ON MAXIMAL OVERDETERMINED HARDY’S INEQUALITY OF SECOND ORDER ON A FINITE INTERVAL

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. A characterization of the weighted Hardy inequality
\[ \|F\|_2 \leq C \|F''v\|_2, \quad F(0) = F'(0) = F(1) = F'(1) = 0 \]
is given.

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INTRODUCTION

Let \( I = [0, 1], \ 1 < p, q < \infty \), let \( k \geq 1 \) be an integer and let \( AC^k_p \) denote the space of all functions on \( I \) with absolutely continuous \((k-1)\)-th derivative \( F^{(k-1)}(x) \) and such that

\[ \|F\|_{AC^k_p} := \|F^{(k)}v\|_p < \infty, \]

\( F(0) = F'(0) = \ldots = F^{(k-1)}(0) = F(1) = \ldots = F^{(k-1)}(1) = 0, \)

where \( v(x) \) is a locally integrable weight function and \( \|g\|_p := (\int_0^1 |g(x)|^p \, dx)^{1/p} \).

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We consider the characterization problem for the inequality

\[(1) \quad \|Fu\|_q \leq C\|F^{(k)}v\|_p, \quad F \in AC^k.\]

The case \(k = 1\) has been solved by P. Gurka [2] (see also [13]) and many works have been performed in this area by A. Kufner [6] and by A. Kufner with co-authors [1], [5], [7–10]. In particular, following Kufner’s terminology we call the inequality (1) “maximal overdetermined Hardy’s inequality”, that is when a function \(F\) and its derivatives vanish at both ends of the interval up to \((k-1)\)-th order. A part of analysis related to the weighted Hardy inequality for functions vanishing at both ends of an interval was also given by G. Sinnamon [15] and the authors [11], [12]. In particular, the maximal inequality (1) on semiaxis was characterized in [11], [12].

The aim of the present paper is twofold. At first we prove an alternative version of (1) (see Theorem 1) and it allows, using the results of [4], to characterize the inequality (1), when \(p = q = 2, k = 2\) (Theorem 3).

Without loss of generality we assume throughout the paper that the undeterminates of the form 0 · ∞, 0/0, ∞/∞ are equal to zero.

**AN ALTERNATE VERSION**

Denote \(I_k f(x)\) and \(J_k f(x)\) the Riemann-Liouville operators of the form

\[I_k f(x) = \frac{1}{\Gamma(k)} \int_0^x (x - y)^{k-1} f(y) \, dy, \quad x \in I,\]

\[J_k f(x) = \frac{1}{\Gamma(k)} \int_x^1 (y - x)^{k-1} f(y) \, dy, \quad x \in I.\]

Then the maximal inequality (1) is equivalent either to

\[(2) \quad \|(I_k f)u\|_q \leq C \|fv\|_p, \quad f \in P_{k-1}^\perp\]

or to

\[(3) \quad \|(J_k f)u\|_q \leq C \|fv\|_p, \quad f \in P_{k-1}^\perp,\]

where \(P_{k-1}\) is the \(k\)-dimensional space of all polynomials \(g(t) = c_0 + c_1 t + \ldots + c_{k-1} t^{k-1}, t \in I,\) and \(P_{k-1}^\perp \subset L_{p,v} := \{f: \|fv\|_p < \infty\}\) denotes the closed subspace of \(L_{p,v}\) of functions “orthogonal” to \(P_{k-1}\) in the sense that

\[\int_0^1 f(x)g(x) \, dx = 0 \quad \text{for all} \quad g \in P_{k-1}, \quad f \in P_{k-1}^\perp.\]
In particular, \( f \in P_{k-1}^\perp \) if, and only if,
\[
\int_0^1 f(x) \, dx = \int_0^1 xf(x) \, dx = \ldots = \int_0^1 x^{k-1} f(x) \, dx = 0
\]
and, obviously,
\[
I_k f(x) = J_k f(x), \ f \in P_{k-1}^\perp.
\]
We need the following

**Lemma 1.** ([14], Chapter 4, Exercise 19). Let \( X \) be a Banach space and \( Y \subset X \) the closed subspace. Let \( X^\ast \) be the dual space and
\[
Y^\perp = \{ \varphi \in X^\ast : \varphi(y) = 0 \text{ for all } y \in Y \}.
\]
Then
\[
(4) \quad \text{dist}(e, Y) := \inf_{y \in Y} \| e - y \|_X = \sup_{\varphi \in Y^\perp} \frac{|\varphi(e)|}{\| \varphi \|_{X^\ast}}
\]
for all \( e \notin Y \).

