DISJOINT SEQUENCES IN BOOLEAN ALGEBRAS

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Abstract. We deal with the system Conv $B$ of all sequential convergences on a Boolean algebra $B$. We prove that if $\alpha$ is a sequential convergence on $B$ which is generated by a set of disjoint sequences and if $\beta$ is any element of Conv $B$, then the join $\alpha \lor \beta$ exists in the partially ordered set Conv $B$. Further we show that each interval of Conv $B$ is a Brouwerian lattice.

Keywords: Boolean algebra, sequential convergence, disjoint sequence

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1. Introduction

Some types of sequential convergences on Boolean algebras were investigated by Lőwig [3], Novák and Novotný [4] and Papangelou [5].

This note is a continuation of [1]. Throughout the paper we assume that $B$ is a Boolean algebra which has more than one element. Conv $B$ is the system of all sequential convergences on $B$ which are compatible with the structure of $B$. For the sake of completeness, the definition of Conv $B$ as given in [1] is recalled in Section 2.

The system Conv $B$ is partially ordered by the set-theoretical inclusion. It is a $\land$-semilattice with the least element (the discrete convergence on $B$). In general, Conv $B$ fails to be a lattice; i.e., for $\alpha$ and $\beta$ in Conv $B$, the join $\alpha \lor \beta$ need not exist in the partially ordered set Conv $B$.

A sufficient condition for Conv $B$ to be a lattice was found in [2].

We denote by $D(B)$ the system of all sequences $(x_n)$ in $B$ such that

(i) $x_{n(1)} \land x_{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct positive integers;
(ii) $x_n > 0$ for each positive integer $n$. 
The sequences belonging to $D(B)$ will be called disjoint.

We prove that for each subset $A$ of $D(B)$ there exists a sequential convergence $\alpha \in \text{Conv } B$ which is generated by $A$ and that for any $\beta \in \text{Conv } B$ the join $\alpha \lor \beta$ exists in the partially ordered set $\text{Conv } B$.

Further we show that each interval of $\text{Conv } B$ is a complete lattice satisfying the identity

$$\left( \bigvee_{i \in I} \alpha_i \right) \land \beta = \bigvee_{i \in I} (\alpha_i \land \beta).$$

This implies that each interval of $\text{Conv } B$ is a Brouwerian lattice.

2. Preliminaries

We denote by $S$ the system of all sequences in $B$. Let $\alpha \subseteq S \times B$. If $((x_n), x) \in \alpha$, then we denote this fact by writing $x_n \rightarrow^{\alpha} x$. For $a \in B$, const $a$ denotes the sequence $(a)$ such that $x_n = a$ for each $n \in \mathbb{N}$.

We recall the definitions of $\text{Conv } B$ and $\text{Conv}_0 B$ from [1].

2.1. Definition. A subset of $S \times B$ is said to be a convergence on $B$ if the following conditions are satisfied:

(i) If $x_n \rightarrow^{\alpha} x$ and $(y_n)$ is a subsequence of $(x_n)$, then $y_n \rightarrow^{\alpha} x$.

(ii) If $(x_n) \in S$, $x \in B$ and if for each subsequence $(y_n)$ of $(x_n)$ there is a subsequence $(z_n)$ of $(y_n)$ such that $z_n \rightarrow^{\alpha} x$, then $x_n \rightarrow^{\alpha} x$.

(iii) If $a \in B$ and $(x_n) = \text{const } a$, then $x_n \rightarrow^{\alpha} a$.

(iv) If $x_n \rightarrow^{\alpha} x$ and $x_n \rightarrow^{\alpha} y$, then $x = y$.

(v) If $x_n \rightarrow^{\alpha} x$ and $y_n \rightarrow^{\alpha} y$, then $x_n \lor y_n \rightarrow^{\alpha} x \lor y$, $x_n \land y_n \rightarrow^{\alpha} x \land y$ and $x' \rightarrow^{\alpha} x'$.

(vi) If $x_n \leq y_n \leq z_n$ is valid for each $n \in \mathbb{N}$ and $x_n \rightarrow^{\alpha} x$, $z_n \rightarrow^{\alpha} x$, then $y_n \rightarrow^{\alpha} x$.

