LINEAR STIELTJES INTEGRAL EQUATIONS IN BANACH SPACES II; OPERATOR VALUED SOLUTIONS

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Abstract. This paper is a continuation of [9]. In [9] results concerning equations of the form
\[
x(t) = x(a) + \int_a^t d[A(s)]x(s) + f(t) - f(a)
\]
were presented. The Kurzweil type Stieltjes integration in the setting of [6] for Banach space valued functions was used.

Here we consider operator valued solutions of the homogeneous problem
\[
\Phi(t) = I + \int_a^t d[A(s)]\Phi(s)
\]
as well as the variation-of-constants formula for the former equation.

Keywords: linear Stieltjes integral equations, generalized linear differential equation, equation in Banach space

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Assume that \( X \) is a Banach space and that \( L(X) \) is the Banach space of all bounded linear operators \( A: X \rightarrow X \) with the uniform operator topology. Defining the bilinear form \( B: L(X) \times X \rightarrow X \) by \( B(A, x) = Ax \in X \) for \( A \in L(X) \) and \( x \in X \), we obtain in a natural way the bilinear triple \( B = (L(X), X, X) \) (see [6]) because using the usual operator norm we have
\[
\|B(A, x)\|_X \leq \|A\|_{L(X)} \|x\|_X.
\]

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Similarly, if we define the bilinear form $B^*: L(X) \times L(X) \to L(X)$ by the relation
$B^*(A, C) = AC \in L(X)$ for $A, C \in L(X)$ where $AC$ is the composition of the linear operators $A$ and $C$ we get the bilinear triple $B^* = (L(X), L(X), L(X))$ because we have

$$\|B^*(A, C)\|_{L(X)} \leq \|AC\|_{L(X)} \leq \|A\|_{L(X)} \|C\|_{L(X)}.$$

Assume that $[a, b] \subset \mathbb{R}$ is a bounded interval.

Given $A: [a, b] \to L(X)$, the function $A$ is of bounded variation on $[a, b]$ if

$$\text{var}(A) = \sup \left\{ \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} \right\} < \infty,$$

where the supremum is taken over all finite partitions

$$D: a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a, b]$. The set of all functions $A: [a, b] \to L(X)$ with $\text{var}(A) < \infty$ will be denoted by $BV([a, b]; L(X))$.

For $A: [a, b] \to L(X)$ and a partition $D$ of the interval $[a, b]$ define

$$V^b_a(A, D) = \sup \left\{ \left\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]y_j \right\|_X \right\},$$

where the supremum is taken over all possible choices of $y_j \in X, j = 1, \ldots, k$ with $\|y_j\| \leq 1$ and similarly

$$\hat{V}^b_a(A, D) = \sup \left\{ \left\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]C_j \right\|_{L(X)} \right\},$$

where the supremum is taken over all possible choices of $C_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leq 1$.

Define

$$(B) \text{var}(A) = \sup_{[a, b]} V^b_a(A, D)$$

and

$$(B^*) \text{var}(A) = \sup_{[a, b]} \hat{V}^b_a(A, D)$$

where the supremum is taken over all finite partitions

$$D: a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b.$$
of the interval \([a, b]\).

The function \(A : [a, b] \to L(X)\) with \((B) \var{A} < \infty\) is called a function with bounded \(B\)-variation on \([a, b]\) and similarly if \((B^*) \var{A} < \infty\) then \(A\) is of bounded \(B^*\)-variation on \([a, b]\) ([3]).

We denote by \((B)BV([a, b]; L(X))\) the set of all functions \(A : [a, b] \to L(X)\) with \((B) \var{A} < \infty\) and by \((B^*)BV([a, b]; L(X))\) the set of all functions \(A : [a, b] \to L(X)\) with \((B^*) \var{A} < \infty\).

In [9, Prop. 1.1 and 1.2] it is shown that

\[
BV([a, b]; L(X)) \subset (B)BV([a, b]; L(X)) = (B^*)BV([a, b]; L(X))
\]

holds.

Given \(x : [a, b] \to X\), the function \(x\) is called regulated on \([a, b]\) if it has one-sided limits at every point of \([a, b]\), i.e. if for every \(s \in [a, b]\) there is a value \(x(s+) \in X\) such that

\[
\lim_{t \to s^+} \|x(t) - x(s+)\|_X = 0
\]

and if for every \(s \in (a, b]\) there is a value \(x(s-) \in X\) such that

\[
\lim_{t \to s^-} \|x(t) - x(s-)\|_X = 0.
\]

The set of all regulated functions \(x : [a, b] \to X\) will be denoted by \(G([a, b]; X)\) and similarly we denote the set of all regulated functions \(A : [a, b] \to L(X)\) by \(G([a, b]; L(X))\).

If \(B = (L(X), X, X)\) is the bilinear triple of Banach spaces mentioned above then a function \(A : [a, b] \to L(X)\) is called \(B\)-regulated on \([a, b]\) if for every \(y \in X, \|y\|_X \leq 1\), the function \(Ay : [a, b] \to X\) given by \(t \in [a, b] \mapsto A(t)y \in X\) for \(t \in [a, b]\) is regulated, i.e. \(Ay \in G([a, b]; X)\) for every \(y \in X, \|y\|_X \leq 1\).

We denote by \((B)G([a, b]; L(X))\) the set of all \(B\)-regulated functions \(A : [a, b] \to L(X)\).

1. Equations with operator valued solutions

For \([a, b] = [0, 1]\) we denote shortly

\[
BV(L(X)) = BV([0, 1]; L(X)), (B)BV(L(X)) = (B)BV([0, 1]; L(X)),
\]

\[
G(L(X)) = G([0, 1]; L(X)) \text{ and } (B)G(L(X)) = (B)G([0, 1]; L(X)).
\]
Assume that $A: [0, 1] \to L(X)$ satisfies

\[(1.1) \quad A \in (\mathcal{B})BV(L(X)) \cap (\mathcal{B})G(L(X))\]

and the following condition (E) (see [9]):

for every $d \in [0, 1]$ there are $0 < \varrho = \varrho(d) < 1$ and $\Delta = \Delta(d) > 0$ such that

\[(E+) \quad (\mathcal{B})\operatorname{var}_{(d,d+\Delta\cap[0,1])} (A) < \varrho\]

and

\[(E-) \quad (\mathcal{B})\operatorname{var}_{[d-\Delta,d\cap[0,1])} (A) < \varrho.\]

