THE DIRECTED DISTANCE DIMENSION OF ORIENTED GRAPHS

GARY CHARTRAND, MICHAEL RAINES, PING ZHANG, Kalamazoo

(Received January 26, 1998)

Abstract. For a vertex \( v \) of a connected oriented graph \( D \) and an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \) of vertices of \( D \), the (directed distance) representation of \( v \) with respect to \( W \) is the ordered \( k \)-tuple \( r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)) \), where \( d(v, w_i) \) is the directed distance from \( v \) to \( w_i \). The set \( W \) is a resolving set for \( D \) if every two distinct vertices of \( D \) have distinct representations. The minimum cardinality of a resolving set for \( D \) is the (directed distance) dimension \( \dim(D) \) of \( D \). The dimension of a connected oriented graph need not be defined. Those oriented graphs with dimension 1 are characterized. We discuss the problem of determining the largest dimension of an oriented graph with a fixed order. It is shown that if the outdegree of every vertex of a connected oriented graph \( D \) of order \( n \) is at least 2 and \( \dim(D) \) is defined, then \( \dim(D) \leq n - 3 \) and this bound is sharp.

Keywords: oriented graphs, directed distance, resolving sets, dimension

MSC 2000: 05C12, 05C20

1. Introduction

For an oriented graph \( D \) of order \( n \), an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \) of vertices of \( D \), and a vertex \( v \) of \( D \), the \( k \)-vector (ordered \( k \)-tuple)

\[
    r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))
\]

is referred to as the (directed distance) representation of \( v \) with respect to \( W \), where \( d(x, y) \) denotes the directed distance from \( x \) to \( y \), that is, the length of a shortest directed \( x - y \) path in \( D \). Since directed \( x - y \) paths need not exist in \( D \), even if \( D \) is connected (its underlying graph is connected), the vector \( r(v \mid W) \) need not exist as well. If \( r(v \mid W) \) exists for every vertex \( v \) of \( D \), then the set \( W \) is called a resolving set for \( D \) if every two distinct vertices of \( D \) have distinct representations. A resolving set of minimum cardinality is called a basis for \( D \) and this cardinality is
the (directed distance) dimension $\dim(D)$ of $D$. Of course, not every oriented graph has a dimension. An oriented graph of dimension $k$ is also called $k$-dimensional.

To determine whether an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in an oriented graph $D$ is a resolving set, we need only show that the representations of the vertices of $V(D) - W$ are distinct since $r(w_i \mid W)$ is the only representation whose $i$th coordinate is 0.

The directed distance dimension of an oriented graph is a natural analogue of the metric dimension of a graph that was introduced independently by Harary and Melter [2] and Slater [3], [4]. This concept was also investigated in [1] as a result of studying a problem in pharmaceutical chemistry.

In the oriented graph $D$ of Figure 1, let $W_1 = \{u, v\}$. The five representations of the vertices of $D$ with respect to $W_1$ are $r(u \mid W_1) = (0, 2)$, $r(v \mid W_1) = (1, 0)$, $r(w \mid W_1) = (2, 1)$, $r(x \mid W_1) = (2, 1)$, and $r(y \mid W_1) = (1, 3)$. Since $x$ and $w$ have the same representation, $W_1$ is not a resolving set for $D$.

The five representations of the vertices of $D$ with respect to $W_2 = \{u, v, w\}$ are
\[
\begin{align*}
    r(u \mid W_2) &= (0, 2, 2), & r(v \mid W_2) &= (1, 0, 3), & r(w \mid W_2) &= (2, 1, 0), \\
    r(x \mid W_2) &= (2, 1, 1), & r(y \mid W_2) &= (1, 3, 3).
\end{align*}
\]
Since these five 3-vectors are distinct, $W_2$ is a resolving set for $D$. However, $W_2$ is not a basis for $D$. To see this, let $W_3 = \{x, y\}$. Then $r(u \mid W_3) = (1, 3)$, $r(v \mid W_3) = (2, 1)$, $r(w \mid W_3) = (3, 1)$, $r(x \mid W_3) = (0, 2)$, and $r(y \mid W_3) = (2, 0)$, which are distinct as well. So $W_3$ is a resolving set for $D$. Since there is no 1-element resolving set for $D$, it follows that $W_3$ is a basis and $\dim(D) = 2$.

Now let $T$ be the tournament shown in Figure 2. Table 1 gives all 2-element choices for $W$ and shows that for each such choice, there exist two equal 2-vectors, thus showing that $\dim(T) \geq 3$. However, $\dim(T) = 3$ since $\{v_1, v_3, v_6\}$ is a basis for $T$. Figure 3 shows an oriented graph $D$ containing $T$ as an induced subdigraph.

