AVERAGES OF QUASI-CONTINUOUS FUNCTIONS

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Abstract. The goal of this paper is to characterize the family of averages of comparable (Darboux) quasi-continuous functions.

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Preliminaries

The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the real line, the set of rationals and the set of positive integers, respectively. The word function denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$. We say that functions $\varphi$ and $\psi$ are comparable if either $\varphi < \psi$ on $\mathbb{R}$ or $\varphi > \psi$ on $\mathbb{R}$.

For each $A \subset \mathbb{R}$ we use the symbols $clA$ and $bdA$ to denote the closure and the boundary of $A$, respectively.

Let $f$ be a function. If $A \subset \mathbb{R}$ is nonvoid, then let $\omega(f, A)$ be the oscillation of $f$ on $A$, i.e., $\omega(f, A) = \sup \{|f(x) - f(t)| : x, t \in A\}$. For each $x \in \mathbb{R}$ let $\omega(f, x)$ be the oscillation of $f$ at $x$, i.e., $\omega(f, x) = \lim_{\delta \to 0^+} \omega(f, (x - \delta, x + \delta))$. The symbol $C_f$ denotes the set of points of continuity of $f$.

We say that a function $f$ is quasi-continuous in the sense of Kempisty [4] (cliquish [10]) at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and each open set $U \ni x$ there is a nonvoid open set $V \subset U$ such that $\omega(f, \{x\} \cup V) < \varepsilon$ ($\omega(f, V) < \varepsilon$ respectively). We say that $f$ is quasi-continuous (cliquish) if it is quasi-continuous (cliquish) at each point $x \in \mathbb{R}$. Cliquish functions are also known as pointwise discontinuous.

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Let $I$ be an interval and $f: I \to \mathbb{R}$. We say that $f$ is Darboux if it has the intermediate value property. We say that $f$ is strong Świątkowski [6] if whenever $a, b \in I$, $a < b$, and $y$ is a number between $f(a)$ and $f(b)$, there is an $x \in (a, b) \cap \mathbb{Q}$ with $f(x) = y$. One can easily verify that strong Świątkowski functions are both Darboux and quasi-continuous, and that the converse is not true.

For brevity, if $f$ is a cliquish function and $x \in \mathbb{R}$, then we define

$$\text{LIM}(f, x) = \lim_{t \to x, t \in \mathbb{Q}} f(t).$$

The symbols $\text{LIM}(f, x^-)$ and $\text{LIM}(f, x^+)$ are defined analogously.

**Introduction**

In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson characterized the averages of comparable Darboux functions [1, Theorem 2]. In this paper we solve an analogous problem, namely we characterize the averages of comparable quasi-continuous functions.

A similar problem is to determine a necessary and sufficient condition that for a function $f$ there exists a quasi-continuous function $\psi$ such that $\psi > f$ on $\mathbb{R}$. (The answer to this question for Darboux functions can be easily obtained using the proof of [1, Theorem 2].) In both cases we ask whether there is a positive function $g$ such that both $f + g$ and $-f + g$ are quasi-continuous (the first problem) or such that $f + g$ is quasi-continuous (the second problem). This suggests a similar problem for larger classes of functions. Theorem 4.1 contains a solution of this problem for finite classes of cliquish functions. Recall that by [5, Example 2], we cannot in general allow infinite families in Theorem 4.1. Unlike [7, Theorem 4], we cannot conclude in condition (ii) of Theorem 4.1 that $g$ is a Baire one function; actually, we cannot even conclude that $g$ is Borel measurable (Corollary 4.5).

The Baire class one case makes no difficulty if we require only quasi-continuity of the sums, but it needs a separate argument if we require both the Darboux property and the quasi-continuity. Notice that by Proposition 4.3, the necessary and sufficient condition for Darboux quasi-continuous Baire one functions is essentially stronger.

**Auxiliary lemmas**

**Lemma 3.1.** If $f$ is a cliquish function, then the mapping $x \mapsto \text{LIM}(f, x)$ is lower semicontinuous, while the mapping $x \mapsto \text{LIM}(f, x^-)$ belongs to Baire class two.