Taking the bilinear triple $\mathcal{B}^* = (L(X), L(X), L(X))$, by Proposition 1.1 in [9] we have

\[(\mathcal{B})BV(L(X)) = (\mathcal{B}^*)BV(L(X))\]

and

\[(\mathcal{B})\operatorname{var}_{[a,b]} (A) = (\mathcal{B}^*)\operatorname{var}_{[a,b]} (A)\]

for every $[a, b] \subset [0, 1]$. Therefore condition (1.1) reads

\[(1.1) \quad A \in (\mathcal{B}^*)BV(L(X)) \cap (\mathcal{B})G(L(X)),\]

and in condition (E) the symbol $\mathcal{B}$ can also be replaced by $\mathcal{B}^*$, i.e. condition (E) reads

for every $d \in [0, 1]$ there are $0 < \varrho = \varrho(d) < 1$ and $\Delta = \Delta(d) > 0$ such that

\[(E+) \quad (\mathcal{B}^*)\operatorname{var}_{(d,d+\Delta\cap[0,1])} (A) < \varrho\]

and

\[(E-) \quad (\mathcal{B}^*)\operatorname{var}_{[d-\Delta,d\cap[0,1])} (A) < \varrho.\]

Hence the results presented in Section 2 from [9] can be used for equations of the form

\[(1.2) \quad Y(t) = \bar{Y} + \int_{d}^{t} d[A(s)]Y(s) + F(t) - F(d)\]

for every $t \in [0, 1]$ where $F \in G(L(X))$, $d \in [0, 1]$ and $\bar{Y} \in L(X)$. 434
The operator valued function $Y: [\alpha, \beta] \rightarrow L(X)$ is called a solution of (1.2) on an interval $[\alpha, \beta] \subset [0, 1]$ if $Y$ satisfies (1.2) for every $t \in [\alpha, \beta]$. If $d \in [\alpha, \beta]$ then of course we have $Y(d) = \tilde{Y}$ for this solution.

With regard to the above mentioned facts we obtain by a simple reformulation of Proposition 2.4 and Theorem 2.10 from [9] the following

**1.1. Theorem.** Assume that $A: [0, 1] \rightarrow L(X)$ satisfies (1.1) and condition (E). Then for every $d \in [0, 1]$, $\tilde{Y} \in X$, $F \in G(L(X))$ there is a $\Delta > 0$ such that for the interval $J_d = [d - \Delta, d + \Delta] \cap [0, 1]$ there is a unique function $Y \in G(J_d; L(X))$ such that

$$Y(t) = \tilde{Y} + \int_d^t d[A(s)]Y(s) + F(t) - F(d), \quad t \in J_d,$$

i.e. $Y(t)$ is a local solution of the operator valued equation (1.2) on $J_d = [d - \Delta, d + \Delta] \cap [0, 1]$.

If

$$A \in (B)BV(L(X)) \cap G(L(X)),$$

condition (U):

$$(U+) \quad [I + \Delta^+ A(t)]^{-1} \in L(X) \text{ exists for every } t \in [0, 1)$$

and

$$(U−) \quad [I - \Delta^- A(t)]^{-1} \in L(X) \text{ exists for every } t \in [0, 1]$$

and (E) hold, then for every choice of $d \in [0, 1]$, $\tilde{Y} \in L(X)$, $F \in G([0, 1]; L(X))$ there exists a unique $Y \in G([0, 1]; X)$ which is a (global) solution of (1.2) on $[0, 1]$.

Let us consider the special case of the equation (1.2) with $F$ a constant, i.e. the so called homogeneous equation

$$Y(t) = \tilde{Y} + \int_d^t d[A(s)]Y(s).$$

Theorem 1.1 applies to this equation and therefore there is a unique (global) solution to this equation and this operator valued solution is regulated provided $A: [0, 1] \rightarrow L(X)$ satisfies (1.3), (E) and (U).

Together with (1.4) let us consider the equation

$$\Phi(t) = I + \int_d^t d[A(s)]\Phi(s)$$

(1.5)
where $I \in L(X)$ is the identity operator.

Clearly every solution $Y: [0, 1] \rightarrow L(X)$ of (1.4) can be written in the form

$$Y(t) = \Phi(t)\hat{Y}, \quad t \in [0, 1].$$

Let us now consider the properties of the solution $\Phi: [0, 1] \rightarrow L(X)$ of (1.5).

1.2. Lemma. Assume that $A: [0, 1] \rightarrow L(X)$ satisfies (1.3), (E) and (U). Then for the solution $\Phi: [0, 1] \rightarrow L(X)$ of (1.5) we have

$$\Phi \in (B)BV(L(X)) \cap G(L(X))$$

and there is a constant $K > 0$ such that $\|\Phi(t)\| \leq K$ for every $t \in [0, 1]$.

Proof. By Theorem 1.1 $\Phi \in G([0, 1]; L(X))$ and therefore there exists a $K > 0$ such that $\|\Phi(t)\| \leq K$ for every $t \in [0, 1]$. It remains to show that $\Phi \in (B)BV([0, 1]; L(X))$.

Assume that

$$D: 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$$

is an arbitrary partition of the interval $[0, 1]$.

For any $y_j \in X, j = 1, \ldots, k$ with $\|y_j\| \leq 1$ we have

$$\left\| \sum_{j=1}^{k} [\Phi(\alpha_j) - \Phi(\alpha_{j-1})]y_j \right\|_X = \left\| \sum_{j=1}^{k} \int_{\alpha_{j-1}}^{\alpha_j} d[A(s)]\Phi(s)y_j \right\|_X.$$

Define

$$\varphi(s) = \Phi(s)y_j \text{ for } s \in (\alpha_{j-1}, \alpha_j) \text{ and } \varphi(s) = 0 \text{ for } s = \alpha_j.$$

Evidently $\|\varphi(s)\| \leq K$.

Then by 1.18 from [9] we get

$$\int_{\alpha_{j-1}}^{\alpha_j} d[A(s)]\Phi(s)y_j = \int_{\alpha_{j-1}}^{\alpha_j} d[A(s)]\varphi(s)$$

$$+ [A(\alpha_{j-1}+) - A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j + [A(\alpha_j) - A(\alpha_{j-1})]\Phi(\alpha_j)y_j$$

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and
\[
\left\| \sum_{j=1}^{k} \int_{\alpha_{j-1}}^{\alpha_j} d[A(s)\Phi(s)]y_j \right\|_X = \left\| \sum_{j=1}^{k} \int_{\alpha_{j-1}}^{\alpha_j} d[A(s)]\varphi(s) + [A(\alpha_{j-1}) - A(\alpha_j)]\Phi(\alpha_j) y_j \right\|_X
\]
\[
+ \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] \Phi(\alpha_j) y_j \leq \left\| \int_0^1 d[A(s)]\varphi(s) \right\|_X
\]
\[
+ \left\| \sum_{j=1}^{k} [A(\alpha_{j-1}) - A(\alpha_j)] \Phi(\alpha_j) y_j \right\|_X + \left\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] \Phi(\alpha_j) y_j \right\|_X.
\]

