ON A HIGHER-ORDER HARDY INEQUALITY

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Dedicated to Professor A. Kufner on the occasion of his 65th birthday

Abstract. The Hardy inequality

\[ \int_{\Omega} |u(x)|^p d(x)^{-p} dx \leq c \int_{\Omega} |\nabla u(x)|^p dx \]

with \( d(x) = \text{dist}(x, \partial \Omega) \) holds for \( u \in C_0^\infty(\Omega) \) if \( \Omega \subset \mathbb{R}^n \) is an open set with a sufficiently smooth boundary and if \( 1 < p < \infty \). P. Hajlasz proved the pointwise counterpart to this inequality involving a maximal function of Hardy-Littlewood type on the right hand side and, as a consequence, obtained the integral Hardy inequality. We extend these results for gradients of higher order and also for \( p = 1 \).

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1. Introduction

Let \( \Omega \) be a proper subdomain of \( \mathbb{R}^n \) and let \( d(x) = \text{dist}(x, \partial \Omega), \ x \in \Omega, \) be the corresponding distance function.

It is well known that the Hardy inequality

\[ \int_{\Omega} |u(x)|^p d(x)^{-p} dx \leq c \int_{\Omega} |\nabla u(x)|^p dx, \]

holds for \( u \in C_0^\infty(\Omega) \) if \( 1 < p < \infty \) and the boundary of \( \Omega \) satisfies the Lipschitz condition or similar regularity conditions. For these results and further references we refer to [8], [10], [12].

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Different authors introduced the notions of capacity and of thick sets in various ways (see, e.g. [1], [4]–[9], etc.) in order to find weaker sufficient conditions for inequalities of Hardy, Poincaré and other types. We shall concentrate mainly on [4] and [6].

Let $K$ be a compact subset of $\Omega$ and let $1 \leq p < \infty$. The variational $(1, p)$-capacity $C_{1,p}(K, \Omega)$ of the condenser $(K, \Omega)$ is defined to be

$$C_{1,p}(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla u(x)|^p \, dx : u \in C_0^\infty(\Omega), u(x) \geq 1 \text{ for } x \in K \right\}.$$

By $B(x, r)$ we denote the open ball in $\mathbb{R}^n$ of radius $r$, $0 < r < \infty$, centered at $x \in \mathbb{R}^n$.

**Definition 1.** A closed set $K \subset \mathbb{R}^n$ is locally uniformly $(1, p)$-thick, if there exist numbers $b > 0$ and $r_0$, $0 < r_0 \leq \infty$ such that

$$C_{1,p}(B(x, r) \cap K, B(x, 2r)) \geq b C_{1,p}(B(x, r), B(x, 2r))$$

for all $x \in K$ and $0 < r < r_0$. If $r_0 = \infty$, then the set $K$ is called uniformly $(1, p)$-thick.

Note that a scaling argument yields

$$(1.3) \quad C_{1,p}(B(x, r), B(x, 2r)) = c(n, p) r^{n-p}.$$

P. Hajłasz [4] used the Hardy-Littlewood maximal operator $M$ and showed that for a domain $\Omega$ with a locally uniformly $(1, p)$-thick complement there exists $q \in (1, p)$ such that every function $u \in C_0^\infty(\Omega)$ satisfies the pointwise analogue of the Hardy inequality, which in a slightly simplified formulation reads

$$|u(x)| \leq cd(x) \left[ M(|\nabla u|^q)(x) \right]^{1/q}.$$

As a corollary he obtained the integral Hardy inequality

$$\int_{\Omega} |u(x)|^p d(x)^{n-p} \, dx \leq c \int_{\Omega} |\nabla u(x)|^p d(x)^n \, dx,$$

for small positive numbers $a$. Similar results were obtained also by J. Kinnunen and O. Martio [6].