**Proof.** Let $y \in \mathbb{R}$. For every $x \in \mathbb{R}$, if $\text{LIM}(f, x) > y$, then there exist an open interval $I_x \ni x$ and a rational $q_x > y$ such that $f > q_x$ on $\mathbb{Q} \cap I_x$, whence $\text{LIM}(f, t) \geq q_x > y$ for each $t \in I_x$. Thus the set $\{x \in \mathbb{R} : \text{LIM}(f, x) > y\}$ is open.
To prove the other assertion put \( A_y = \{ x \in \mathbb{R} : \liminf (f, x^-) > y \} \) for each \( y \in \mathbb{R} \). Let \( y \in \mathbb{R} \). If \( x \in A_y \), then proceeding as above we can find a closed interval \( I_x \subset A_y \) with \( x \in I_x \). So \( A_y \cap \text{bd} A_y \) is at most countable. Hence \( A_y \) is an \( F_\sigma \) set, while \( \{ x \in \mathbb{R} : \liminf (f, x^-) < y \} = \bigcup \{ \mathbb{R} \setminus A_y : q < y, q \in \mathbb{Q} \} \) is the difference of an \( F_\sigma \) set and a countable one.

\[
\text{Lemma 3.2. Let } I = [a, b] \text{ and } n \in \mathbb{N}. \text{ Suppose that functions } f_1, \ldots, f_k \text{ are cliquish and } \max \{ \omega(f_i, I), \ldots, \omega(f_k, I) \} < 1. \text{ There is a positive Baire one function } g \text{ such that } g = 1 \text{ on bd} I, \mathcal{C}_g \supseteq \bigcap_{i=1}^{k} \mathcal{C}_{f_i}, \text{ and for each } i \text{ the function } (f_i + g) | I \text{ is strong Šwiątkowski and}
\]

\[
(f_i + g) \bigg[ I \cap \bigcap_{i=1}^{k} \mathcal{C}_{f_i} \bigg] \supseteq \bigg[ \inf f_i[I] + 1, \max \{ \inf f_i[I] + 1, n \} \bigg].
\]

\textbf{Proof.} Put } T = \max \{ |n - \inf f_i[I]| : i \in \{ 1, \ldots, k \} \} + 1. \text{ Construct a nonnegative continuous function } \varphi \text{ such that } \varphi[I] = [0, T] \text{ and } \varphi = 0 \text{ outside of } I. \text{ For each } i \text{ define } \tilde{f}_i(x) = (f_i + \varphi)(x) \text{ if } x \in I, \text{ and let } \tilde{f}_i \text{ be constant on } (-\infty, a] \text{ and } [b, \infty). \text{ By } [7, \text{ Theorem 4}], \text{ there is a Baire one function } \tilde{g} \text{ such that } \tilde{f}_i + \tilde{g} \text{ is strong Šwiątkowski for each } i \text{ (see condition (8) in the proof of } [7, \text{ Theorem 4}], \mathcal{C}_g \supseteq \bigcap_{i=1}^{k} \mathcal{C}_{f_i}, \text{ and } |\tilde{g}| < 1 \text{ on } \mathbb{R}; \text{ by its proof, we can conclude that } \tilde{g} = 0 \text{ on } \{a, b\}. \text{ Put } g = \varphi + \tilde{g} + 1. \text{ Then for each } i, \text{ since } f_i + \tilde{g} \text{ is strong Šwiątkowski and } f_i + g = f_i + \tilde{g} + 1 \text{ on } I, \text{ we have}
\]

\[
(f_i + g) \bigg[ I \cap \bigcap_{i=1}^{k} \mathcal{C}_{f_i} \bigg] \supseteq \bigg( \inf(f_i + g)[I], \sup(f_i + g)[I] \bigg)
\]

\[
\supseteq \bigg( f_i(a), \inf f_i[I] + \sup g[I] \bigg)
\]

\[
\supseteq \bigg[ \inf f_i[I] + 1, \max \{ \inf f_i[I] + 1, n \} \bigg].
\]

The other requirements are evident. \hfill \Box

\textbf{Main results}

\textbf{Theorem 4.1.} Let } \mathcal{F} \text{ be one of the following classes of functions: all cliquish functions, Lebesgue measurable cliquish functions, cliquish functions in Baire class } \alpha (\alpha \geq 1), \text{ and suppose } f_1, \ldots, f_k \in \mathcal{F}. \text{ The following are equivalent:}