The set $W = \{x, y\}$ is a basis of $D$, so $\dim(D) = 2$. Hence we have the possibly
unexpected property that the 3-dimensional tournament $T$ is an induced subdigraph of the 2-dimensional oriented graph $D$.

![Figure 2. The tournament $T$](image)

There is a fundamental question here—one whose answer is not known to us, but one which deserves further study. What is a necessary and sufficient condition for the dimension of a digraph $D$ to be defined? Certainly, if $D$ is strong, then dim($D$) is defined. Also, if $D$ is connected and contains a vertex such that $D - v$ is strong, then dim($D$) is defined. This last statement follows because if od $v > 0$, then $V(D) - \{v\}$ is a resolving set; while if id $v > 0$, then $V(D)$ is a resolving set. There are numerous other sufficient conditions for dim($D$) to be defined.
then, for each pair if and only if there exists a vertex describesome properties of bases for 1-dimensional oriented graphs.

\[
\text{Table 1.}
\]

2. 1-DIMENSIONAL ORIENTED GRAPHS

In this section we characterize those oriented graphs having dimension 1. We also describe some properties of bases for 1-dimensional oriented graphs.

**Theorem 2.1.** Let \( D \) be a nontrivial oriented graph of order \( n \). Then \( \dim(D) = 1 \) if and only if there exists a vertex \( v \) in \( D \) such that

(i) \( D \) contains a hamiltonian path \( P \) with terminal vertex \( v \) such that \( \text{id}_D(v) = 1 \); and

(ii) if the hamiltonian path \( P \) in (i) is of the form

\[v_{n-1}, v_{n-2}, \ldots, v_1, v,\]

then, for each pair \( i, j \) of integers with \( 1 \leq i < j \leq n - 1 \), the digraph \( D - E(P) \) contains no arc of the form \((v_j, v_i)\).
Let $W = \{v\}, \ v \in V(D),$ be a basis of $D$. Then the distance $d(u,v)$ from $u$ to $v$ is defined for each vertex $u$ in $D$ and the set $\{d(u,v); \ u \in V(D)\} = \{0,1,\ldots,n-1\}$. Thus, we may assume that $V(D) = \{v,v_1,v_2,\ldots,v_{n-1}\}$ where $d(v_i,v) = i \ (1 \leq i \leq n - 1)$. Clearly, id $v = 1$. Since $d(v_{n-1},v) = n - 1$, there exists a hamiltonian path in $D$, namely $P$: $v_{n-1}, v_{n-2}, \ldots, v_1, v$, so (i) holds. Furthermore, if there exists a pair $i, j$ of integers $(1 \leq i < j \leq n - 1)$ such that the arc $(v_j,v_i)$ is in $D - E(P)$, then $j \neq i + 1$ and $d(v_j,v) = d(v_{i+1},v)$ (shown in Figure 4). This contradicts the fact that $\{d(u,v); \ u \in V(D)\}$ consists of $n$ distinct integers, so (ii) holds.

Conversely, assume that there is a vertex $v$ in $D$ such that (i) and (ii) hold. We show that $W = \{v\}$ is a resolving set of $D$. Since $d(u,v)$ is defined for each $u \in V(D)$, it suffices to show that the set $\{d(v_i,v); \ 1 \leq i \leq n - 1\}$ consists of $n - 1$ distinct integers. Suppose that this is not the case. Then there exist integers $i, j$ $(1 \leq i < j \leq n - 1)$ such that $d(v_j,v) = d(v_i,v) = \ell$. Let $P_1$ be a $v_i - v$ path and $P_2$ a $v_j - v$ path in $D$ such that $P_1$ and $P_2$ have the same length $\ell$. Since id $v = 1$, there exists a vertex $v_k \neq v$ in $D$ that belongs to both $P_1$ and $P_2$. Assume that $v_k$ is the vertex with largest index $k$ such that the path $v_k,v_{k-1},\ldots,v_1,v$ is on both $P_1$ and $P_2$ (see Figure 5).

\[\begin{array}{c}
\text{Figure 4.}
\end{array}\]

\[\begin{array}{c}
\text{Figure 5.}
\end{array}\]

Let $(v_{k_1},v_k) \in E(P_1)$ and $(v_{k_2},v_k) \in E(P_2)$ where $(v_{k_1},v_k) \neq (v_{k_2},v_k)$. Clearly, $k_1 > k$ and $k_2 > k$. It follows that at least one of these arcs is in $D - E(P)$, but this is a contradiction to (ii). \hfill \Box

We now present some facts concerning bases in 1-dimensional oriented graphs.

