We have \(\gamma(2,G) = f(V(G)) = \frac{1}{2}|V^+| = 3q + 1\). Hence 
\(\gamma(1,G) - \gamma(2,G) = q.\) \(\Box\)
Problem. Can Theorem 10 be generalized to a theorem analogous to Theorem 9?

A final remark. The $Y(k)$-domination number of a graph can be defined in another way, without using the concept of a $Y(k)$-dominating function:

A subset $D$ of $V(G)$ is called $k$-tuply dominating in $G$, if for each $x \in V(G) - D$ there exist $k$ vertices $y_1, \ldots, y_k$ od $D$ adjacent to $x$ and for each $y \in D$ there exist $k - 1$ vertices $z_1, \ldots, z_{k-1}$ adjacent to $y$. The minimum number of vertices of a $k$-tuply dominating set in $G$ is called the $Y(k)$-domination number of $G$.

A $k$-tuply dominating set was defined and used also in [5], but in a weaker form: the requirement of existence of $z_1, \ldots, z_{k-1}$ for $y \in D$ was not used there.

References


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