(i) there is a positive function } g \text{ such that } f_i + g \text{ is quasi-continuous for each } i;

(ii) there is a positive function } g \in \mathcal{F} \text{ such that } \mathcal{C}_g \supseteq \bigcap_{i=1}^{k} \mathcal{C}_{f_i} \text{ and } f_i + g \text{ is quasi-continuous for each } i;

\]

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(iii) for each $x \in \mathbb{R}$ and each $i$ we have $\lim(f_i, x) < \infty$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii). Let $x \in \mathbb{R}$ and $i \in \{1, \ldots, k\}$. Since $f_i + g$ is quasi-continuous, so by [2] (see also [3, Lemma 2]) we obtain

$$\lim(f_i, x) \leq \lim(f_i + g, x) \leq (f_i + g)(x) < \infty.$$  

(iii) $\Rightarrow$ (ii). Put $A = \bigcup_{i=1}^{k} \{x \in \mathbb{R}: \omega(f_i, x) \geq 1\}$. Then $A$ is closed and nowhere dense. Find a family $\{I_n: n \in \mathbb{N}\}$ consisting of nonoverlapping compact intervals, such that $\bigcup_{n \in \mathbb{N}} I_n = \mathbb{R} \setminus A$ and each $x \notin A$ is an interior point of $I_n \cup I_m$ for some $n, m \in \mathbb{N}$. Since each $I_n$ is compact and $\omega(f_i, x) < 1$ for each $x \in I_n$ and $i \in \{1, \ldots, k\}$, so we may assume that $\omega(f_i, I_n) < 1$ for each $i$ and $n$. For each $n \in \mathbb{N}$ use Lemma 3.2 to construct a positive Baire one function $g_n$ such that $g_n = 1$ on $\partial I_n$, $\partial g_n \supset \bigcap_{i=1}^{k} \partial f_i$, and for each $i$ the function $(f_i + g_n) \upharpoonright I_n$ is strong Świątkowski and

$$(*) \quad (f_i + g_n) \left[I_n \cap \bigcap_{i=1}^{k} \partial f_i \right] \supset [\inf f_i[I_n] + 1, \max\{\inf f_i[I_n] + 1, n\}].$$

Define $g(x) = g_n(x)$ if $x \in I_n$ for some $n \in \mathbb{N}$, and

$$g(x) = \max\{\max\{\lim(f_i, x) - f_i(x): i \in \{1, \ldots, k\}, 0\} + 1$$

if $x \in A$. By Lemma 3.1, each mapping $x \mapsto \lim(f_i, x)$ is Baire one, so $g \in \mathcal{F}$.

Fix an $i \in \{1, \ldots, k\}$. Clearly $f_i + g$ is quasi-continuous outside of $A$. On the other hand, if $x \in A$, then by $(*)$, for each $\delta > 0$ we have

$$(f_i + g)[(x - \delta, x + \delta) \cap \partial f_{i+g}] \supset (\lim(f_i, x) + 1, \infty).$$

Hence $f_i + g$ is quasi-continuous.

Theorem 4.2. Let $\mathcal{F}$ be one of the following classes of functions: all cliquish functions, Lebesgue measurable cliquish functions, cliquish functions in Baire class $\alpha$ ($\alpha \geq 2$), and suppose $f_1, \ldots, f_k \in \mathcal{F}$. The following are equivalent:

(i) there is a positive function $g$ such that $f_i + g$ is both Darboux and quasi-continuous for each $i$;

(ii) there is a positive function $g \in \mathcal{F}$ such that $\partial g \supset \bigcap_{i=1}^{k} \partial f_i$, and $f_i + g$ is strong Świątkowski for each $i$;

(iii) for each $x \in \mathbb{R}$ and each $i$ we have $\max\{\lim(f_i, x^-), \lim(f_i, x^+)\} < \infty$.  

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The proof of the implication (iii) ⇒ (ii) is a repetition of the argument used in Theorem 4.1, and the implication (ii) ⇒ (i) is obvious.