Our aim is to extend these results for derivatives of higher order.
If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of non-negative integers, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $\alpha! = \alpha_1! \ldots \alpha_n!$, and for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we set $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The corresponding partial derivative operators will be denoted by

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n} = \frac{\partial|\alpha|}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

and the gradient of a real-valued function of order $k$, $k \in \mathbb{N}$, will be the vector $\nabla^k u = \{D^\alpha u\}_{|\alpha|=k}$. For $k = 1$, $\nabla^1 u = \nabla u$ is the usual gradient.

Given a measurable set $E \subset \mathbb{R}^n$, we denote its Lebesgue $n$-measure by $|E|$ and the characteristic function of $E$ by $\chi_E$. Constants $c$ in estimates may vary during calculations but they always remain independent of all non-fixed entities.

2. THE POINTWISE HARDY INEQUALITY

The fractional maximal function $M_{\gamma,R}u$, $0 \leq \gamma \leq n$, is defined by $0 < R \leq \infty$, is defined for every $u \in L^1_{loc}(\mathbb{R}^n)$ by

$$M_{\gamma,R}u(x) = \sup_{0 < r < R} |B(x,r)|^{\gamma/n - 1} \int_{B(x,r)} |u(y)| \, dy, \quad x \in \mathbb{R}^n.$$  

Note that $M_{0,\infty}u = Mu$ is the classical Hardy-Littlewood maximal function.

**Theorem 1.** Let $1 \leq p < \infty$, let $k$ be a positive integer and $0 \leq \gamma < k$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,p)$-thick and let $b$ be the constant from Definition 1. Then there exists a constant $c = c(k,p,n,b) > 0$ such that every function $u \in C_0^\infty(\Omega)$ satisfies the inequality

$$|u(x)| \leq c d(x)^{k-\gamma/p} \left[ M_{\gamma,4d(x)}(|\nabla^k u|^p \chi_{B(2d(x))})(x) \right]^{1/p},$$

where $x \in \Omega$, $d(x) < r_0$, and $\overline{x} \in \partial \Omega$ is such that $|x - \overline{x}| = d(x)$.

This is the main result of this section which extends Theorem 2 of [4]. To prove it we shall need several auxiliary assertions. The first one is a generalization of [3, Lemma 7.16].

**Lemma 1.** Let $k$ be a natural number. There exists a constant $c = c(k,n) > 0$ such that for every ball $B \subset \mathbb{R}^n$ and for every function $u \in C^k(B)$ the inequality

$$|u(x) - |B|^{-1} \int_B P(x,y) \, dy| \leq c \int_B \frac{|\nabla^k u(y)|}{|x - y|^{n-k}} \, dy, \quad x \in B,$$
holds, where \( P \) is the polynomial of order \( \leq k - 1 \) given by

\[
P(x, y) = \sum_{|\alpha| \leq k - 1} \frac{(-1)^{|\alpha|}}{\alpha!} D^\alpha u(y - x)^\alpha, \quad x, y \in B.
\]

Lemma 1 can be proved in a way similar to the proof of Lemma 7.16 in [3] using the Taylor expansion of the function \( v(r) = u(x + r\theta) \), where \( r = |x - y|, \theta = (y - x)/r, x, y \in \Omega \). Note that assertions of this type can be found for instance in [1, §8.1] and [8, §1.1.10].

The next assertion is a variation of a well-known result of L. I. Hedberg.

**Lemma 2.** Let \( 0 \leq \gamma < \kappa \) and let \( B \subset \mathbb{R}^n \) be a ball of radius \( R \). Then there exists a constant \( c = c(n, \gamma, \kappa) > 0 \) such that every function \( g \in L^1_{\text{loc}}(B) \) satisfies the inequality

\[
\int_B \frac{|g(y)| \, dy}{|x - y|^{n-\kappa}} \leq c R^{\kappa - \gamma} M_{\gamma, 2R}(g)(x), \quad x \in B.
\]