For a given \( \eta > 0 \) let us choose a \( \theta > 0 \) such that
\[
\|A(\alpha_{j-1} + \theta) - A(\alpha_{j-1})\|_{L(X)} < \frac{\eta}{k + 1}
\]
and
\[
\|A(\alpha_j - \theta) - A(\alpha_j)\|_{L(X)} < \frac{\eta}{k + 1}
\]
for all \( j = 1, \ldots, k \). Then
\[
\left\| \sum_{j=1}^{k} [A(\alpha_{j-1} + \theta) - A(\alpha_{j-1})] \Phi(\alpha_j) y_j \right\|_X
\]
\[
= \left\| \sum_{j=1}^{k} [A(\alpha_{j-1} + \theta) - A(\alpha_{j-1} + \theta) + A(\alpha_{j-1} + \theta) - A(\alpha_{j-1})] \Phi(\alpha_j) y_j \right\|_X
\]
\[
\leq \left\| \sum_{j=1}^{k} [A(\alpha_{j-1} + \theta) - A(\alpha_{j-1} + \theta)] \Phi(\alpha_j) y_j \right\|_X
\]
\[
+ \left\| \sum_{j=1}^{k} [A(\alpha_{j-1} + \theta) - A(\alpha_{j-1})] \Phi(\alpha_j) y_j \right\|_X
\]
\[
\leq \sum_{j=1}^{k} \frac{K\eta}{k + 1} + \sum_{j=1}^{k} \left\| A(\alpha_{j-1} + \theta) - A(\alpha_{j-1}) \Phi(\alpha_j) y_j \right\|_X
\]
\[
< K\eta + K(B) \operatorname{var}_{[0,1]}(A)
\]
and similarly also
\[
\left\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] \Phi(\alpha_j) y_j \right\|_X < K\eta + K(B) \operatorname{var}_{[0,1]}(A).
\]
By 1.11 from [9] we have further
\[ \left\| \int_0^1 d[A(s)]\varphi(s) \right\|_X \leq K(B) \text{var}(A) \]
and finally we obtain
\[ \left\| \sum_{j=1}^k [\Phi(\alpha_j) - \Phi(\alpha_{j-1})]y_j \right\|_X = \left\| \sum_{j=1}^k \int_{\alpha_{j-1}}^{\alpha_j} d[A(s)]\Phi(s)y_j \right\|_X < 2K\eta + 3K(B) \text{var}(A). \]

Passing to the corresponding suprema we arrive easily at
\[ (B) \text{var}(\Phi) \leq 3K(B) \text{var}(A) < \infty, \]
i.e. \( \Phi \in (B)BV([0, 1]; L(X)) \).

**1.3. Lemma.** Assume that \( A : [0, 1] \to L(X) \) satisfies (1.3), (E) and (U).

Then the solution \( \Phi : [0, 1] \to L(X) \) of (1.5) has an inverse \( [\Phi(t)]^{-1} \in L(X) \) for every \( t \in [0, 1] \).

**Proof.** For \( t = d \) we have \( \Phi(t) = \Phi(d) = I \) and the inverse \( [\Phi(t)]^{-1} \) evidently exists for this value.

Assume that there is a point \( t^* \in [0, 1] \) such that the inverse \( [\Phi(t^*)]^{-1} \) does not exist. Then there exists \( y \in X \) such that the equation
\[ \Phi(t^*)z = y \]
has no solution in \( X \). Assume that \( \Psi : [0, 1] \to L(X) \) is a solution of the operator valued equation
\[ \Psi(t) = I + \int_t^{t^*} d[A(s)]\Psi(s); \]
this solution exists and is uniquely determined by the second part of Theorem 1.1.

Let us set \( z = \Psi(d)y \). The function \( x : [0, 1] \to X \) given by \( x(t) = \Psi(t)y \) is a solution of the equation
\[ x(t) = y + \int_t^{t^*} d[A(s)]x(s) \]
with \( x(t^*) = y \) and \( x(d) = \Psi(d)y \). On the other hand, \( \varphi(t) = \Phi(t)z \) is a solution of
\[ \varphi(t) = z + \int_d^t d[A(s)]\varphi(s) \]
where \( \varphi(d) = z = \Psi(d)y = x(d) \) and

\[
x(t) = x(d) + \int_d^t \, d[A(s)]x(s).
\]

Hence by the uniqueness of a solution stated in Theorem 2.10 from [9] we have \( x(t) = \varphi(t) \) for all \( t \in [0, 1] \). Therefore

\[
x(t^*) = y = \varphi(t^*) = \Phi(t^*)z = \Phi(t^*)\Psi(d)y,
\]
i.e. \( z = \Psi(d)y \in X \) is a solution of the equation \( \Phi(t^*)z = y \). This contradicts the assumption and proves that the operator \( \Phi(t) \in L(X) \) has an inverse for every \( t \in [0, 1] \).

\[
\square
\]

1.4. Lemma. Assume that \( A : [0, 1] \to L(X) \) satisfies (1.3), (E) and (U).

Then the inverse \( [\Phi(t)]^{-1} = \Phi^{-1}(t) \) to the solution \( \Phi : [0, 1] \to L(X) \) of (1.5) belongs to \( G(L(X)) \) and there is a constant \( L > 0 \) such that

\[
||\Phi^{-1}(t)||_{L(X)} \leq L
\]

for every \( t \in [0, 1] \).

Proof. By Theorem 1.1 we have \( \Phi \in G(L(X)) \) and therefore the one-sided limits of this function exist at every point of \( [0, 1] \). E. g., the limit \( \lim_{t \to t^+} \Phi(r) \) exists for every \( t \in [0, 1) \) and by 1.18 from [9] we have

\[
\lim_{r \to t^+} \Phi(r) = I + \lim_{r \to t^+} \int_d^r \, d[A(s)]\Phi(s) = I + \int_d^t \, d[A(s)]\Phi(s)
\]

\[
+ \lim_{r \to t^+} \int_t^r \, d[A(s)]\Phi(s) = \Phi(t) + \lim_{r \to t^+} \int_t^r \, d[A(s)]\Phi(s)
\]

\[
= \Phi(t) + [A(t^+) - A(t)]\Phi(t) = \Phi(t) + [A(t^+) - A(t)]\Phi(t).
\]

Hence \( \Phi(t^+) = [I + \Delta^+A(t)]\Phi(t) \) and because \( \Phi^{-1}(t) \) exists by Lemma 1.3 and the inverse \( [I + \Delta^+A(t)]^{-1} \) exists by (U+) from the assumption (U) the inverse \( [\Phi(t^+)]^{-1} = \Phi^{-1}(t^+) \) also exists and we have the relation

\[
[\Phi(t^+)]^{-1} = \Phi^{-1}(t^+) = \Phi^{-1}(t) \cdot [I + \Delta^+A(t)]^{-1}, \quad t \in [0, 1).
\]

Similarly we have also

\[
\Phi^{-1}(t-) = \Phi^{-1}(t) \cdot [I - \Delta^-A(t)]^{-1}, \quad t \in (0, 1]
\]

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where $\Phi^{-1}(t-) = [\Phi(t-)]^{-1}$.