The system of all convergences on $B$ is denoted by $\text{Conv } B$.

For each $\alpha \in \text{Conv } B$ we put

$$\alpha_0 = \{(x_n) \in S : x_n \rightarrow^{\alpha} 0\}.$$

Further we define

$$\text{Conv}_0 B = \{\alpha_0 : \alpha \in \text{Conv } B\}.$$

Both the systems $\text{Conv } B$ and $\text{Conv}_0 B$ are partially ordered by the set-theoretical inclusion; the suprema and infima (if they exist) in $\text{Conv } B$ or in $\text{Conv}_0 B$ are denoted by the symbol $\lor$ or $\land$, respectively.
Next, we denote by $d$ the system of all $((x_n), x) \in S \times B$ such that the set 
$n \in \mathbb{N}: x_n \neq x$ is finite. Then $d$ is the least element of Conv $B$.

For each $\alpha \in \text{Conv } B$ we put $f(\alpha) = \alpha_0$.

2.2. Lemma. The mapping $f$ is an isomorphism of the partially ordered set Conv $B$ onto the partially ordered set Conv$_0 B$.

Proof. We have $f(\text{Conv } B) = \text{Conv}_0 B$. In view of 1.4 in [1], $f$ is a monomorphism.

Let $\alpha, \beta \in \text{Conv } B$, $\alpha \leq \beta$. Further let $(x_n) \in \alpha$. Hence $((x_n), 0) \in \alpha$, thus
$((x_n), 0) \in \beta$ and then $(x_n) \in \beta_0$. Thus $\alpha_0 \leq \beta_0$.

Now let $\alpha, \beta \in \text{Conv } B$, $\alpha_0 \leq \beta_0$. Assume that $((x_n), x) \in \alpha$. In view of 1.3 in [1] we have
$x_n \wedge x' =_{\alpha} 0$, $x_n' \wedge x =_{\alpha} 0$.

Thus from the relation $\alpha_0 \leq \beta_0$ we obtain
$x_n \wedge x' =_{\beta} 0$, $x_n' \wedge x =_{\beta} 0$.

Then by applying 1.3 in [1] again we get $x_n \rightarrow_{\beta} x$. Hence $\alpha \leq \beta$. \hfill $\square$

As a consequence we obtain that $d_0$ is the least element of Conv$_0 B$.

2.3. Lemma. (Cf. [1].) (i) Conv$_0 B$ is a $\wedge$-semilattice and each interval of
Conv$_0 B$ is a complete lattice.

(ii) If $\emptyset \neq \{\alpha^0_i\}_{i \in I} \subseteq \text{Conv}_0 B$, then
$$\bigwedge_{i \in I} \alpha^0_i = \bigcap_{i \in I} \alpha^0_i.$$  

(iii) There exists a Boolean algebra $B_1$ such that Conv$_0 B_1$ fails to be a lattice.

From 2.2 and 2.3 we infer

2.4. Proposition. Conv $B$ is a $\wedge$-semilattice and each interval of Conv $B$ is
a complete lattice. There exists a Boolean algebra $B_1$ such that Conv $B_1$ is not a lattice.
3. On the set \( D(B) \)

We apply the notation as in the previous sections. A subset \( T \) of \( S \) is called regular if there exists \( \alpha_0 \in \text{Conv}_0 B \) such that \( T \subseteq \alpha_0 \).

Let \( T \) be a regular subset of \( S \) and let \( \alpha_0 \) be as above. Then in view of 2.3 there exists an element \( \alpha^0(T) \) of \( \text{Conv}_0 B \) such that \( \alpha^0(T) \) is the least element of \( \text{Conv}_0 B \) having \( T \) as a subset. We say that \( \alpha^0(T) \) is the element of \( \text{Conv}_0 B \) which is generated by \( T \). We also say that \( T \) generates the convergence \( \alpha_0 \), where \( \alpha_0 = \alpha^0(T) \).

If \( T \) is regular, then clearly each subset of \( T \) is regular.