**Proof.** Let \( y \in Y \), \( \varphi \in Y^\perp \). Then
\[
\varphi(e) = \varphi(e) - \varphi(y) = \varphi(e - y)
\]
and
\[
|\varphi(e)| = |\varphi(e - y)| \leq \| \varphi \|_{X^\ast} \| e - y \|.
\]
Consequently,
\[
\sup_{\varphi \in Y^\perp} \frac{|\varphi(e)|}{\| \varphi \|_{X^\ast}} \leq \| e - y \|
\]
and
\[
(5) \quad \sup_{\varphi \in Y^\perp} \frac{|\varphi(e)|}{\| \varphi \|_{X^\ast}} \leq \text{dist}(e, Y).
\]

Now suppose \( e \notin Y \), \( y \in Y \). Then \( e - y \notin Y \) and by the Hahn-Banach theorem there exists \( \varphi \in X^\ast \) such that \( \varphi(y) = 0 \) for all \( y \in Y \), \( \| \varphi \|_{X^\ast} = 1 \) and \( \varphi(e - y) = \| e - y \| \). This implies that \( \varphi \in Y^\perp \) and
\[
|\varphi(e)| = |\varphi(e - y)| = \| e - y \| \geq \text{dist}(e, Y).
\]
Therefore,
\[
(6) \quad \sup_{\varphi \in Y^*} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}} \geq \text{dist}(e, Y).
\]

Combining the estimates (5) and (6) we obtain (4).

Put
\[
M_k(p, q) := \sup_{AC_p^\infty \ni F \neq 0} \frac{\|Fu\|_q}{\|fv\|_p}.
\]

Because of (2) and (3) we have
\[
(7) \quad M_k(p, q) = \sup_{f \in L_{p',1/u}^{q',1/u}} \|f/u\|_q^{-1} \text{dist} \left( I_kf, P_{k-1} \right).
\]

Denote \( p' = p/(p-1) \) and \( q' = q/(q-1) \) for \( 1 < p, q < \infty \) and observe that \((L_{p,v})^* = L_{p',1/v}\) if and only if \( v \in L_{p,\text{loc}} \) and \( 1/v \in L_{p',\text{loc}} \).

The following result gives an alternative version of the problems to characterize (1), (2), (3) and helps us to realise the desired solution for \( p = q = k = 2 \).

**Theorem 1.** Let \( 1 < p, q < \infty \) and the weight functions \( u \) and \( v \) be such that \((L_{p,v})^* = L_{p',1/v}\), \((L_{q,u})^* = L_{q',1/u}\). Then
\[
(8) \quad M_k(p, q) = \sup_{f \in L_{p',1/u}} \|f/u\|_q^{-1} \text{dist} \left( I_kf, P_{k-1} \right).
\]

**Proof.** Applying Lemma 1 and the duality of \( L_{p,v} \) and \( L_{p',1/v} \), \( L_{q,u} \) and \( L_{q',1/u} \), \( J_k \) and \( I_k \), we write
\[
M_k(p, q) = \sup_{g \in P_k^{\infty}} \frac{\|(J_kg)f\|_p}{\|gv\|_p} = \sup_{g \in P_k^{\infty}} \frac{\left| \int_0^1 (J_kg)f \right|}{\|f\|_{p'} \|gv\|_p} = \sup_{g \in P_k^{\infty}} \frac{\left| \int_0^1 (J_kf)g \right|}{\|f\|_{p'} \|gv\|_p} = \sup_{f \in L_{p',1/u}} \|f/u\|_q^{-1} \text{dist} \left( I_kf, P_{k-1} \right).
\]

**Remark.** The equality (8) holds for \( J_kf \) instead of \( I_kf \).
The case $p = 2$

The implicit formulae (8) becomes clearer when $p = 2$. Let $d\mu(x) = |v(x)|^{-2} \, dx$ and

$$F_k(x) = I_k(fu)(x) = \frac{1}{\Gamma(k)} \int_0^x (x - y)^{k-1} f(y)u(y) \, dy.$$ 

Then

$$\text{dist}_{L_{2,\mu}}(F_k, P_{k-1}) = \left( \int I \left| F_k(x) - F_{k,0} - \sum_{i=1}^{k-1} F_{k,i}\omega_i(x) \right|^2 \, d\mu(x) \right)^{1/2},$$

where $L_{2,\mu} = \{ f : \| f \|_{L_{2,\mu}} := \left( \int_0^1 |f|^2 \, d\mu \right)^{1/2} < \infty \}$ and

$$F_{k,0} = \frac{1}{\mu(I)} \int_I F_k \, d\mu,$$

$$F_{k,i} = \frac{1}{\mu_i(I)} \int_I F_k \omega_i \, d\mu, \quad i = 1, \ldots, k - 1$$

and polynomials $\{\omega_i(x)\}, i = 1, \ldots, k - 1$, appear from the Gram-Schmidt orthogonalization process of $\{1, t, \ldots, t^{k-1}\}$ in $L_{2,\mu}$ (see [4], Lemma 2).