Using the continuity of the operation of taking an inverse (see [2], p. 624) we obtain

$$\lim_{r \to t^+} \Phi^{-1}(r) = \Phi^{-1}(t^+) \text{ for } t \in [0,1)$$

and

$$\lim_{r \to t^-} \Phi^{-1}(r) = \Phi^{-1}(t^-) \text{ for } t \in (0,1]$$

because $\lim_{r \to t^+} \Phi(r) = \Phi(t^+)$ for $t \in [0,1)$ and $\lim_{r \to t^-} \Phi(r) = \Phi(t^-)$ for $t \in (0,1]$.

Hence the operator valued function $\Phi^{-1} : [0,1] \to \mathcal{L}(X)$ belongs to the space $G(\mathcal{L}(X))$ and it is therefore bounded, i.e. there is an $L \geq 0$ such that

$$\|\Phi^{-1}(t)\|_{\mathcal{L}(X)} \leq L$$

for every $t \in [0,1]$. \hfill \Box

**1.5. Lemma.** Assume that $A : [0,1] \to \mathcal{L}(X)$ satisfies (1.3), (E) and (U).

Assume that $d \in [0,1]$ is fixed and that $\Phi : [0,1] \to \mathcal{L}(X)$ is the solution of (1.5). Then for every $t_0 \in [0,1]$ and $\bar{x} \in X$, the unique solution $x : [0,1] \to X$ of the homogeneous equation

$$x(t) = \bar{x} + \int_{t_0}^{t} d[A(s)]x(s)$$

is given by the relation

$$x(t) = \Phi(t)\Phi^{-1}(t_0)\bar{x}, \quad t \in [0,1].$$

**Proof.** The solution $x$ exists and is unique by Theorem 2.11 in [9]. Using (1.1) we have

$$x(t) = \Phi(t)\Phi^{-1}(t_0)\bar{x} = \left[ I + \int_{t_0}^{t} d[A(s)]\Phi(s) \right] \Phi^{-1}(t_0)\bar{x}$$

$$= \left[ I + \int_{d}^{t_0} d[A(s)]\Phi(s) + \int_{t_0}^{t} d[A(s)]\Phi(s) \right] \Phi^{-1}(t_0)\bar{x}$$

$$= \Phi(t_0)\Phi^{-1}(t_0)\bar{x} + \int_{t_0}^{t} d[A(s)]\Phi(s)\Phi^{-1}(t_0)\bar{x} = \bar{x} + \int_{t_0}^{t} d[A(s)]x(s)$$

and the lemma is proved. \hfill \Box
2. Variation of constants

2.1. Lemma. Assume that \( A : [0, 1] \to L(X) \) satisfies (1.3), (E) and (U). Let \( \Phi : [0, 1] \to L(X) \) be the solution of (1.5) and assume that its inverse \( \Phi^{-1} : [0, 1] \to L(X) \) given by Lemma 1.3 is such that \( \Phi^{-1} \in (B)BV(L(X)) \).

Then for every \( g \in G(X) \), \( t \in [0, 1] \) the equality

\[
(2.1) \quad \int_t^r \frac{d}{dA(r)} \Phi(r) \frac{d}{d\Phi^{-1}(s)} g(s) = \Phi(t) \int_d^t \frac{d}{d\Phi^{-1}(s)} g(s) + \int_d^t \frac{d}{dA(s)} g(s)
\]

holds.

Proof. Since \( g \in G(X) \) and \( \Phi^{-1} \in (B)BV(L(X)) \), the integrals on both sides of (2.1) exist by [6, Theorem 11] (see also [9, 1.12]).

To show that the equality (2.1) is valid for every regulated function \( g : [0, 1] \to X \) it is sufficient to prove it for an arbitrary finite step function, because the finite step functions are dense in the space \( G(X) \) (see [2]).

For a given \( \alpha \in [0, 1] \), \( c \in X \) and for \( s \in [0, 1] \) we define

\[
\psi_{1+}^\alpha(s) = 0 \text{ if } s \leq \alpha, \quad \psi_{1+}^\alpha(s) = c \text{ if } s > \alpha
\]

and

\[
\psi_{1-}^\alpha(s) = 0 \text{ if } s < \alpha, \quad \psi_{1-}^\alpha(s) = c \text{ if } s \geq \alpha.
\]

It is a matter of routine to verify that every finite step function can be expressed in the form of a finite sum of functions of the type \( \psi_{1+}^\alpha \) and \( \psi_{1-}^\alpha \). Hence by the linearity of the integral it suffices to show that (2.1) holds for functions of this type.

Let us prove e.g. that (2.1) is satisfied for the function \( \psi_{1+}^\alpha \).

Assume that \( \alpha < d \). Then

\[
\int_d^r \frac{d}{ds} [\Phi^{-1}(s)] \psi_{1+}^\alpha(s) = [\Phi^{-1}(r) - \Phi^{-1}(d)] c \text{ if } r > \alpha
\]

and

\[
(2.2) \quad \int_d^r \frac{d}{ds} [\Phi^{-1}(s)] \psi_{1+}^\alpha(s) = [\Phi^{-1}(\alpha+) - \Phi^{-1}(d)] c \text{ if } r \leq \alpha.
\]

Hence for \( t > \alpha \) we have

\[
(2.3) \quad \int_d^t \frac{d}{ds} [A(r)] \Phi(r) \frac{d}{d\Phi^{-1}(s)} \psi_{1+}^\alpha(s)
\]

\[
= \int_d^t \frac{d}{ds} [A(r)] \Phi(r) \frac{d}{d\Phi^{-1}(s)} \Phi^{-1}(r) - \Phi^{-1}(d)] c = \int_d^t \frac{d}{ds} [A(r)] [I - \Phi(r) \Phi^{-1}(d)] c
\]

\[
= [A(t) - A(d)] c - [\Phi(t) - \Phi(d)] \Phi^{-1}(d) c = [A(t) - A(d)] c + c - \Phi(t) \Phi^{-1}(d) c.
\]
If \( t \leq \alpha \) then

\[
\int_t^\alpha d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s) = - \int_t^\alpha d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s)
\]

\[
= - \left( \int_t^\alpha d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s) + \int_\alpha^d d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s) \right)
\]

and

\[
\int_\alpha^d d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s)
\]

\[
= [A(\alpha+) - A(\alpha)]|\Phi(\alpha)|\Phi^{-1}(\alpha+) - \Phi^{-1}(d)c
\]

\[
+ \lim_{\delta \to 0+} \int_{\alpha+\delta}^d d[A(r)]\Phi(r)[\Phi^{-1}(r) - \Phi^{-1}(d)]c
\]

\[
= [A(\alpha+) - A(\alpha)]|\Phi(\alpha)|\Phi^{-1}(\alpha+) - \Phi^{-1}(d)c
\]

\[
+ \lim_{\delta \to 0+} \int_{\alpha+\delta}^d d[A(r)]c - \lim_{\delta \to 0+} \int_{\alpha+\delta}^d d[A(r)]\Phi(r)\Phi^{-1}(d)c
\]

\[
= [A(\alpha+) - A(\alpha)]|\Phi(\alpha)|\Phi^{-1}(\alpha+) - \Phi^{-1}(d)c + [A(d) - A(\alpha+)]c
\]

\[
- [\Phi(d) - \Phi(\alpha+)]\Phi^{-1}(d)c.
\]