For \( (x_n), (y_n) \in S \) we put \( (x_n) \leq (y_n) \) if \( x_n \leq y_n \) for each \( n \in \mathbb{N} \). Then \( S \) turns out to be a Boolean algebra. Let \( A \) be a nonempty subset of \( S \). We denote by

\[ A^* \]   the set of all \( (x_n) \in S \) such that for each subsequence \( (y_n) \) of \( (x_n) \) there exists a subsequence \( (z_n) \) of \( (y_n) \) which belongs to \( A \);

\[ [A] \]   the ideal of the Boolean algebra generated by the set \( A \);

\( \delta A \)   the set of all subsequences of sequences belonging to \( A \).

The following assertion is easy to verify.

3.1. Lemma. Let \( A \) be a nonempty subset of \( S \). Then \([A]\) is the set of all sequences \((z_n) \in S\) such that there exist \( k \in \mathbb{N} \) and \((w^1_n), (w^2_n), \ldots, (w^k_n) \in A\) having the property that the relation

\[ z_n \leq w^1_n \lor w^2_n \lor \ldots \lor w^k_n \]

is valid for each \( n \in \mathbb{N} \).

3.2. Lemma. (Cf. [1], 2.9.) Let \( \emptyset \neq A \subseteq S \). Then the following conditions are equivalent:

(i) \( A \) is regular.

(ii) If \((y^1_n), (y^2_n), \ldots, (y^k_n) \) are elements of \( \delta A \) and if \( b \) is an element of \( B \) such that \( b \leq y^1_n \lor y^2_n \lor \ldots \lor y^k_n \) is valid for each \( n \in \mathbb{N} \), then \( b = 0 \).

From the definition of \( \text{Conv}_0 B \) and from [1], 2.5 we conclude

3.3. Lemma. Let \( A \neq \emptyset \) be a regular subset of \( S \). Then \([\delta A]^*\) is an element of \( \text{Conv}_0 B \) which is generated by the set \( A \).

3.4. Lemma. (Cf. [1], 5.2.) Let \( (x_n) \in D(B) \). Then the set \( \{ (x_n) \} \) is regular.

3.5. Lemma. Let \( (x_n) \in D(B) \) and suppose that \((y^1_n), (y^2_n), \ldots, (y^k_n) \) are subsequences of \( (x_n) \). Put \( (z_n) = y^1_n \lor y^2_n \lor \ldots \lor y^k_n \) for each \( n \in \mathbb{N} \). Then there exists a subsequence \( (t_n) \) of \( (z_n) \) such that \( (t_n) \in D(B) \).
Proof. For each $i \in \{1, 2, \ldots, k\}$ and each $n \in \mathbb{N}$ there is a positive integer $j(i, n)$ such that

$$y_n^i = x_{j(i, n)}.$$

Thus for each $i \in \{1, 2, \ldots, k\}$ we have

$$j(i, n) \to \infty \quad \text{as} \quad n \to \infty. \quad (1)$$

We define the sequence $(t_n)$ by induction as follows. We put $t_1 = z_1$. Suppose that $n > 1$ and that $t_1, t_2, \ldots, t_{n-1}$ are defined. Hence there are $\ell(1), \ell(2), \ldots, \ell(n-1) \in \mathbb{N}$ with

$$t_s = z_{\ell(s)} \quad \text{for} \quad s = 1, 2, \ldots, n - 1.$$

In view of (1) there exists the least positive integer $p$ having the property that for each $s \in \{1, 2, \ldots, n - 1\}$ and each $i(1), i(2) \in \{1, 2, \ldots, k\}$ the relation

$$j(i(1), s) < j(i(2), p)$$

is valid. Then we put $t_n = z_p$.

Hence $t_n \wedge t_s = 0$ for $s = 1, 2, \ldots, n - 1$. Thus $(z_n) \in D(B)$. \hfill \Box

3.6. Lemma. Let $\emptyset \neq A_1$ be a regular subset of $S$ and let $(x_n) \in D(B)$. Then the set $A_1 \cup \{(x_n)\}$ is regular.

