MV-ALGEBRAS ARE CATEGORICALLY EQUIVALENT TO A CLASS OF DRL-(i)-SEMIGROUPS

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Abstract. In the paper it is proved that the category of MV-algebras is equivalent to
the category of bounded DRl-semigroups satisfying the identity $1 - (1 - x) = x$. Consequently, by a result of D. Mundici, both categories are equivalent to the category of bounded commutative BCK-algebras.

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The notion of an MV-algebra was introduced by C. C. Chang in [1], [2] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. D. Mundici in [9] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras introduced by S. Tanaka in [12]. The notion of a dually residuated lattice ordered semigroup (DRl-semigroup) was introduced by K. L. N. Swamy in [11] as a common generalization of Brouwerian algebras and commutative lattice ordered groups (l-groups). Some connections between DRl-semigroups and MV-algebras were studied by the author in [10].

In this paper we will show that MV-algebras (and so also bounded commutative BCK-algebras) are categorically equivalent to some DRl-semigroups.

Let us recall the notions of an MV-algebra and a DRl-semigroup.

An MV-algebra is an algebra $A = (A, \boxplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities. (See e.g. [3].)

\begin{align*}
(MV 1) & \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z; \\
(MV 2) & \quad x \oplus y = y \oplus x; \\
(MV 3) & \quad x \oplus 0 = x;
\end{align*}
(MV 4) \( \neg \neg x = x; \)
(MV 5) \( x \oplus 0 = 0; \)
(MV 6) \( \neg(\neg x \oplus y) \oplus y = \neg(x \oplus y) \oplus x. \)

A **DRI-semigroup** is an algebra \( A = (A, +, 0, \vee, \wedge, -) \) of type \( (2, 0, 2, 2, 2) \) such that

1. \( (A, +, 0) \) is a commutative monoid;
2. \( (A, \vee, \wedge) \) is a lattice;
3. \( (A, +, \vee, \wedge) \) is a lattice ordered semigroup \((l\text{-semigroup})\), i.e. \( A \) satisfies the identities
   \[
   x + (y \vee z) = (x + y) \vee (x + z),
   \]
   \[
   x + (y \wedge z) = (x + y) \wedge (x + z).
   \]
4. If \( \leq \) denotes the order on \( A \) induced by the lattice \( (A, \vee, \wedge) \) then for each \( x, y \in A \), the element \( x - y \) is the smallest \( z \in A \) such that \( y + z \geq x \).
5. \( A \) satisfies the identity
   \[
   (x - y) \vee 0 \leq x \vee y.
   \]

As is shown in [11], condition (4) is equivalent to the following system of identities:

\[
(4') \quad x + (y - x) \geq y; \\
   x - y \leq (x \vee z) - y; \\
   (x + y) - y \leq x.
\]

Hence **DRI-semigroups** form a variety of type \( (2, 0, 2, 2, 2) \).

**Note.** In Swamy's original definition of a **DRI-semigroup**, the identity \( x - x \geq 0 \) is also required. But by [6], Theorem 2, in any algebra satisfying (1)-(4) the identity \( x - x = 0 \) is always satisfied.

**DRI-semigroups** can be viewed as intervals of abelian \( l\)-groups. Indeed, let \( G = (G, +, 0, -, (\cdot), \vee, \wedge) \) be an abelian \( l\)-group and let \( 0 \leq u \in G \). For any \( x, y \in [0, u] = \{ x \in G; 0 \leq x \leq u \} \) set \( x \oplus y = (x + y) \wedge u \) and \( \neg x = u - x \). Put \( \Gamma(G, u) = ([0, u], \oplus, \neg, 0) \). Then \( \Gamma(G, u) \) is an **MV-algebra**. The **MV-algebras** in the form \( \Gamma(G, u) \) are sufficiently universal because by [7], if \( A \) is any **MV-algebra** then there exist an abelian \( l\)-group \( G \) and \( 0 \leq u \in G \) such that \( A \) is isomorphic to \( \Gamma(G, u) \).