Further we have

\[
\int_t^\alpha d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s) = [\Phi(\alpha) - \Phi(t)]|\Phi^{-1}(\alpha+) - \Phi^{-1}(d)c
\]

and

\[
\int_d^t d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s)
\]

\[
= - \left\{ [A(\alpha+) - A(\alpha)]|\Phi(\alpha)|\Phi^{-1}(\alpha+) - \Phi^{-1}(d)c + [A(d) - A(\alpha+)]c
\]

\[
- [\Phi(d) - \Phi(\alpha+)]\Phi^{-1}(d)c + [\Phi(\alpha) - \Phi(t)]|\Phi^{-1}(\alpha+) - \Phi^{-1}(d)|c. \right\}
\]

Since \( [A(\alpha+) - A(\alpha)]|\Phi(\alpha) = \Delta^+ A(\alpha)|\Phi(\alpha) = \Phi(\alpha+) - \Phi(\alpha) \) we have

\[
\int_d^t d[A(r)]\Phi(r) \int_d^r d_s[\Phi^{-1}(s)]\psi_\alpha^+(s)
\]

\[
= - \left\{ ([\Phi(\alpha+) - \Phi(\alpha)]|\Phi^{-1}(\alpha+) - \Phi^{-1}(d)] + [A(d) - A(\alpha+)]
\]

\[
- I + \Phi(\alpha+)\Phi^{-1}(d) + \Phi(\alpha)\Phi^{-1}(\alpha+) - \Phi(\alpha)\Phi^{-1}(d)
\]

\[
- \Phi(t)\Phi^{-1}(\alpha+) + \Phi(t)\Phi^{-1}(d)c
\]

\[
= - \left\{ [A(d) - A(\alpha+)] - \Phi(t)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c \right. \}
\]

\[
= [A(\alpha+) - A(d)]c + \Phi(t)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c
\]

\[
(2.4)
\]
for $t \leq \alpha$.

For the right hand side of (2.1) we use (2.2) for obtaining

$$\Phi(t) \int_d^t d[\Phi^{-1}(s)]\psi^+_\alpha(s) = \Phi(t)[\Phi^{-1}(t) - \Phi^{-1}(d)]c \quad \text{if } t > \alpha$$

and

$$(2.5) \quad \Phi(t) \int_d^t d[\Phi^{-1}(s)]\psi^+_\alpha(s) = [\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c \quad \text{if } t \leq \alpha.$$ 

Now it is a matter of routine to show that

$$\int_d^t d[A(s)]\psi^+_\alpha(s) = [A(t) - A(d)]c \quad \text{if } t > \alpha$$

and

$$(2.6) \quad \int_d^t d[A(s)]\psi^+_\alpha(s) = [A(\alpha+) - A(d)]c \quad \text{if } t \leq \alpha.$$ 

Using (2.5) and (2.6) we obtain

$$\Phi(t) \int_d^t d[\Phi^{-1}(s)]\psi^+_\alpha(s) + \int_d^t d[A(s)]\psi^+_\alpha(s)$$

$$= -\Phi(t)[\Phi^{-1}(t) - \Phi^{-1}(d)]c + [A(t) - A(d)]c \quad \text{if } t > \alpha$$

and

$$\Phi(t) \int_d^t d[\Phi^{-1}(s)]\psi^+_\alpha(s) + \int_d^t d[A(s)]\psi^+_\alpha(s)$$

$$= [\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c + [A(\alpha+) - A(d)]c \quad \text{if } t \leq \alpha.$$ 

Looking at (2.3) and (2.4) we can see immediately that the equality (2.1) holds for the function $\psi^+_\alpha$ if $\alpha < d$.

For $\alpha \geq d$ as well as for the case of the function $\psi^-_\alpha$ the result can be proved similarly. The computations are straightforward but slightly tedious.$\square$

Let us assume that $A : [0,1] \to L(X)$ satisfies (1.3), (E) and (U).

Let us consider the equation

$$x(t) = \bar{x} + \int_{t_0}^t d[A(s)]x(s) + f(t) - f(t_0).$$
By [9, Theorem 2.10] we obtain that for every choice of $t_0 \in [0, 1], \bar{x} \in X, f \in G(X)$ there exists $x \in G(X)$ such that

$$x(t) = \bar{x} + \int_{t_0}^{t} d[A(s)]x(s) + f(t) - f(t_0)$$

for every $t \in [0, 1]$.

This solution of (2.7) is determined uniquely.

2.2. Theorem. Assume that $A: [0, 1] \to L(X)$ satisfies (1.3), (E) and (U). Let $\Phi: [0, 1] \to L(X)$ be the solution of (1.5) and assume that its inverse $\Phi^{-1}: [0, 1] \to L(X)$ given by Lemma 1.3 is such that $\Phi^{-1} \in (B)BV(L(X))$.

Then for every $t_0 \in [0, 1], \bar{x} \in X$ and $f \in G(X)$ the formula

$$x(t) = \Phi(t)\Phi^{-1}(t_0)\bar{x} + f(t) - f(t_0) - \Phi(t) \int_{t_0}^{t} d[\Phi^{-1}(s)](f(s) - f(t_0)), t \in [0, 1],$$

represents a solution of (2.7).