**Proof.** Fix \( x \in B \) and for \( i \in \mathbb{N} \) set \( A_i = (B(x, 2^{1-i}R) \setminus B(x, 2^{-i}R)) \cap B \). Then

\[
\int_B \frac{|g(y)| \, dy}{|x - y|^{n-\kappa}} \leq \sum_{i=0}^\infty \int_{A_i} \frac{|g(y)| \, dy}{|x - y|^{n-\kappa}} \leq \max(1, 2^{\kappa-\gamma}) \sum_{i=0}^\infty (2^{-i}R)^{\kappa-\gamma} \int_{B(x, 2^{1-i}R)} |g(y)| \, dy \leq |B(0, 1)|^{-1} \max(1, 2^{\kappa-\gamma}) 2^{n-\gamma} R^{\kappa - \gamma} \sum_{i=0}^\infty 2^{-i(\kappa-\gamma)} M_{\gamma, 2R}(g)(x).
\]

We shall also need the following inequality of Poincaré type which follows from the considerations in [8, Sections 9.3 and 10.1.2].

**Lemma 3.** Let \( 1 \leq p < \infty \). Let \( B = B(x, R) \) be a ball in \( \mathbb{R}^n \) and let \( K \) be a closed subset of \( \overline{B} \). Then every function \( u \in C^\infty(\overline{B}) \) such that \( \text{dist}(\text{supp} \, u, K) > 0 \) satisfies the inequality

\[
\int_{\overline{B}} |u(x)|^p \, dx \leq c \frac{R^n}{C_{1,p}(K, B(x, 2R))} \int_{\overline{B}} |\nabla u(x)|^p \, dx,
\]

where \( c \) is a positive constant independent of \( B, K \) and \( u \).
Proof of Theorem 1. Let \( x \in \Omega \) be such that \( d(x) < r_0 \), where \( r_0 \) is the number from Definition 1. Let \( \overline{x} \in \partial \Omega \) satisfy \( |x - \overline{x}| = d(x) = R \) and let \( u \in C_0^\infty(\Omega) \). Set \( B = B(\overline{x}, 2R) \). Then \( x \in B \) and

(2.3) \[ |u(x)| \leq |u(x) - P_B(x)| + |P_B(x)|, \]

where \( P_B(x) = |B|^{-1} \int_B P(x, y) \, dy \) and \( P \) is the polynomial from Lemma 1. Using Lemma 1, Lemma 2 and the Hölder inequality we obtain

(2.4) \[ |u(x) - P_B(x)| \leq c \int_B \frac{\|\nabla^k u(y)\|}{|x - y|^{n-k}} \, dy \leq c R^{k-\gamma} M_{\gamma, 4R}(\|\nabla^k u\| \chi_B)(x) \]
\[ \leq c R^{k-\gamma/p} \left[ M_{\gamma, 4R}(\|\nabla^k u\|^p \chi_B)(x) \right]^{1/p}. \]

From (2.2) we have

\[ |P_B(x)| \leq |B|^{-1} \int_B |P(x, y)| \, dy \leq c \sum_{i=0}^{k-1} R^i |B|^{-1} \int_B |\nabla^i u(y)| \, dy \]
\[ \leq c \sum_{i=0}^{k-1} R^i \left( |B|^{-1} \int_B |\nabla^i u(y)|^p \, dy \right)^{1/p}. \]

Repeated application of Lemma 3 and of (1.2) and (1.3) yields

\[ \int_B \|\nabla^i u(x)\|^p \, dx \leq c \frac{R^n}{C_{1,p}(\mathbb{R}^n \setminus \Omega) \cap B(\overline{x}, 4R)}} \int_B \|\nabla^{i+1} u(x)\|^p \, dx \]
\[ \leq c R^p \int_B \|\nabla^{i+1} u(x)\|^p \, dx \]
\[ \leq c R^{(k-i)p} \int_B \|\nabla^k u(x)\|^p \, dx, \quad i = 0, \ldots, k-1. \]