Observe, that if $p \neq 2$, $p \in (1, \infty)$ and $k = 1$, then

$$\left( \int I |F_1 - F_{1,0}|^p \, d\mu_p \right)^{1/p} \leq \text{dist}_{L_{p,\mu}}(F_1, P_0) \leq 2 \left( \int I |F_1 - F_{1,0}|^p \, d\mu_p \right)^{1/p},$$

(see [3]), where $d\mu_p(x) = |v(x)|^{-p} \, dx$.

Thus, for $p = 2$ the characterization problems of (1), (2) and (3) are equivalent to the following Poincaré-type inequality

$$\left\| F_k - F_{k,0} - \sum_{i=1}^{k-1} F_{k,i}\omega_i \right\|_{L_{2,\mu}} \leq C \| f \|_{q'}.$$  

(9)
The case \( k = 2 \)

We need the following notation. Let \( k > 1 \), \( 1 < p, q < \infty \), \( 1/r = 1/q - 1/p \) if \( 1 < q < p < \infty \). Put

\[
A_{k,0} = A_{k,0,(a,b),u,v} = \begin{cases} 
\sup_{a < t < b} \left( f^b_a (x-t)^q |u(x)|^q \frac{dx}{x} \right)^{1/q} \left( f^b_a |v|^{-p'} \frac{dx}{x} \right)^{1/p'}, & p \leq q \\
\left( f^b_a \left( \int_a^b (x-t)^q |u(x)|^q \frac{dx}{x} \right)^{r/q} \left( \int_a^b |v|^{-p'} \frac{dx}{x} \right)^{r/p'} \right)^{1/r}, & p > q 
\end{cases}
\]

\[
A_{k,1} = A_{k,1,(a,b),u,v} = \begin{cases} 
\sup_{a < t < b} \left( f^b_a |u|^q \right)^{1/q} \left( f^b_a (t-x)^p |v(x)|^{-p'} \frac{dx}{x} \right)^{1/p'}, & p \leq q \\
\left( f^b_a \left( \int_a^b |u|^q \right)^{r/p} \left( \int_a^b (t-x)^p |v(x)|^{-p'} \frac{dx}{x} \right)^{r/p'} \right)^{1/r}, & p > q 
\end{cases}
\]

\[
B_{k,0} = B_{k,0,(a,b),u,v} = \begin{cases} 
\sup_{a < t < b} \left( f^b_a (x-t)^q |u(x)|^q \frac{dx}{x} \right)^{1/q} \left( f^b_a |v|^{-p'} \frac{dx}{x} \right)^{1/p'}, & p \leq q \\
\left( f^b_a \left( \int_a^b (x-t)^q |u(x)|^q \frac{dx}{x} \right)^{r/q} \left( \int_a^b |v|^{-p'} \frac{dx}{x} \right)^{r/p'} \right)^{1/r}, & p > q 
\end{cases}
\]

\[
B_{k,1} = B_{k,1,(a,b),u,v} = \begin{cases} 
\sup_{a < t < b} \left( f^b_a |u|^q \right)^{1/q} \left( f^b_a (x-t)^p |v(x)|^{-p'} \frac{dx}{x} \right)^{1/p'}, & p \leq q \\
\left( f^b_a \left( \int_a^b |u|^q \right)^{r/p} \left( \int_a^b (x-t)^p |v(x)|^{-p'} \frac{dx}{x} \right)^{r/p'} \right)^{1/r}, & p > q 
\end{cases}
\]

\[
A_k = A_{k,(a,b),u,v} = \max(A_{k,0}, A_{k,1}),
\]

\[
B_k = B_{k,(a,b),u,v} = \max(B_{k,0}, B_{k,1}).
\]

The constants \( A_k \) and \( B_k \) are equivalent to the norms of the Riemann-Liouville operators \( I_k \) and \( J_k \), respectively, from \( L_{p,v}(a,b) \) into \( L_{q,u}(a,b) \) [16–17].