**Theorem 2.2.** Let $D$ be a digraph of order $n$ with dim($D$) = 1. Furthermore, let $v_1$ and $v_2$ be distinct vertices of $D$ with $d(v_1,v_2) = 2$ such that both $\{v_1\}$ and
\{v_2\} are bases of \(D\). If \(v\) is a vertex of \(D\) such that \((v_1, v), (v, v_2) \in E(D)\), then \(\{v\}\) is also a basis of \(D\).

**Proof.** To show that \(\{v\}\) is a basis of \(D\), we show that for each \(u \in V(D)\), the distance \(d(u, v)\) is defined and the set \(\{d(u, v) : u \in V(D)\}\) consists of \(n\) distinct integers.

First notice that \(id_v = 1\), for otherwise there exist distinct vertices \(x\) and \(y\) of \(D\) such that \(d(x, v) = d(y, v) = 1\). Since \(id_{v_2} = 1\), by Theorem 2.1, we have
\[
d(x, v_2) = d(y, v_2) = d(x, v) + 1 = 2
\]
This contradicts the fact that \(\{v_2\}\) is a basis of \(D\).

Furthermore, suppose that there exist vertices \(u, w\) in \(D\) such that \(d(u, v) = d(w, v)\). Since \(id_v = 1\), each \(u - v\) path contains the arc \((v_1, v)\) as its terminal arc, as does each \(w - v\) path, so
\[
d(u, v_1) = d(w, v_1) = d(u, v) - 1
\]
Again, this contradicts the fact that \(\{v_1\}\) is a basis of \(D\). \(\square\)

We now have an immediate consequence of Theorem 2.2.

**Corollary 2.3.** If \(D\) is a 1-dimensional oriented graph of order \(n \geq 3\) such that \(\{v\}\) is a basis of \(D\) for every vertex \(v\) in \(D\), then \(D\) is a directed cycle.

**Proof.** Let \(V(D) = \{v_1, v_2, \ldots, v_n\}\). By Theorem 2.2, \(id_v = 1\) for every vertex \(v\) of \(D\). Moreover, \(D\) contains a hamiltonian path \(P\). We can assume that
\[
P: \ v_n, v_{n-1}, \ldots, v_2, v_1
\]
Next, we show that \(D\) contains the cycle
\[
C_n: \ v_n, v_{n-1}, \ldots, v_2, v_1, v_n
\]
Since \(id_{v_n} = 1\), there exists a unique vertex \(v\) such that \((v, v_n) \in E(D)\). If \(v \neq v_1\), then \((v_i, v_n) \in E(D)\) for some \(i\) \((2 \leq i \leq n - 1)\). Since \(\{v_n\}\) is a basis of \(D\), there exists a hamiltonian path in \(D\) with terminal vertex \(v_n\). However, since every vertex has indegree 1, the only possible path in \(D\) with \(v_n\) as its terminal vertex is
\[
P': \ v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v_i, v_n
\]
Since \(P'\) has length \(n - i\), it is not a hamiltonian path. This contradicts the fact that \(\{v_n\}\) is a basis. So \(D\) contains the cycle \(C_n\). Furthermore, since \(id_v = 1\), \(D\) cannot contain any arc except those in \(C_n\). So \(D = C_n\). \(\square\)
We can improve Corollary 2.3 slightly.

**Corollary 2.4.** If $D$ is a 1-dimensional oriented graph of order $n \geq 3$ such that

$$|\{v; \{v\} \text{ is a basis of } D\}| \geq n - 1$$

then $D$ is a directed cycle.

**Proof.** Let $V(D) = \{v, v_1, v_2, \ldots, v_{n-1}\}$. Without loss of generality, we assume that $\{v_i\}$ is a basis of $D$ for $1 \leq i \leq n - 1$. By Corollary 2.3, it suffices to show that $\{v\}$ is a basis as well.