Proof. Using (2.8) we have for $t \in [0, 1]$

$$\int_{t_0}^{t} d[A(r)]x(r) = \int_{t_0}^{t} d[A(r)]\bigg\{\Phi(r)\Phi^{-1}(t_0)\bar{x} + f(r) - f(t_0) - \Phi(r) \int_{t_0}^{r} d[\Phi^{-1}(s)](f(s) - f(t_0)) \bigg\}$$

$$= \int_{t_0}^{t} d[A(r)]\Phi(r)\Phi^{-1}(t_0)\bar{x} + \int_{t_0}^{t} d[A(r)](f(r) - f(t_0))$$

$$- \int_{t_0}^{t} d[A(r)]\Phi(r) \int_{t_0}^{r} d[\Phi^{-1}(s)](f(s) - f(t_0)).$$

For a solution $\Phi$ of (1.5) we have

$$\int_{t_0}^{t} d[A(r)]\Phi(r) = \Phi(t) - \Phi(t_0)$$

and by Lemma 2.1 we have

$$\int_{t_0}^{t} d[A(r)]\Phi(r) \int_{t_0}^{r} d[\Phi^{-1}(s)](f(s) - f(t_0))$$

$$= \Phi(t) \int_{t_0}^{t} d[\Phi^{-1}(s)](f(s) - f(t_0)) + \int_{t_0}^{t} d[A(s)](f(s) - f(t_0)).$$

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Therefore
\[ \int_{t_0}^t d[A(r)]x(r) = [\Phi(t) - \Phi(t_0)]\Phi^{-1}(t_0)\bar{x} + \int_{t_0}^t d[A(r)](f(r) - f(t_0)) - \Phi(t)\int_{t_0}^t d[\Phi^{-1}(s)](f(s) - f(t_0)) - \int_{t_0}^t d[A(s)](f(s) - f(t_0)) = \Phi(t)\Phi^{-1}(t_0)\bar{x} - \bar{x} - \Phi(t_0)\int_{t_0}^t d[A(s)](f(s) - f(t_0)).\]

Hence
\[ \int_{t_0}^t d[A(r)]x(r) = x(t) - x_0 - (f(t) - f(t_0))\]
for every \( t \in [0, 1] \) and this means that the function \( x: [0, 1] \to X \) given by (2.8) is a solution of the equation (2.7). \( \square \)

Remark. From the point of view of the variation-of-constants formula (2.8) presented in Theorem 2.2 the assumption that the inverse \( \Phi^{-1}: [0, 1] \to L(X) \) to \( \Phi: [0, 1] \to L(X) \) given by Lemma 1.3 is such that \( \Phi^{-1} \in (B)BV(L(X)) \) is very unnatural. It would be nice if the property \( \Phi^{-1} \in (B)BV(L(X)) \) could be derived from the general assumptions, i.e. from the fact that \( A: [0, 1] \to L(X) \) satisfies (1.3), (E) and (U).

In the next section we will show that in the special situation of \( A \in BV(L(X)) \) the variation-of-constants formula (2.8) holds without any further assumption.

3. The variation-of-constants formula for the case \( A \in BV(L(X)) \)

Assume throughout this section that \( A \in BV(L(X)) \).

First of all it should be mentioned that by [9, 1.5] we have \( A \in G(L(X)) \) and therefore \( A: [0, 1] \to L(X) \) evidently satisfies (1.3) because, as was already mentioned in the introductory part of this note, we have \( BV(L(X)) \subset (B)BV(L(X)) \) by [9, Prop. 1.1 and 1.2].

As was mentioned in the last Remark in [9], if \( A \in BV(L(X)) \) then \( A \) satisfies also condition (E).

Let us now prove the following proposition.

3.1. Proposition. Assume that \( A: [0, 1] \to L(X) \).
Then $A \in BV(L(X))$ if and only if

\[
\text{(3.1)} \quad \sup_P \left\{ \sup_{C_j, D_j} \left\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \right\|_{L(X)} \right\} < \infty
\]

where $P: 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$ is a partition of $[0, 1]$, $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leq 1$, $\|D_j\|_{L(X)} \leq 1$, $j = 1, \ldots, k$, and

\[
\text{var}(A) = \sup_{[0,1]} \left\{ \sup_{C_j, D_j} \left\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \right\|_{L(X)} \right\}.
\]

**Proof.** Assume that

\[
P: 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1
\]

is an arbitrary partition of $[0, 1]$.

If $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leq 1$, $\|D_j\|_{L(X)} \leq 1$, $j = 1, \ldots, k$ then

\[
\left\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \right\|_{L(X)} \\
\leq \sum_{j=1}^{k} \|D_j\|_{L(X)} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} \|C_j\|_{L(X)} \\
\leq \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)}.
\]

Hence

\[
\sup_{C_j, D_j} \left\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \right\|_{L(X)} \leq \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)}
\]

where the supremum on the left hand side is taken over all $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leq 1$, $\|D_j\|_{L(X)} \leq 1$. Consequently,

\[
\text{(3.2)} \quad \sup_P \left\{ \sup_{C_j, D_j} \left\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \right\|_{L(X)} \right\} \\
\leq \sup_P \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} = \text{var}(A).
\]

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Assume that $\hat{D}_j \in L(X)$ with $\|\hat{D}_j\|_{L(X)} \leq 1$ and $x_j \in X$ with $\|x_j\|_X \leq 1$, $j = 1, \ldots, k$. Let us take $w \in X$ such that $\|w\|_X = 1$. Then for all $j = 1, \ldots, k$ there exist $\tilde{C}_j \in L(X)$ with $\|\tilde{C}_j\|_{L(X)} \leq 1$ such that $\tilde{C}_j w = x_j$. Hence

$$\| \sum_{j=1}^{k} \hat{D}_j[A(\alpha_j) - A(\alpha_{j-1})]x_j \|_X = \| \sum_{j=1}^{k} \hat{D}_j[A(\alpha_j) - A(\alpha_{j-1})] \tilde{C}_j w \|_X \leq \sup_{\|y\|_X \leq 1} \| \sum_{j=1}^{k} \hat{D}_j[A(\alpha_j) - A(\alpha_{j-1})] \tilde{C}_j y \|_X$$

$$= \| \sum_{j=1}^{k} \hat{D}_j[A(\alpha_j) - A(\alpha_{j-1})] \tilde{C}_j \|_{L(X)} \leq \sup_{C_j, \hat{D}_j} \| \sum_{j=1}^{k} D_j[A(\alpha_j) - A(\alpha_{j-1})] \tilde{C}_j \|_{L(X)}$$

where the supremum on the right hand side is taken over all $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leq 1, \|D_j\|_{L(X)} \leq 1$. Passing to the supremum over all $\hat{D}_j \in L(X)$ with $\|\hat{D}_j\|_{L(X)} \leq 1$ and $x_j \in X$ with $\|x_j\|_X \leq 1$, $j = 1, \ldots, k$ we get

$$\| \sum_{j=1}^{k} D_j[A(\alpha_j) - A(\alpha_{j-1})]x_j \|_X \leq \sup_{C_j, \hat{D}_j} \| \sum_{j=1}^{k} D_j[A(\alpha_j) - A(\alpha_{j-1})] C_j \|_{L(X)}.$$  \hfill (3.3)