The intervals of type \([0, u]\) of abelian \( l\)-groups can be also considered as (bounded) **DRI-semigroups**. Indeed, by [10], Theorem 1, if \( G = (G, +, 0, -, (\cdot), \vee, \wedge) \) is an abelian \( l\)-group, \( 0 \leq u \in G \), \( B = [0, u] \), and if \( x \oplus y = (x + y) \wedge u \) and \( x \oplus y = (x - y) \vee 0 \) for any
$x, y \in B$, then $(B, \oplus, 0, \lor, \land, \oslash)$ is a bounded $DRI$-semigroup in which, moreover, $u \ominus (u \ominus x) = x$ for each $x \in B$. So we have ([10], Corollary 2) that if $A = (A, \oplus, \neg, 0)$ is an MV-algebra and if we set $x \leq y \Longleftrightarrow (\neg x \oplus y) \ominus y = y$ for any $x, y \in A$, then $\leq$ is a lattice order on $A$ (with the lattice operations $x \lor y = \neg((\neg x) \oplus y) \ominus y$ and $x \land y = \neg((\neg x) \vee y)$), for any $r, s \in A$ there exists a least element $r \ominus s$ with the property $s \ominus (r \ominus s) \geq r$, and $(A, \oplus, 0, \lor, \land, \oslash)$ is a bounded $DRI$-semigroup with the smallest element 0 and the greatest element $\neg 0$ in which $\neg 0 \ominus (\neg 0 \ominus x) = x$ for any $x \in A$. Further ([10], Theorem 3), if $(B, +, 0, \lor, \land, \neg)$ is a bounded $DRI$-semigroup with the greatest element 1 in which $1 - (1 - x) = x$ for any $x \in B$, and if we set $\neg x = 1 - x$ for any $x \in B$, then $(B, +, \neg, 0)$ is an MV-algebra.

**Note.** In [10], Theorem 3, the validity of the identity $x + (y - x) = y + (x - y)$ is also required. By [5], Theorem 1.2.3, if a $DRI$-semigroup $A$ has the greatest element, then $A$ is bounded also below and, moreover, 0 is the smallest element in $A$. And if this is the case then by [11], Lemma 2, $x + (y - x) = x \lor y$ for any $x, y \in A$, hence the identity $x + (y - x) = y + (x - y)$ is valid in $A$.

The following two propositions will make it possible to prove the main result of the paper. (The homomorphisms will be always meant with respect to the types and signatures mentioned.)

**Proposition 1.** Let $A = (A, \oplus, \neg, 0)$ and $B = (B, \oplus, \neg, 0')$ be MV-algebras and $f: A \rightarrow B$ a homomorphism of MV-algebras. Then $f$ is also a homomorphism of the induced $DRI$-semigroups $(A, \oplus, 0, \lor, \land, \oslash)$ and $(B, \oplus, 0', \lor, \land, \oslash)$.

**Proof.** Let $G$ and $H$ be abelian $l$-groups with elements $0 \leq u \in G$ and $0 \leq v \in H$ such that $A$ is isomorphic to the MV-algebra $\Gamma(G, u)$ and $B$ is isomorphic to the MV-algebra $\Gamma(H, v)$. In [10], Proposition 11, it is proved that if $\tilde{f}$ is a homomorphism of the abelian $l$-group $G$ into an abelian $l$-group $H$ then its restriction $f = \tilde{f} | \Gamma(G, u)$ is a homomorphism of the MV-algebra $\Gamma(G, u)$ into the MV-algebra $\Gamma(H, \tilde{f}(u))$. Further, by [8], Proposition 3.5, if $G'$ and $H'$ are abelian $l$-groups, $u' \in G'$ and $v' \in H'$ are strong order units in $G'$ and $H'$, respectively, and $f: \Gamma(G', u') \rightarrow \Gamma(H', v')$ is a homomorphism of MV-algebras such that $f(u') = v'$, then there exists a homomorphism $\tilde{f}$ of the $l$-group $G'$ into the $l$-group $H'$ such that $f$ is the restriction of $\tilde{f}$ on $\Gamma(G', u')$. (Recall that an element $u$ of an $l$-group $G$ is called a strong order unit if $0 \leq u$ and for each $x \in G$ there exists $n \in \mathbb{N}$ such that $x \leq nu$.) If we consider in our case the convex $l$-subgroup of $G$ generated by $u$ and the convex $l$-subgroup of $H$ generated by $v$ instead of $G$ and $H$, respectively, we get that $f$ is a homomorphism of the $DRI$-semigroup $(A, \oplus, 0, \lor, \land, \oslash)$ into the $DRI$-semigroup $(B, \oplus, 0', \lor, \land, \oslash)$. \qed
For a $DRI$-semigroup with the greatest element $1$ we can consider the identity