We claim that $\text{od } v > 0$. Suppose that this is not the case. Then for each vertex $u (\neq v)$, the distance $d(v, u)$ is not defined, which contradicts the fact that $\{u\}$ is a basis of $D$. Hence, there is a vertex $x (\neq v)$ such that $(v, x) \in E(D)$. Since $\{x\}$ is also a basis of $D$, then by Theorem 2.1(i), $D$ contains a hamiltonian path with terminal vertex $x$ and $\text{id } x = 1$. This implies that there exists a vertex $y$ distinct from $x$ and $v$ such that $(y, v) \in E(D)$. It follows that $d(y, x) = 2$ and by Theorem 2.2, $\{v\}$ is also a basis of $D$. □

The bound in Corollary 2.4 cannot be improved in general. For example, consider the oriented graph $D$ of order $n$ in Figure 6. Since $\{v_i\}$ is a basis for $D$ for $1 \leq i \leq n - 2$, $\dim(D) = 1$. However, neither $\{v_{n-1}\}$ nor $\{v_n\}$ is a basis $D$. So $|\{v; \{v\} \text{ is a basis of } D\}| = n - 2$ and $D$ is not a directed cycle.

![Figure 6. An oriented graph with $(n-2)$ 1-element bases](image)

There is only one 1-dimensional oriented tree of every order.

**Theorem 2.5.** For every oriented tree $T$, $\dim(T) = 1$ or $\dim(T)$ is undefined. Furthermore, if $\dim(T) = 1$, then $T$ is a directed hamiltonian path.
There are certainly oriented trees whose dimension is undefined, for example, any orientation of a star $K_{1,t}$, where $t \geq 3$. Now let $T$ be an oriented tree whose dimension is defined. Since $T$ contains no cycles, for every pair $x, y$ of vertices, whenever $d(x, y)$ is defined, $d(y, x)$ is undefined. Thus $\text{dim}(T) = 1$.

If $\text{dim}(T) = 1$, then, by Theorem 2.1, $T$ contains a hamiltonian path $P$ and so $T = P$. □

3. On oriented graphs with large dimension

We have characterized those oriented graphs with dimension 1. But how large can the dimension of an oriented graph of order $n$ be? In this section, we describe upper bounds for the dimension of a connected oriented graph in terms of lower bounds for the outdegrees of its vertices. The outdegree of every vertex in the oriented graph $D$ of Figure 7 is 2, yet $\text{dim}(D)$ is undefined. Such examples exist regardless of the outdegrees.

![Figure 7. The oriented graph $D$](image)

**Theorem 3.1.** If $D$ is a connected oriented graph of order $n \geq 3$ with $\text{od} v \geq 1$ for all $v \in V(D)$ such that $\text{dim}(D)$ is defined, then $\text{dim}(D) \leq n - 2$.

**Proof.** Let $D$ be an oriented graph satisfying the hypothesis of the theorem. Certainly $\text{dim}(D) \leq n - 1$. Assume, to the contrary, that $\text{dim}(D) = n - 1$. Let $W = \{v_1, v_2, \ldots, v_{n-1}\}$ be a basis for $D$ and let $V(D) - W = \{x\}$. Since $\text{od} x \geq 1$, assume, without loss of generality, that $x$ is adjacent to $v_1$. Also, since $\text{od} v_1 \geq 1$, we may assume that $v_1$ is adjacent to $v_2$. Since $\text{dim}(D) = n - 1$, $r(v_i \mid W - \{v_i\}) = r(x \mid W - \{v_i\})$ for $1 \leq i \leq n - 1$. Since $x$ is adjacent to $v_1$, it follows that $v_2$ is adjacent to $v_1$, but this contradicts the fact that $D$ is an oriented graph. □

We now describe a class of oriented graphs. For $k \geq 2$, let $D_k$ be an oriented graph with vertex set

$$V(D_k) = \{u, v, w_1, w_2, \ldots, w_k\}$$

and let $E(D_k)$ consist of the arc $(u, v)$ and the arcs $(v, w_j)$ and $(w_j, u)$ for $1 \leq j \leq k$. The oriented graph $D_k$ is shown in Figure 8. Then $D_k$ has order $n = k + 2$ and $\text{od} v \geq 1$ for all $v \in V(D_k)$. We claim that $\text{dim}(D_k) = n - 3$. 162
First we show that \(\dim(D_k) \leq n - 3\). Let \(W = \{w_2, w_3, \ldots, w_k\}\), where then \(|W| = n - 3\). The distances \(d(u, w_2) = 2\), \(d(v, w_2) = 1\), and \(d(w_1, w_2) = 3\) show that \(W\) is a resolving set for \(D_k\) and so \(\dim(D_k) \leq n - 3\). On the other hand, at least \(k - 1\) of the vertices \(w_1, w_2, \ldots, w_k\) must belong to every resolving set of \(D_k\) since the distance from any two of these vertices to every other vertex of \(D_k\) is the same. Hence \(\dim(D_k) \geq n - 3\) and so \(\dim(D) = n - 3\). Of course, this does not show that sharpness of the bound in Theorem 3.1, except that if \(D_1\) is the directed 3-cycle, then \(\dim(D_1) = 1 = n - 2\).