**Theorem 2.** Let \( 1 < p, q < \infty \), \( k = 2 \) and let the hypothesis of Theorem 1 be fulfilled. Then

\[
M_2(p,q) \leq \inf_{0 < \tau < \lambda < \sigma < 1} \left( A_{2,(0,\tau),u,v} + A_{1,(\tau,\lambda),u,(x-\tau)^{-1}v}(x) + B_{1,(\tau,\lambda),(x-\tau)\mu(x),v} + D\tau,\lambda + D\tau,\lambda + B_{2,(\sigma,1),u,v} + A_{1,(\lambda,\sigma),u,(\sigma-x)^{-1}v}(x) + B_{1,(\lambda,\sigma),(\sigma-x)\mu(x),v} + D\lambda,\sigma + D\lambda,\sigma \right),
\]

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where
\[
D_{\tau,\lambda} = \left( \int_{\tau}^{\lambda} |u|^q \right)^{1/q} \left( \int_{\lambda}^{\tau} (\tau - x)^{p'} |v(x)|^{-p'} \, dx \right)^{1/p'},
\]
\[
D_{\lambda,\sigma} = \left( \int_{\lambda}^{\sigma} (\sigma - x)^q |u(x)|^q \, dx \right)^{1/q} \left( \int_{\lambda}^{\sigma} |v|^p \, dx \right)^{1/p'},
\]
\[
D_{\tau}^{\ast,\lambda} = \left( \int_{\lambda}^{\tau} (x - \tau)^q |u(x)|^q \, dx \right)^{1/q} \left( \int_{\lambda}^{1} |v|^p \, dx \right)^{1/p'},
\]
\[
D_{\lambda}^{\ast,\sigma} = \left( \int_{\lambda}^{\sigma} |u|^q \right)^{1/q} \left( \int_{\sigma}^{1} (x - \sigma)^p |v(x)|^{-p'} \, dx \right)^{1/p'}.
\]

Proof. If \( f \in P_1^+ \), then for all \( x \in [0, 1] \) we have

\begin{equation}
I_2 f(x) = J_2 f(x).
\end{equation}

Let \( \lambda \in (0, 1) \) and for any \( \tau \in (0, \lambda) \) and \( x \in (\tau, \lambda) \) we find
\[
I_2 f(x) = \int_0^x \left( \int_0^s f \right) \, ds = \int_0^\tau \left( \int_0^s f \right) \, ds + \int_\tau^x \left( \int_0^s f \right) \, ds
\]
\[
= \int_0^\tau (\tau - y) f(y) \, dy - \int_\tau^x \left( \int_y^\tau f \right) \, dy
\]
\[
= \int_0^\tau (\tau - y) f(y) \, dy - \int_\tau^x f(y) \left( \int_y^\tau \, ds \right) \, dy
\]
\[
- \int_\tau^x f(y) \left( \int_y^\tau \, ds \right) \, dy = \int_0^\tau (\tau - y) f(y) \, dy - \int_\tau^x f(y) \left( \int_y^\tau \, ds \right) \, dy
\]
\[
- (x - \tau) \int_\tau^\lambda f \, dy - (x - \tau) \int_\lambda^x f.
\]