$$1 - (1 - x) = x.$$ 

**Proposition 2.** ([10], Proposition 12) Let $A = (A, +, 0, \lor, \land, -)$ and $B = (B, +, 0', \lor, \land, -)$ be $DRI$-semigroups with the greatest elements $1$ and $1'$, respectively, satisfying identity (i) and let $g: A \rightarrow B$ be a homomorphism of $DRI$-semigroups such that $g(1) = 1'$. Then $g$ is a homomorphism of the induced MV-algebras.

Consequently, in what follows, for the class of bounded $DRI$-semigroups, we will consider the greatest element $1$ as a nullary operation and so we will extend the signature of such $DRI$-semigroups to $\langle +, 0, \lor, \land, - \rangle$ of type $\langle 2, 0, 2, 2, 0 \rangle$. Further, the morphisms of the categories of algebras considered will be always all homomorphisms of the corresponding signatures. Then we get the following theorem.

**Theorem 3.** MV-algebras are categorically equivalent to bounded $DRI$-semigroups satisfying identity (i).

**Proof.** If $A = (A, \odot, \neg, 0)$ is an MV-algebra, set $\mathcal{F}(A) = (A, \odot, 0, \lor, \land, \circ, \neg, \otimes, -0)$. For any MV-algebras $A$ and $B$ and any MV-homomorphism $f: A \rightarrow B$ set $\mathcal{F}(f) = f$. If we denote by $\mathcal{MV}$ the category of all MV-algebras and by $\mathcal{DRI}_{1(i)}$ the category of all bounded $DRI$-semigroups satisfying (i) then Propositions 1 and 2 imply that $\mathcal{F}: \mathcal{MV} \rightarrow \mathcal{DRI}_{1(i)}$ is a functor which is an equivalence. \qed

Now, let us recall the notion of a bounded commutative $BCK$-algebra.

A bounded commutative $BCK$-algebra is an algebra $A = (A, *, 0, 1)$ of type $\langle 2, 0, 0 \rangle$ satisfying the following identities:

1. $(x * y) * z = (x * z) * y$;
2. $x * (x * y) = y * (y * x)$;
3. $x * x = 0$;
4. $x * 0 = x$;
5. $x * 1 = 0$.

Bounded commutative $BCK$-algebras were introduced in [12] and, as varieties, in [14]. In [4] it was proved that such a $BCK$-algebra forms a lattice with respect to the order relation $x \leq y \iff x * y = 0$ and in [13] it was proved that this lattice
is distributive. Mundici in [9] showed that MV-algebras and bounded commutative BCK-algebras are categorically equivalent. If we denote by $\mathcal{BCK}_0$ the category of bounded commutative BCK-algebras, the following theorem is an immediate consequence of [9] and our Theorem 3.

**Theorem 4.** The following three categories are equivalent:

a) The category $\mathcal{M}$ of MV-algebras.

b) The category $\mathcal{DR}_1(1)$ of bounded DRI-semigroups satisfying condition (1).

c) The category $\mathcal{BCK}_0$ of bounded commutative BCK-algebras.

**References**


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