We can, however, improve the bound in Theorem 3.1 if we require that the outdegree of every vertex is at least 2.

**Theorem 3.2.** If \(D\) is a connected oriented graph of order \(n \geq 5\) with od \(v \geq 2\) for all \(v \in V(D)\) such that \(\dim(D)\) is defined, then \(\dim(D) \leq n - 3\).

**Proof.** Suppose, to the contrary, that \(D\) contains a basis \(B\) of cardinality \(n - 2\). Let \(B = \{v_1, v_2, \ldots, v_{n-2}\}\), and \(V(D) - B = \{x, y\}\). For each \(i (1 \leq i \leq n - 2)\), \(B - \{v_i\}\) is not a resolving set. Hence for each such \(i\), some two of the three vertices \(x, y, v_i\) have the same representations with respect to \(B - \{v_i\}\). We consider two cases.

**Case 1:** For some \(i (1 \leq i \leq n - 2)\), \(x\) and \(y\) have the same representations with respect to \(B - \{v_i\}\). Assume, without loss of generality, that \(x\) and \(y\) have the same representations with respect to \(W = B - \{v_{n-2}\}\). Then \(x\) and \(y\) have the same out-neighbors in \(W\). Since \(x\) and \(y\) have distinct representations with respect to \(B\), exactly one of \(x\) and \(y\) is adjacent to \(v_{n-2}\); for if neither \(x\) nor \(y\) is adjacent to \(v_{n-2}\), then \(d(x, v_{n-2}) = d(y, v_{n-2})\). Therefore, we may assume that \(y\) is adjacent to \(v_{n-2}\).

Let \(W' = \{v_1, v_2, \ldots, v_{n-4}, v_{n-2}\}\). Two of \(x, y, v_{n-3}\) have the same representations with respect to \(W'\). However, \(y\) is adjacent to \(v_{n-2}\) and \(x\) is not, so \(x\) and \(y\) do not have the same representations with respect to \(W'\). Thus there are two possibilities.
Subcase 1.1: \( r(x \mid W') = r(v_{n-3} \mid W') \). We claim that \( x \) is adjacent to at most one of \( v_1, v_2, \ldots, v_{n-2} \). Suppose that this is not the case. Then we can assume without loss of generality that \( x \) is adjacent to \( v_1 \) and \( v_2 \). Then \( r(v_1 \mid B - \{v_1\}) = r(x \mid B - \{v_1\}) \) or \( r(v_1 \mid B - \{v_1\}) = r(y \mid B - \{v_1\}) \). Similarly, \( r(v_2 \mid B - \{v_2\}) = r(x \mid B - \{v_2\}) \) or \( r(v_2 \mid B - \{v_2\}) = r(y \mid B - \{v_2\}) \). Since the out-neighbors of \( y \) in \( W \) are the same as the out-neighbors of \( x \) in \( W \), we have that \( v_2 \) is an out-neighbor of \( v_1 \) and that \( v_1 \) is an out-neighbor of \( v_2 \). Since \( D \) is an oriented graph, this is impossible, so, as claimed, \( x \) is adjacent to at most one of \( v_1, v_2, \ldots, v_{n-2} \). Now, since \( od(x) \geq 2 \), it follows that \( x \) is adjacent to \( y \) and exactly one vertex from \( v_1, v_2, \ldots, v_{n-2} \), say \( v_1 \). However, since for \( 1 \leq i \leq n - 3 \), \( r(v_i \mid B - \{v_i\}) = r(x \mid B - \{v_i\}) \) or \( r(v_i \mid B - \{v_i\}) = r(y \mid B - \{v_i\}) \), it follows that \( v_1 \) is an out-neighbor of every vertex in the set \( \{x, y, v_2, v_3, \ldots, v_{n-3}, v_{n-2}\} \), so \( od(v_1) = 0 \), which is a contradiction. Therefore, \( x \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4}, v_{n-2} \). Thus, since \( od(x) \geq 2 \), it follows that \( x \) must be adjacent to both \( y \) and \( v_{n-3} \). But \( y \) is adjacent to \( v_{n-3} \) as well, because \( x \) and \( y \) have the same representations with respect to \( W \). Since \( x \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4} \), it follows that \( y \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4} \). Now \( r(y \mid W') = r(v_{n-3} \mid W') \), so it follows that \( v_{n-3} \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4} \). All of this implies that \( od(v_{n-3}) = 1 \), which is a contradiction.