(i) ⇒ (iii). Let \( x \in \mathbb{R} \) and \( i \in \{1, \ldots, k\} \). Since \( f_i + g \) is both Darboux and quasi-continuous, so by [9, Lemma 2] we obtain

\[
\lim (f_i, x^-) \leq \lim (f_i + g, x^-) \leq (f_i + g)(x) < \infty.
\]

Similarly \( \lim (f_i, x^+) < \infty \).

**Proposition 4.3.** There is a Baire one function \( f \) such that \( f + g \) is strong Świetkowski for some positive function \( g \) in Baire class two, but \( f + g \) is Darboux for no positive Baire one function \( g \).

**Proof.** Let \( F \) be the Cantor ternary set and let \( \mathscr{I} = \{(a_n, b_n): n \in \mathbb{N}\} \) and \( \mathscr{J} \) be disjoint families of components of \( \mathbb{R} \setminus F \) such that \( F = (\text{cl} \cup \mathscr{I}) \cap (\text{cl} \cup \mathscr{J}) \). Define \( f(x) = n \) if \( x \in (a_n, b_n) \) for some \( n \in \mathbb{N} \) and \( f(x) = 0 \) otherwise. Clearly \( f \) belongs to Baire class one.

Let \( x \in \mathbb{R} \). If \( x \in (a_n, b_n) \) for some \( n \in \mathbb{N} \), then \( \lim (f, x^-) = n \), otherwise \( \lim (f, x^-) = 0 \). Similarly \( \lim (f, x^+) < \infty \). By Theorem 4.2 there is a positive Baire two function \( g \) such that \( f + g \) is strong Świetkowski.

On the other hand, by [8, Proposition 6.10], \( f + g \) is Darboux for no positive Baire one function \( g \).

In Proposition 4.4 the symbol \( c \) denotes the first ordinal equipollent with \( \mathbb{R} \).

**Proposition 4.4.** Given a family of positive functions, \( \{g_\xi: \xi < c\} \), we can find a cliquish function \( f \) which fulfills condition (iii) of Theorem 4.2 and such that \( f + g_\xi \) is not quasi-continuous for each \( \xi < c \).

**Proof.** Let \( F \) be the Cantor ternary set and let \( \{x_\xi: \xi < c\} \) be an enumeration of \( F \). Define \( f(x) = -g_\xi(x) - 1 \) if \( x = x_\xi \) for some \( \xi < c \), and \( f(x) = 0 \) otherwise. Clearly \( f \) is cliquish, and for each \( x \in \mathbb{R} \) we have \( \lim (f, x^-) = \lim (f, x^+) = 0 \).

Let \( \xi < c \). Then \( (f + g_\xi)(x_\xi) = -1 \) and \( f + g_\xi \) is positive on a dense open set. Thus \( f + g_\xi \) is not quasi-continuous at \( x_\xi \).

**Corollary 4.5.** There is a cliquish function \( f \) which fulfills condition (iii) of Theorem 4.2 and such that \( f + g \) is not quasi-continuous for each positive Borel measurable function \( g \).

**Theorem 4.6.** Let \( f_1, \ldots, f_k \) be Baire one functions. The following are equivalent:

(i) there is a positive Baire one function \( g \) such that \( f_i + g \) is both Darboux and quasi-continuous for each \( i \);
(ii) there is a positive Baire one function \( g \) such that \( C_g \supset \bigcap_{i=1}^{k} C_{f_i} \) and \( f_i + g \) is strong Świątkowski for each \( i \);

(iii) there is a Baire one function \( h \) such that for each \( x \in \mathbb{R} \) and each \( i \) we have

\[
\max \{ \lim \left( f_i(x)^-, x\right), \lim \left( f_i(x)^+, x\right) \} \leq h(x).
\]

Proof. The implication (i) \( \Rightarrow \) (iii) can be proved similarly as in Theorem 4.2 (we let \( h = \max\{f_1, \ldots, f_k\} + g \)), and the implication (ii) \( \Rightarrow \) (i) is obvious.

(iii) \( \Rightarrow \) (ii). The proof of this implication is a repetition of the argument used in Theorem 4.1. The only difference is in the definition of the function \( g \) on the set \( A \). More precisely, we put

\[
g(x) = \max \{ \max \{ h(x) - f_i(x) : i \in \{1, \ldots, k\} \}, 0 \} + 1
\]

if \( x \in A \). Then clearly \( g \) is a Baire one function. \( \square \)

References


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