Proof. We denote by $a_0$ the element of Conv $B$ which is generated by the set $A_1$. Put $A = A_1 \cup \{(x_n)\}$. By way of contradiction, suppose that $A$ fails to be regular. Then in view of 3.2 there are $(y_n^1), (y_n^2), \ldots, (y_n^m) \in \delta A$ and $0 < b \in B$ such that the relation

$$0 < b \leq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$$

is valid for each $n \in \mathbb{N}$. Put

$$M_1 = \{i \in \{1, 2, \ldots, m\} : (y_n^i) \in A_1\},$$

$$M_2 = \{1, 2, \ldots, m\} \setminus M_1.$$

Since the set $A_1$ is regular, in view of 3.2 the relation $M_2 = \emptyset$ cannot hold. Further, according to 3.4 and 3.2, the set $M_1$ cannot be empty. Denote

$$z_n^1 = \bigvee y_n^i \quad (i \in M_1), \quad z_n^2 = \bigvee y_n^i \quad (i \in M_2).$$

Then $(z_n^1) \in a_0$.  

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According to 3.5 there exists a mapping \( \varphi : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \varphi \) is increasing and the sequence \( (z_{\varphi(n)})^2 \) belongs to \( D(B) \). We have
\[
0 < b \leq z_{\varphi(n)}^1 \lor z_{\varphi(n)}^2 \quad \text{for each } n \in \mathbb{N}.
\]
Put
\[
b \land z_{\varphi(n)}^1 = q_n^1, \quad b \land z_{\varphi(n)}^2 = q_n^2.
\]
Then
\[
b = q_n^1 \lor q_n^2
\]
for each \( n \in \mathbb{N} \). We have \( (q_n^1) \in \alpha_0 \) and \( (q_n^2) \in D(B) \).
Since \( b = q_{n+1}^1 \lor q_{n+1}^2 \) we get
\[
q_n^2 = q_n^2 \land b = q_n^2 \land (q_{n+1}^1 \lor q_{n+1}^2) = (q_n^2 \land q_{n+1}^1) \lor (q_n^2 \land q_{n+1}^2) = q_n^2 \land q_{n+1}^2
\]
and clearly \( (q_n^2 \land q_{n+1}^2) \in \alpha_0 \). Therefore \( (q_n^1 \lor q_n^2) \in \alpha_0 \) yielding that \( \text{const } b \in \alpha_0 \), which is impossible.

By the obvious induction, from 3.6 we obtain

3.7. Lemma. Let \( \emptyset \neq A_1 \) be a regular subset of \( S \), \( m \in \mathbb{N} \), \( (x_1^n), (x_2^n), \ldots, (x_m^n) \in D(B) \). Then the set \( A_1 \cup \{(x_1^n), (x_2^n), \ldots, (x_m^n)\} \) is regular.

Since the system of sequences which is dealt with in the condition (ii) of 3.2 is finite, from 3.7 we conclude

3.8. Proposition. Let \( \emptyset \neq A_1 \) be a regular subset of \( S \). Then the set \( A_1 \cup D(B) \) is regular.

It is obvious that if \( \emptyset \neq A_2 \subseteq S \), then \( A_2 \) is regular if and only if the set \( \{\text{const } 0\} \cup A_2 \) is regular. Hence by putting \( A_1 = \{\text{const } 0\} \), from 3.8 we obtain

3.9. Proposition. The set \( D(B) \) is regular.

In view of 3.9, there exists \( \gamma \in \text{Conv } B \) which is generated by the set \( D(B) \).
Let \( \alpha_0 \in \text{Conv } B \). According to 3.8, the set \( \alpha_0 \cup D(B) \) is regular. Hence there exists \( \beta_0 \in \text{Conv } B \) such that \( \beta_0 \) is generated by the set \( \alpha_0 \cup D(B) \).

In view of 3.3, we have \( \alpha_0 \leq \beta_0 \) and \( \gamma_0 \leq \beta_0 \). Let \( \beta_1 \in \text{Conv } B \), \( \beta_1 \geq \alpha_0 \), \( \beta_1 \geq \gamma_0 \). Thus \( D(B) \subseteq \beta_1 \) and hence \( \alpha_0 \cup D(B) \subseteq \beta_1 \). By using 3.3 again we get \( \beta_0 \leq \beta_1 \).
Therefore \( \beta_0 = \alpha_0 \lor \gamma_0 \). We obtain

3.10. Proposition. Let \( \alpha_0 \in \text{Conv } B \). Then the join \( \alpha_0 \lor \gamma_0 \) exists in the partially ordered set \( \text{Conv } B \).