Subcase 1.2: \( r(y \mid W') = r(v_{n-3} \mid W') \). We first suppose that \( x \) is adjacent to some vertex in \( W' \), say \( v_1 \). Because of the assumptions in Case 1 and Subcase 1.2, it follows that \( y \) and \( v_{n-3} \) are also adjacent to \( v_1 \). However, since for \( 2 \leq i \leq n - 3 \), \( r(v_i \mid B - \{v_i\}) = r(x \mid B - \{v_i\}) \) or \( r(v_i \mid B - \{v_i\}) = r(y \mid B - \{v_i\}) \), it follows that \( v_1 \) is an out-neighbor of every vertex in the set \( \{x, y, v_2, v_3, \ldots, v_{n-3}, v_{n-2}\} \), so \( od(v_1) = 0 \), which is a contradiction. Therefore, \( x \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4}, v_{n-2} \). Thus, since \( od(x) \geq 2 \), it follows that \( x \) must be adjacent to both \( y \) and \( v_{n-3} \). But \( y \) is adjacent to \( v_{n-3} \) as well, because \( x \) and \( y \) have the same representations with respect to \( W \). Since \( x \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4} \), it follows that \( y \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4} \). Now \( r(y \mid W') = r(v_{n-3} \mid W') \), so it follows that \( v_{n-3} \) is not adjacent to any of \( v_1, v_2, \ldots, v_{n-4} \). All of this implies that \( od(v_{n-3}) = 1 \), which is a contradiction.

Case 2: For every \( i \) (\( 1 \leq i \leq n - 2 \)), \( x \) and \( y \) have distinct representations with respect to \( B - \{v_i\} \). We next prove that every vertex of \( B \) is an out-neighbor of \( x \) or \( y \) but at most one vertex of \( B \) is an out-neighbor of both \( x \) and \( y \). To prove this, we first show that among the out-neighbors \( y_1, y_2, \ldots, y_k \) of \( y \) in \( B \), at most one \( y_i \) has the same representation as \( y \) with respect to \( B - \{y_i\} \). Suppose that this is not the case. Then we may assume that \( r(y_1 \mid B - \{y_1\}) = r(y \mid B - \{y_1\}) \) and \( r(y_2 \mid B - \{y_2\}) = r(y \mid B - \{y_2\}) \). The first equality tells us that \( y_2 \) is an out-neighbor of \( y_1 \) and the second equality tells us that \( y_1 \) is an out-neighbor of \( y_2 \), contradicting the fact that \( D \) is an oriented graph. Similarly, among the out-neighbors \( x_1, x_2, \ldots, x_l \) of \( x \) in \( B \), at most one \( x_j \) has the same representation as \( x \) with respect to \( B - \{x_j\} \).

Next, we show that for each \( i \) (\( 1 \leq i \leq n - 2 \)), at least one of \( x \) and \( y \) is adjacent to \( v_i \). This follows from the fact that if neither \( x \) nor \( y \) is adjacent to \( v_i \), then no other
vertex $v_j$ from $B - \{v_i\}$ can be adjacent to $v_i$ since $r(v_j | B - \{v_j\}) = r(x | B - \{v_j\})$ or $r(v_j | B - \{v_j\}) = r(y | B - \{v_j\})$. Thus id $v_i = 0$, which is impossible since $d(z, v_i)$ must be defined for all $z \in V(D)$. Finally, $x$ and $y$ are simultaneously adjacent to at most one vertex $v_i$ ($1 \leq i \leq n - 2$), for if $v_a$ and $v_b$ are distinct out-neighbors of both $x$ and $y$, then $v_a$ and $v_b$ are out-neighbors of each other, which is impossible.

This creates a natural partition of the vertices of $B$ into either two or three subsets, depending on whether there exists a vertex to which $x$ and $y$ are simultaneously adjacent. We now consider these two subcases.

**Subcase 2.1: There exists a unique common out-neighbor of $x$ and $y$.**

We assume, without loss of generality, that $v_{n-2}$ is an out-neighbor of both $x$ and $y$. Furthermore, we can assume, without loss of generality, that the set $X = \{v_1, v_2, \ldots, v_k\}$ consists of the out-neighbors of $x$ and not $y$, and that the set $Y = \{v_{k+1}, v_{k+2}, \ldots, v_{n-3}\}$ consists of the out-neighbors of $y$ and not $x$. We further assume, without loss of generality, that the representations of $y$ and $v_{n-2}$ with respect to $B - \{v_{n-2}\}$ are the same. Therefore, there is no vertex in $v_j \in Y$ for which the representations of $y$ and $v_j$ with respect to $B - \{v_j\}$ are the same. Therefore, for every $v_j \in Y$, the representations of $x$ and $v_j$ with respect to $B - \{v_j\}$ are the same.