Hence,

(2.5) \[ |P_B(x)| \leq c R^k \left( |B|^{-1} \int_B \|\nabla^k u(x)\|^p \, dx \right)^{1/p} \]
\[ \leq c R^{k-\gamma/p} \left[ M_{\gamma, 4R}(\|\nabla^k u\|^p \chi_B)(x) \right]^{1/p}. \]

The inequality (2.1) follows from (2.3)–(2.5). \( \square \)

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In this section we shall use Theorem 1 to obtain higher-order analogues of the classical Hardy inequality. As in [4] and [6], in further considerations we shall essentially use the openness of the \((1,p)\)-thickness with respect to \(p\). This deep property was originally proved by J.L. Lewis [7, Theorem 1] and later on in another way by P. Mikkonen [9, Theorem 8.2]. The following lemma can be obtained as a particular case of Lewis’ and Mikkonen’s results. It is not important for our purpose that Lewis dealt with another type of capacity.

**Lemma 4.** Let \(1 < p < \infty\) and let \(K \subset \mathbb{R}^n\) be a closed locally uniformly \((k,p)\)-thick set. Then there exists \(q, 1 < q < p\), depending only on \(n, k, p\) and \(b\), such that \(K\) is locally uniformly \((k,q)\)-thick with the same value of \(r_0\) as for \(p\).

For \(r > 0\) we set
\[
\Omega_r = \{x \in \Omega: d(x) < r\}.
\]

**Theorem 2.** Let \(1 < p < \infty\) and let \(k\) be a positive integer. Let \(\Omega\) be an open subset of \(\mathbb{R}^n\) such that \(\mathbb{R}^n \setminus \Omega\) is locally uniformly \((1,p)\)-thick. Then there exists a positive constant \(c = c(k,p,n,b)\) such that the inequality
\[
(2.6) \int_{\Omega_r} \left( \frac{|u(x)|}{d(x)^k} \right)^p \, dx \leq c \int_{\Omega_r} |\nabla^k u(x)|^p \, dx
\]
holds for every function \(u \in C_0^\infty(\Omega)\) and for every \(r \in (0, r_0)\), where \(r_0\) is the parameter given in Definition 1.

**Proof.** Let \(p > 1\) and let \(q \in (1,p)\) be from Lemma 4, and suppose that \(r \in (0, r_0)\). It follows from (2.1) that for all \(u \in C_0^\infty(\Omega),\)
\[
(2.7) |u(x)|d(x)^{-k} \leq c \left[ M(|\nabla^k u|^q \chi_{\Omega_r})(x) \right]^{1/q}, \quad x \in \Omega_r.
\]
We use the boundedness of \(M: L^{p/q} \to L^{p/q}\) and the Hölder inequality to obtain
\[
(2.8) \int_{\Omega_r} \left( \frac{|u(x)|}{d(x)^k} \right)^p \, dx \leq c \int_{\Omega_r} \left[ M(|\nabla^k u|^q \chi_{\Omega_r})(x) \right]^{p/q} \, dx \leq c \int_{\Omega_r} |\nabla^k u(x)|^p \, dx.
\]
Note that the norm of the maximal operator \(M\) and, consequently, also the constant \(c\) depend on the value of \(p/q\). \(\square\)
If \( p = 1 \), we cannot use Lemma 4. Instead we use the fact that for \( \Omega \) with \( |\Omega| < \infty \) the maximal operator \( M \) is a bounded mapping of \( L \log L(\Omega) \) in \( L^1(\Omega) \) (see [2], p. 74). Recall that \( L \log L(\Omega) \) is the Zygmund space which consists of all measurable functions \( u \) with \( \int_{\Omega} |u(x)| \log_+ |u(x)| \, dx < \infty \), endowed with the norm
\[
\|u\|_{L \log L(\Omega)} = \int_0^{\Omega} u^*(t) \log \frac{|\Omega|}{t} \, dt,
\]
where \( u^* \) is the non-increasing rearrangement of \( u \).