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In view of 2.2 we conclude

3.11. Corollary. Let $\alpha \in \text{Conv } B$. Then the join $\alpha \lor \gamma$ exists in the partially ordered set $\text{Conv } B$.

If $A_0$ is a nonempty subset of $D(B)$, then it is regular and thus there exists $\gamma_1 \in \text{Conv } B$ which is generated by $A_0$. Clearly $\gamma_1 \leq \gamma$; from 3.11 and 2.4 we obtain

3.12. Corollary. Under the notation as above, for each $\alpha \in \text{Conv } B$ there exists $\alpha \lor \gamma_1$ in $\text{Conv } B$.

4. A DISTRIBUTIVE IDENTITY

Suppose that $\mu_1$ and $\mu_2$ are elements of $\text{Conv}_0 B$ such that $\mu_1 \leq \mu_2$. Consider the interval $[\mu_1, \mu_2]$ of the partially ordered set $\text{Conv}_0 B$. In view of 2.3, this interval is a complete lattice.

Let $\emptyset \neq \{\alpha_i\}_{i \in I} \subseteq [\mu_1, \mu_2]$ and $\beta \in [\mu_1, \mu_2]$. Then the elements

$$\nu_1 = \bigvee_{i \in I} \alpha_i \land \beta, \quad \nu_2 = \bigvee_{i \in I} (\alpha_i \land \beta)$$

exist in $[\mu_1, \mu_2]$ and $\nu_1 \geq \nu_2$. Put

$$A_1 = \bigcup_{i \in I} \alpha_i, \quad A_2 = \bigcup_{i \in I} (\alpha_i \cap \beta).$$

Suppose that $(v_n) \in \nu_1$. Hence according to 2.3 we have

$$(v_n) \in \beta \quad \text{and} \quad (v_n) \in \bigvee_{i \in I} \alpha_i.$$

From the second relation and from Lemma 3.3 in [1] we obtain

$$(v_n) \in [A_1]^*.$$

Hence for each subsequence $(t^1_n)$ of $(v_n)$ there is a subsequence $(t^2_n)$ of $(t^1_n)$ such that $(t^2_n) \in [A_1]$.

Let $(t^1_n)$ and $(t^2_n)$ have the mentioned properties. Therefore in view of 3.1 there are $(w^1_n), (w^2_n), \ldots, (w^k_n)$ in $A$ such that the relation

$$t^2_n \leq w^1_n \lor w^2_n \lor \ldots \lor w^k_n$$

holds for each $n$.
is valid for each \( n \in \mathbb{N} \). Put
\[
q_n^j = t_n^2 \wedge w_n^j
\]
for each \( n \in \mathbb{N} \) and each \( j \in \{1, 2, \ldots, k\} \). Thus
\[
t_n^2 = q_n^1 \vee q_n^2 \vee \ldots \vee q_n^k
\]
for each \( n \in \mathbb{N} \),

and \( (q_n^1), (q_n^2), \ldots, (q_n^k) \in A_1 \). At the same time we have \( (q_n^1), (q_n^2), \ldots, (q_n^k) \in \beta \).

Hence for each \( j \in \{1, 2, \ldots, k\} \) there is \( i(j) \in I \) such that
\[
(q_n^j) \in \alpha_{i(j)} \cap \beta.
\]

In view of 3.1, this yields that \( t_n^2 \) belongs to \([A_2]\). Therefore \((v_n) \in [A_2]^*\). Thus by applying Lemma 3.3 in [1] we get \((v_n) \in \nu_2\).

Summarizing, we have

4.1. Proposition. Let \([\mu_1, \mu_2] \) be an interval of \( \text{Conv}_0 B \), \( \beta \in [\mu_1, \mu_2] \), \( \emptyset \neq \{\alpha_i\}_{i \in I} \subseteq [\mu_1, \mu_2] \). Then

\[
\left( \bigvee_{i \in I} \alpha_i \right) \wedge \beta = \bigvee_{i \in I} (\alpha_i \wedge \beta).
\]

4.2. Corollary. Each interval of \( \text{Conv}_0 B \) is Brouwerian.

From 4.1 and 2.2 we obtain

4.3. Corollary. Each interval of \( \text{Conv} B \) satisfies the identity (1).

References


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