Since $x$ is adjacent to every vertex in $X$, every vertex in $Y$ is adjacent to every vertex in $X \cup \{v_{n-2}\}$. Now, there is at most one $v_i \in X$ for which the representations of $x$ and $v_i$ are the same with respect to $B - \{v_i\}$. Therefore, if $|X| \geq 2$, there exists at least one vertex $v_i \in X$ for which the representations of $y$ and $v_i$ with respect to $B - \{v_i\}$ are the same. Hence, such a vertex $v_i$ is adjacent to every vertex in $Y$, but this implies that $D$ is not an oriented graph since for any $v_j \in Y$, there is an arc from $v_i$ to $v_j$ and an arc from $v_j$ to $v_i$. Therefore, $|X| \leq 1$. But if $|X| = 1$, then $v_1$ is the only vertex that could possibly be an out-neighbor of $v_{n-2}$. This contradicts the assumption that the out-degree of every vertex in $D$ is at least 2, so $|X| = 0$. We have already seen that every vertex in $Y \cup \{x\}$ is adjacent to vertex $v_{n-2}$, so even if $|X| = 0$, we have that od $v_{n-2} = 0$, which cannot occur.

**Subcase 2.2: No vertex is a common out-neighbor of $x$ and $y$.**

We assume, without loss of generality, that the set $X = \{v_1, v_2, \ldots, v_k\}$ consists of the out-neighbors of $x$ and not $y$, and that the set $Y = \{v_{k+1}, v_{k+2}, \ldots, v_{n-2}\}$ consists of the out-neighbors of $y$ and not $x$. Recall that there is at most one $v_i \in X$ such that the representations of $v_i$ and $x$ with respect to $B - \{v_i\}$ are equal and at most one $v_j \in Y$ such that the representations of $v_j$ and $y$ with respect to $B - \{v_j\}$ are equal. This produces three possibilities to consider.

**Subcase 2.2.1: For every $v_i \in X$ and $v_j \in Y$, the representations of $v_i$ and $y$ with respect to $B - \{v_i\}$ are the same and the representations of $v_j$ and $x$ with respect to
\(B - \{v_j\}\) are the same. Then every vertex in \(Y\) is adjacent to every vertex in \(X\), and every vertex in \(X\) is adjacent to every vertex in \(Y\). This contradicts the fact that \(D\) is an oriented graph as long as \(X\) and \(Y\) are both nonempty. However, if \(X\) or \(Y\) is empty, then \(\text{od} x \leq 1\) or \(\text{od} y \leq 1\), respectively, which is a contradiction.

**Subcase 2.2.2:** There is exactly one \(v_i \in X\) for which the representations of \(v_i\) and \(x\) with respect to \(B - \{v_i\}\) are equal and there is no \(v_j \in Y\) for which \(v_j\) and \(y\) have the same representations with respect to \(B - \{v_j\}\). (Note that this subcase is symmetric to the case when there is exactly one \(v_j \in Y\) for which the representations of \(v_j\) and \(y\) with respect to \(B - \{v_j\}\) are equal and for which there is no \(v_i \in X\) such that \(v_i\) and \(x\) have the same representations with respect to \(B - \{v_i\}\).) Now every vertex in \(Y\) has the same out-neighbors as \(x\), namely the vertices in the set \(X\). So if \(Y \neq \emptyset\), then every vertex in \(Y\) is adjacent to every vertex in \(X\). Furthermore, every vertex in \(X - \{v_i\}\) has the same out-neighbors as \(y\). So if \(|X| \geq 2\), then there is at least one vertex in \(X\) which is adjacent to every vertex in \(Y\). But this produces a contradiction since \(D\) is an oriented graph. Note that if \(Y = \emptyset\), then \(y\) is adjacent to at most one vertex, namely \(x\), and this is a contradiction.

Assume now that \(|X| \leq 1\) (so \(|Y| \geq 2\)). If \(|X| = 1\), then \(v_i = v_1\) and since every vertex in \(Y\) is adjacent to \(v_i\), the vertex \(v_i\) is adjacent to no vertex except possibly \(y\). Hence, \(\text{od} v_i \leq 1\), which is a contradiction. If \(X = \emptyset\), then \(x\) has no out-neighbors except possibly for \(y\), but this contradicts the assumption that the out-degree of \(x\) is at least 2.