**Theorem 3.** Let \( p = 1 \) and let \( k \) be a positive integer. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) such that \( \mathbb{R}^n \setminus \Omega \) is locally uniformly \((1, 1)\)-thick. Then there exists a positive constant \( c = c(k, n, b) \) such that the inequality
\[
(2.9) \quad \int_{\Omega_r} \frac{|u(x)|}{d(x)^k} \, dx \leq c \|\nabla^k u\|_{L \log L(\Omega_r)}
\]
holds for every function \( u \in C_0^\infty(\Omega) \) and for every \( r \in (0, r_0) \), where \( r_0 \) is the parameter given in Definition 1.

**Proof.** From the estimate (2.1) we have
\[
|u(x)|d(x)^{-k} \leq c M(\nabla^k u, \chi_{\Omega_r})(x), \quad x \in \Omega_r.
\]
Integrating both sides of the inequality over \( \Omega_r \) and using the boundedness of \( M : L \log L(\Omega) \to L^1(\Omega) \) we arrive at the inequality (2.9). \( \square \)

**Corollary 1.** Let \( 1 < p < \infty \) and let \( k \) be a positive integer. Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) such that \( \mathbb{R}^n \setminus \Omega \) is locally uniformly \((1, p)\)-thick. Then there exists a number \( \varepsilon_0 > 0 \) such that the inequality
\[
(2.10) \quad \int_{\Omega_r} \left( \frac{|u(x)|}{d(x)^k} \right)^p d(x)^{xp} \, dx \leq c \int_{\Omega_r} |\nabla^k u(x)|^p d(x)^{xp} \, dx
\]
holds for all \( u \in C_0^\infty(\Omega) \), \( r \in (0, r_0) \) and \( 0 \leq \varepsilon < \varepsilon_0 \). The constant \( c > 0 \) depends on \( n, p, k, b \) and on the number \( q \) from Lemma 4.

**Proof.** Fix \( \varepsilon > 0 \) and let \( u \in C_0^\infty(\Omega) \) be such that the integral on the right hand side of (2.10) is finite.

If \( k = 1 \), we set \( v(x) = |u(x)|d(x)^{\varepsilon} \). Then
\[
(2.11) \quad |\nabla v(x)| \leq |\nabla u(x)|d(x)^{\varepsilon} + \varepsilon |u(x)|d(x)^{\varepsilon - 1} \quad \text{for a.e. } x \in \Omega,
\]
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and (2.10) implies that \( v \) belongs to the Sobolev space \( W^{1,p}_0(\Omega) \). Applying Theorem 2 to functions from \( C_0^\infty(\Omega) \) which approximate \( v \) in \( W^{1,p}_0(\Omega) \) and passing to the limit we obtain
\[
\int_{\Omega_r} \left( \frac{|u(x)|}{d(x)} \right)^p d(x)^p dx = \int_{\Omega_r} \left( \frac{|v(x)|}{d(x)} \right)^p dx \leq c \int_{\Omega_r} |\nabla v(x)|^p dx
\]
for \( 0 \leq \varepsilon < \varepsilon_0 \). By (2.11), we have
\[
\int_{\Omega_r} \left( \frac{|u(x)|}{d(x)} \right)^p d(x)^p dx \\
\leq c \left( \int_{\Omega_r} |\nabla u(x)|^p d(x)^p dx + c^p \int_{\Omega_r} \left( \frac{|u(x)|}{d(x)} \right)^p d(x)^p dx \right).
\]
Thus, the inequality (2.10) holds for \( 0 \leq \varepsilon < \varepsilon_0 = c^{-1/p} \).