Assume that $\varepsilon > 0$ is given. Choose vectors $x_j \in X$ with $\|x_j\|_X \leq 1$, $j = 1, \ldots, k$ such that

$$\| [A(\alpha_j) - A(\alpha_{j-1})] x_j \|_X > \| [A(\alpha_j) - A(\alpha_{j-1})] \|_{L(X)} - \frac{\varepsilon}{k}.$$  \hfill (3.4)

Let us set

$$v_j = \frac{[A(\alpha_j) - A(\alpha_{j-1})] x_j}{\| [A(\alpha_j) - A(\alpha_{j-1})] x_j \|_X} \text{ if } [A(\alpha_j) - A(\alpha_{j-1})] x_j \neq 0$$

and

$$v_j = 0 \text{ if } [A(\alpha_j) - A(\alpha_{j-1})] x_j = 0.$$

For $v_j \neq 0$ let $Y_j$ be the onedimensional subspace of $X$ given by

$$Y_j = \{ \lambda v_j; \lambda \in \mathbb{R} \}$$
and assume that $\tilde{f}_j$ is a bounded linear functional on $Y_j$ such that $\tilde{f}_j(v_j) = 1$ and denote by $f_j \in X^*$ its extension onto $X$ with $\|f_j\| = 1$.

Assume that $w \in X$ is fixed such that $\|w\|_X = 1$ and define the linear operator $D_j \in L(X)$ by the relation

$$D_j x = f_j(x)w, \ x \in X, \ j = 1, \ldots, k.$$  

Then certainly

$$\|D_j\|_{L(X)} = \|f_j\| \|w\| = 1$$

and

$$D_j[A(\alpha_j) - A(\alpha_{j-1})]x_j = \|A(\alpha_j) - A(\alpha_{j-1})\|_X D_j v_j$$

$$= \|A(\alpha_j) - A(\alpha_{j-1})\|_X f_j(v_j)w = \|A(\alpha_j) - A(\alpha_{j-1})\|_X w.$$  

Hence by (3.4) we get

$$\left\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]x_j \right\|_X = \left\| \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})\|_X w \right\|_X$$

$$= \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})\|_X w > \sum_{j=1}^k (\|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} - \epsilon)$$

$$= \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} - \epsilon.$$

Taking the supremum over all $D_j \in L(X)$ with $\|D_j\|_{L(X)} \leq 1$ and $x_j \in X$ with $\|x_j\|_X \leq 1, \ j = 1, \ldots, k$ we get

$$\sup_{x_j, D_j} \left\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]x_j \right\|_X > \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} - \epsilon$$

and using (3.3) we finally obtain

$$\sup_{C_j, D_j} \left\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]C_j \right\|_{L(X)} \geq \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} - \epsilon.$$

Taking the supremum over all partitions $P$ of $[0, 1]$ we obtain together with (3.2) for every $\epsilon > 0$ the inequality

$$\var(A) - \epsilon < \sup_{[0, 1]} \left\{ \sup_{C_j, D_j} \left\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]C_j \right\|_{L(X)} \right\} \leq \var(A)$$

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and therefore

$$\text{var}(A) = \sup_{[0,1]} \left\{ \sup_{C_j,D_j} \left\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \right\|_{L(X)} \right\}. $$

\[\square\]

Remark. It has to be mentioned that the characterization of the space $BV(L(X))$ given by Proposition 3.1 is interesting independently of the context of the equations studied in this paper.

3.2. Lemma. Assume that $A: [0,1] \to L(X)$ satisfies $A \in BV(L(X))$ and (U). Then for the solution $\Phi: [0,1] \to L(X)$ of (1.5) we have $\Phi \in BV(L(X))$.

Proof. Since $BV(L(X)) \subset (B^*) BV(L(X))$ the conclusion of Lemma 1.2 holds and there exists a $K > 0$ such that $\|\Phi(t)\| \leq K$ for every $t \in [0,1]$. It remains to show that the relation $\Phi \in BV(L(X))$ holds.

Assume that

$$ P: 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1 $$

is an arbitrary partition of the interval $[0,1]$ and that $C_j, D_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leq 1, \|D_j\|_{L(X)} \leq 1$ are given.

The fact that $\Phi \in G(L(X))$ yields by [6, Prop.15] the existence of the integral $\int_0^1 d[A(r)] \Phi(r)$ and therefore by definition for every $\varepsilon > 0$ there is a gauge $\delta: [0,1] \to (0, \infty)$ such that

$$ \left\| \sum_{i=1}^{l} [A(\beta_i) - A(\beta_{i-1})] \Phi(\sigma_i) - \int_{\alpha_{i-1}}^{\alpha_i} d[A(r)] \Phi(r) \right\|_{L(X)} \leq \frac{\varepsilon}{k+1} $$

for every $\delta$-fine $P$-partition

$$ \{\beta_0, \sigma_1, \beta_1, \ldots, \beta_{l-1}, \sigma_l, \beta_l\} $$

of the interval $[0,1]$.

By the Saks-Henstock Lemma (see [6, Lemma 16]) we have

$$ \left\| \sum_{i=1}^{l_j} [A(\beta_{i,j}) - A(\beta_{i,j-1})] \Phi(\sigma_{i,j}) - \int_{\alpha_{i,j-1}}^{\alpha_{i,j}} d[A(r)] \Phi(r) \right\|_{L(X)} \leq \frac{\varepsilon}{k+1} $$

for every $\delta$-fine $P$-partition

$$ \{\beta_{0,j}, \sigma_{1,j}, \beta_{1,j}, \ldots, \beta_{l_{j-1},j}, \sigma_{l_{j-1},j}, \beta_{l_{j-1},j}\} $$

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of the interval \([\alpha_{j-1}, \alpha_j]\), \(j = 1, \ldots, k\).

Further, we have
\[
\Phi(\alpha_j) - \Phi(\alpha_{j-1}) = \int_{\alpha_{j-1}}^{\alpha_j} d[A(r)] \Phi(r)
\]
for every \(j = 1, \ldots, k\) by the definition of a solution of (1.5) and therefore
\[
\left\| \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})]C_j \right\|_{L(X)} = \left\| \sum_{j=1}^{k} D_j \left[ \int_{\alpha_{j-1}}^{\alpha_j} d[A(r)] \Phi(r) \right] C_j \right\|_{L(X)}
\]
\[
= \left\| \sum_{j=1}^{k} \left\{ D_j \left[ \int_{\alpha_{j-1}}^{\alpha_j} d[A(r)] \Phi(r) - \sum_{i=1}^{l_j} [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) \right] C_j \right\}
\]
\[
+ \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) C_j \right\|_{L(X)}
\]
\[
\leq \sum_{j=1}^{k} \left\| \int_{\alpha_{j-1}}^{\alpha_j} d[A(r)] \Phi(r) - \sum_{i=1}^{l_j} [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) \right\|_{L(X)}
\]
\[
+ \left\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) C_j \right\|_{L(X)}
\]
\[
\leq \sum_{j=1}^{k} \left\| \int_{\alpha_{j-1}}^{\alpha_j} d[A(r)] \Phi(r) - \sum_{i=1}^{l_j} [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) \right\|_{L(X)}
\]
\[
+ \left\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) C_j \right\|_{L(X)}
\]
provided
\[
\{\beta^j_0, \sigma^j_1, \beta^j_1, \ldots, \beta^j_{l_j-1}, \sigma^j_{l_j}, \beta^j_{l_j}\}
\]
is a \(\delta\)-fine \(P\)-partition of the interval \([\alpha_{j-1}, \alpha_j]\), \(j = 1, \ldots, k\). Hence using (3.5) we obtain by the last inequalities
\[
\left\| \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})]C_j \right\|_{L(X)}
\]
\[
\leq \sum_{j=1}^{k} \frac{\varepsilon}{k+1} + \left\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) C_j \right\|_{L(X)}
\]
\[
< \varepsilon + \left\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) C_j \right\|_{L(X)}.
\]