**Subcase 2.2.3:** There exists exactly one \(v_i \in X\) for which the representations of \(v_i\) and \(x\) with respect to \(B - \{v_i\}\) are the same and exactly one \(v_j \in Y\) for which the representations of \(v_j\) and \(y\) with respect to \(B - \{v_j\}\) are the same. First, suppose that \(|X| \geq 2\) and \(|Y| \geq 2\). Then there exists at least one vertex \(v \in X\) for which the representations of \(v\) and \(y\) with respect to \(B - \{v\}\) are the same. Therefore, \(v\) is adjacent to every vertex in \(Y\). Similarly, there is at least one vertex \(w \in Y\) for which the representations of \(w\) and \(x\) with respect to \(B - \{w\}\) are the same. Therefore, \(w\) is adjacent to every vertex in \(X\). However, since \(v \in X\) and \(w \in Y\), it follows that \(v\) is adjacent to \(w\) and \(w\) is adjacent to \(v\). This contradicts the fact that \(D\) is an oriented graph.

Next suppose that \(|X| = 1\), that \(|Y| \geq 2\), and that \(X = \{v_1\}\). Then the out-neighbors of \(x\) are \(y\) and \(v_1\). Furthermore, \(v_1\) is an out-neighbor of every vertex in \(Y - \{v_j\}\). The only possible out-neighbors of \(v_1\) are \(y\) and \(v_j\). However, if \(v_i\) is adjacent to \(v_j\), then \(x\) is adjacent to \(v_j\), which contradicts the fact that \(v_j \notin X\). Therefore, \(\text{od} v_1 \leq 1\), contradicting the fact that every vertex in \(D\) has out-degree at least 2. The case where \(|Y| = 1\) and \(|X| \geq 2\) is similar. \(\square\)
The sharpness of the bound in Theorem 3.1 is not illustrated by the digraph $D_k$ shown in Figure 8 since the outdegrees of most vertices of $D_k$ are 1. We can, however, show that the upper bound in Theorem 3.2 is sharp. Let $F_k$ be an oriented graph with vertex set

$$V(F_k) = \{u_1, u_2, v_1, v_2, w_2, w_3, \ldots, w_k\}$$

and let $E(F_k)$ consist of (1) the arcs $(u_i, v_j)$ for $1 \leq i, j \leq 2$ and (2) the arcs $(v_i, w_j)$ and $(w_j, u_i)$ for $1 \leq i \leq 2$ and $1 \leq j \leq k$. The oriented graph $F_k$ is shown in Figure 9.

Then $F_k$ has order $n = k + 4$ and the property that $\text{od} v \geq 2$ for all $v \in V(F_k)$. We claim that $\dim(F_k) = n - 3$.

![Figure 9. The oriented graph $F_k$ with minimum outdegree 2](image)

First we show that $\dim(F_k) \leq n - 3$. Let $W = \{u_1, u_2, w_3, \ldots, w_k\}$, where then $|W| = n - 3$. The distances $d(u_2, w_2) = 2$, $d(v_2, w_2) = 1$, and $d(w_1, w_2) = 3$ show that $W$ is a resolving set for $F_k$ and so $\dim(F_k) \leq n - 3$. Next we show that $\dim(F_k) \geq n - 3$. Let $W$ be a resolving set for $F_k$. Certainly at least $k - 1$ of the vertices $w_1, w_2, \ldots, w_k$ must belong to $W$ since the distance from any two of these vertices to every other vertex of $F_k$ is the same. Moreover, at least one of $u_1$ and $u_2$ must belong to $W$ since the distance from $u_1$ and $u_2$ every other vertex of $F_k$ is the same. For the same reason, at least one of $v_1$ and $v_2$ must belong to $W$. Hence $\dim(F_k) \geq n - 3$ and so $\dim(F_k) = n - 3$.

No additional restriction on the outdegrees of the vertices of an oriented graph yields an improved bound, however. Let $r \geq 2$ be an integer. In the oriented graph of Figure 8, replace $u_1, u_2$ by the $r$ vertices $u_1, u_2, \ldots, u_r$ and $v_1, v_2$ by the $r$ vertices $v_1, v_2, \ldots, v_r$ and add the appropriate arcs. The resulting oriented graph $H_k$ has $\text{od} v \geq r$ for all $v \in V(H_k)$, but $\dim(H_k) = n - 3$. 

167
References


Authors’ addresses: Gary Chartrand, Michael Raines, Ping Zhang, Department of Mathematics and Statistics Western Michigan University Kalamazoo, MI 49008, USA, e-mail: zhang@math-stat.wmich.edu.