Let \( k > 1 \) and suppose that the inequality (2.10) holds for \( j = 1, 2, \ldots, k-1 \) and \( 0 \leq \varepsilon < \varepsilon_0 \). Let \( \varrho \) be the regularized distance function equivalent to \( d \) and satisfying the estimate
\[
|\nabla^j \varrho(x)| \leq c_j d(x)^{1-j}, \quad x \in \Omega, \quad j = 1, 2, \ldots,
\]
(see, e.g., [11, p. 171]). Set \( v(x) = |u(x)|\varrho(x)^\varepsilon \). Then
\[
|\nabla^k v(x)| \leq |\nabla^k u(x)|\varrho(x)^\varepsilon + \varepsilon \sum_{j=1}^k Q_j(\varepsilon)|\nabla^{k-j} u(x)|\varrho(x)^{\varepsilon-j},
\]
where \( Q_j \) are polynomials of degree \( j \). Thus, we have
\[
\int_{\Omega_r} \left( \frac{|u(x)|}{d(x)} \right)^p d(x)^p dx \leq c \int_{\Omega_r} \left( \frac{|v(x)|}{d(x)^k} \right)^p d(x)^p dx \\
\leq c \left( \int_{\Omega_r} |\nabla^k u(x)|^p \varrho(x)^p dx + c^p \sum_{j=1}^k |Q_j(\varepsilon)|^p \int_{\Omega_r} \left( \frac{|u(x)|}{\varrho(x)^{k-j}} \right)^p \varrho(x)^p dx \right) \\
\leq c \left( \int_{\Omega_r} |\nabla^k u(x)|^p \varrho(x)^p dx + c^p \int_{\Omega_r} \left( \frac{|u(x)|}{\varrho(x)^k} \right)^p \varrho(x)^{\varepsilon-p} dx \right) \\
\leq c \left( \int_{\Omega_r} |\nabla^k u(x)|^p d(x)^p dx + c^p \int_{\Omega_r} \left( \frac{|u(x)|}{d(x)^k} \right)^p d(x)^{\varepsilon-p} dx \right),
\]
and the inequality (2.10) holds for \( 0 \leq \varepsilon < c^{-1/p} \). \( \square \)

**Corollary 2.** Let \( \Omega \) be such that \( \mathbb{R}^n \setminus \Omega \) is locally uniformly \((1,p)\)-thick with \( r_0 > \frac{1}{2} \text{diam}(\Omega) \). Then the inequality (2.1) holds for every \( x \in \Omega \) and the assertions of Theorem 2, Theorem 3 and Corollary 1 hold with \( \Omega \) in place of \( \Omega_r \) and for all functions \( u \) from the corresponding Sobolev spaces \( W^{1,p}_0 \) on \( \Omega \).

**Proof.** It suffices to observe that \( \Omega_r = \Omega \) for \( r > \frac{1}{2} \text{diam}(\Omega) \) and that the constant \( c \) does not depend on the parameter \( r_0 \). \( \square \)
Note that the assumption of Corollary 2 holds, in particular, if $\mathbb{R}^n \setminus \Omega$ is uniformly $(1,p)$-thick (i.e., $r_0 = \infty$).

An open problem. Additional weights could be introduced into the inequality (2.6) by applying a weighted inequality for the maximal function. Following the proof of Theorem 2 we can multiply both sides of inequality (2.7) (or, more precisely, of inequality (2.1)) by $d(x)^e$ and integrate over $\Omega_r$. However, to make the final step in (2.8) we have to know that the maximal function satisfies the weighted inequality

$$\int_{\Omega_r} \left[ M^k (|\nabla u|^q \chi_{\Omega_r}(x)) \right]^{p/q} d(x)^{ep} \, dx \leq c \int_{\Omega_r} |\nabla u(x)|^p d(x)^{ep} \, dx.$$ 

Note that we are dealing with the global maximal function (the balls in the construction of $M^k_d(x)$ from inequality (2.1) cross the complement of $\Omega$) and so to use the known weighted inequalities for $M^k$ we would have to consider $d(x)$ extended properly outside $\Omega$. The question is, if the sufficient conditions for such weighted estimate would not override the condition of $(1,p)$-thickness of $\mathbb{R}^n \setminus \Omega$.

References


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