Analogously, with \( \sigma \in (\lambda, 1) \) for \( x \in (\lambda, \sigma) \) we write
\[
I_2 f(x) = J_2 f(x) = \int_x^1 \left( \int_s^1 f \right) \, ds
\]
\[
= \int_0^1 \left( \int_s^1 f \right) \, ds + \int_0^x \left( \int_s^1 f \right) \, ds
\]
\[
= \int_0^1 (y - \sigma) f(y) \, dy - \int_x^\sigma (\sigma - y) f(y) \, dy
\]
\[
- (\sigma - x) \int_\lambda^\sigma f \, dy - (\sigma - x) \int_0^\lambda f.
\]
Now we estimate the norm of each term on the right hand side. Using \cite{16–17} we obtain
\[
\|\chi_{[0,\tau]} (I_2 f) u\|_q \leq A_{2; (0, \tau), u, v} \|\chi_{[0,\tau]} f v\|_p \leq A_{2; (0, \tau), u, v} \|f v\|_p.
\]
Plainly
\[
\|\chi_{[\tau, \lambda]} (I_2 f) u\|_q \leq \|\chi_{[\tau, \lambda]} (x) u(x) \int_0^\tau (\tau - y) f(y) dy\|_q
\]
\[
+ \|\chi_{[\tau, \lambda]} (x) u(x) \int_\tau^x (y - \tau) f(y) dy\|_q + \|\chi_{[\tau, \lambda]} (x) u(x) (x - \tau) \int_x^\lambda f\|_q
\]
(we use the Hölder inequality for the first and the fourth term and the upper estimates which follow from the weighted Hardy inequalities \cite{13} for the second and the third term)
\[
\leq (D_{\tau, \lambda} + A_{1; (\tau, \lambda), u, (x - \tau)^{-1} v(x)} + B_{1; (\tau, \lambda), (x - \tau) u(x), v} + D'_{\tau, \lambda}) \|f v\|_p.
\]
Similarly, applying (11),
\[
\|\chi_{[\lambda, \sigma]} (I_2 f) u\|_q \leq (D'_{\lambda, \sigma} + B_{1; (\lambda, \sigma), u, (\sigma - x)^{-1} v(x)} + A_{1; (\lambda, \sigma), (\sigma - x) u(x), v} + D_{\lambda, \sigma}) \|f v\|_p.
\]
\[
\|\chi_{[\sigma, 1]} (I_2 f) u\|_q = \|\chi_{[\sigma, 1]} (J_2 f) u\|_q \leq B_{2; (\sigma, 1), u, v} \|f v\|_p.
\]
Finally we obtain
\[
\| (I_2 f) u\|_q \leq \|\chi_{[0, \tau]} (I_2 f) u\|_q + \|\chi_{[\tau, \lambda]} (I_2 f) u\|_q
\]
\[
+ \|\chi_{[\lambda, \sigma]} (I_2 f) u\|_q + \|\chi_{[\sigma, 1]} (I_2 f) u\|_q
\]
\[
\leq (A_{2; (0, \tau), u, v} + D_{\tau, \lambda} + A_{1; (\tau, \lambda), u, (x - \tau)^{-1} v(x)} + B_{1; (\tau, \lambda), (x - \tau) u(x), v}
\]
\[
+ D'_{\tau, \lambda} + D'_{\lambda, \sigma} + B_{1; (\lambda, \sigma), (\sigma - x)^{-1} v(x)}
\]
\[
+ A_{1; (\lambda, \sigma), (\sigma - x) u(x), v} + D_{\lambda, \sigma} + B_{2; (\sigma, 1), u, v}) \|f v\|_p.
\]
Since \(\tau, \lambda\) and \(\sigma\) were arbitrary the upper bound (10) of \(M_2(p, q)\) follows.

Remark. Theorem 2 gives the upper bound for \(M_k(p, q)\), when \(k = 2\). Obviously the similar upper estimates can be proved by the same method for \(k > 2\). We omit the details.

Denote \(\mathcal{E}\) the right hand side of (10) when \(p = q = 2\). The following result brings the characterization of (1) for \(p = q = k = 2\).
Theorem 3. Let the hypothesis of Theorem 1 be fulfilled for \( p = q = 2 \). Then
\[
\frac{1}{40} \kappa E \leq M_2(2, 2) \leq E,
\]
where \( \kappa = \kappa(v) \).

Proof. The upper bound is an immediate corollary of Theorem 2. To prove the lower bound we use Theorem 1 and the arguments from Lemma 7 [4]. Let
\[
d\mu(x) = |v(x)|^{-2} \, dx; \quad \mu(I) = \int_I d\mu(y);
\]
\[
\omega(x) = \int_I (x - y) \, d\mu(y); \quad d\mu_1(x) = |\omega(x)|^2 \, d\mu(x); \quad \mu_1(I) = \int_I d\mu_1(y).
\]
If we take the point \( \lambda \in I \) such that \( \omega(\lambda) = 0 \) and choose \( \tau, \sigma \) so that
\[
0 < \tau < \lambda < \sigma < 1, \quad \mu(0, \tau) = \mu(\tau, \lambda) \text{ and } \mu(\lambda, \sigma) = \mu(\sigma, b),
\]
then there exist positive numbers \( \delta_i = \delta_i(v) \in (0, 1) \), \( i = 1, \ldots, 5 \) for which
\[
\mu(0, \lambda) = \delta_1 \mu(I), \quad \mu_1(\tau, \lambda) = \delta_2 \mu_1(I), \quad \mu_1(\lambda, \sigma) = \delta_3 \mu_1(I),
\]
\[
\int_0^\tau (\tau - s)^2 \, d\mu(s) = \delta_4 \frac{\mu_1(I)}{\mu(I)^2},
\]
\[
\int_\sigma^1 (s - \sigma)^2 \, d\mu(s) = \delta_5 \frac{\mu_1(I)}{\mu(I)^2}.
\]
Set \( \delta = \min \delta_i \) and \( \kappa = (\delta)^{3/2} \). Then Lemma 7 [4] gives us the required lower bound
\[
M_2(2, 2) \geq \frac{1}{40} \kappa E.
\]

References


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