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For the second term on the right hand side we have

\[
\left\lVert \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta^j_i) - A(\beta^j_{i-1})] \Phi(\sigma^j_i) C_j \right\rVert_{L(X)}
\]

\[
\leq \sum_{j=1}^{k} \sum_{i=1}^{l_j} \|D_j\|_{L(X)} \|A(\beta^j_i) - A(\beta^j_{i-1})\|_{L(X)} \|\Phi(\sigma^j_i)\|_{L(X)} \|C_j\|_{L(X)}
\]

\[
\leq K \cdot \sum_{j=1}^{k} \sum_{i=1}^{l_j} \|A(\beta^j_i) - A(\beta^j_{i-1})\|_{L(X)} \leq K \cdot \text{var}(A).
\]

Hence

\[
\left\lVert \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})] C_j \right\rVert_{L(X)} < \varepsilon + K \cdot \text{var}(A)
\]

and since \( \varepsilon > 0 \) can be taken arbitrarily small, we get

\[
\left\lVert \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})] C_j \right\rVert_{L(X)} \leq K \cdot \text{var}(A)
\]

for any partition

\( P: 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1 \)

of the interval \([0, 1]\) and any choice of \( C_j, D_j \in L(X), j = 1, \ldots, k \) with \( \|C_j\|_{L(X)} \leq 1, \|D_j\|_{L(X)} \leq 1 \).

Passing to the suprema over all \( C_j, D_j \in L(X), j = 1, \ldots, k \) with \( \|C_j\|_{L(X)} \leq 1, \|D_j\|_{L(X)} \leq 1 \) and all partitions \( P \) of \([0, 1]\) we obtain

\[
\sup_{P} \sup_{C_j, D_j} \left\lVert \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})] C_j \right\rVert_{L(X)} \leq K \cdot \text{var}(A)
\]

and this together with Proposition 3.1 yields the result.

\[ \square \]

3.3. Lemma. Assume that \( A: [0, 1] \to L(X) \) satisfies \( A \in BV(L(X)) \) and \( (U) \).

Then the inverse \( [\Phi(t)]^{-1} = \Phi^{-1}(t) \) to the solution \( \Phi: [0, 1] \to L(X) \) of (1.5) exists for every \( t \in [0, 1] \) and we have \( \Phi^{-1} \in BV(L(X)) \).

Proof. By the results given in Lemma 1.3 and 1.4 the inverse \( \Phi^{-1} \) exists and \( \Phi^{-1} \in G(L(X)) \). Hence there is a constant \( L > 0 \) such that

\[
\|\Phi^{-1}(t)\|_{L(X)} \leq L
\]

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for every \( t \in [0,1] \).

It remains to show that \( \Phi^{-1} \in BV(L(X)) \).

Assume that

\[
P : 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1
\]

is an arbitrary partition of the interval \([0,1]\) and that \( C_j, D_j \in L(X), j = 1, \ldots, k \) with \( \|C_j\|_{L(X)} \leq 1 \), \( \|D_j\|_{L(X)} \leq 1 \) are given.

We have

\[
\begin{align*}
\left\| \sum_{j=1}^{k} D_j [\Phi^{-1}(\alpha_j) - \Phi^{-1}(\alpha_{j-1})] C_j \right\| &= \left\| \sum_{j=1}^{k} D_j [I - \Phi(\alpha_j)\Phi^{-1}(\alpha_{j-1})] C_j \right\| \\
&= \left\| \sum_{j=1}^{k} D_j \Phi^{-1}(\alpha_j) [\Phi(\alpha_{j-1}) - \Phi(\alpha_j)] C_j \right\| \\
&\leq L^2 \cdot \text{var}(\Phi) \leq L^2 \cdot K \cdot \text{var}(A).
\end{align*}
\]

Passing to the suprema over all \( C_j, D_j \in L(X), j = 1, \ldots, k \) with \( \|C_j\|_{L(X)} \leq 1 \), \( \|D_j\|_{L(X)} \leq 1 \) and all partitions \( P \) of \([0,1]\) we obtain

\[
\sup_{P} \sup_{C_j, D_j} \left\| \sum_{j=1}^{k} D_j [\Phi^{-1}(\alpha_j) - \Phi^{-1}(\alpha_{j-1})] C_j \right\|_{L(X)} \leq L^2 \cdot K \cdot \text{var}(A).
\]

and this together with Proposition 3.1 yields \( \Phi^{-1} \in BV(L(X)) \).

\[\square\]

3.4. Theorem. Assume that \( A : [0,1] \to L(X) \) satisfies \( A \in BV(L(X)) \) and

\((U)\). Let \( \Phi : [0,1] \to L(X) \) be the solution of (1.5).

Then for every \( t_0 \in [0,1], \bar{x} \in X \) and \( f \in G(X) \) the formula

\[
(2.8) \quad x(t) = \Phi(t)\Phi^{-1}(t_0)\bar{x} + f(t) - f(t_0) - \Phi(t) \int_{t_0}^{t} d[\Phi^{-1}(s)](f(s) - f(t_0)),
\]

\( t \in [0,1], \) represents a solution of (2.7).

Proof. By Lemma 3.3 the inverse \( \Phi^{-1} : [0,1] \to L(X) \) given by Lemma 1.3 belongs to \( BV(L(X)) \) and therefore we have also \( \Phi^{-1} \in (B)BV(L(X)) \). All the assumptions of Theorem 2.2 being satisfied we obtain the result by this theorem. \(\square\)
3.5 Example. Let us consider the abstract linear differential equation

\[
\frac{dx}{dt} = a(t)x + \varphi(t)
\]
on \[0,1\] where \(a: [0, 1] \rightarrow L(X), \varphi: [0, 1] \rightarrow X\) and both \(a\) and \(\varphi\) are Bochner integrable. For equations of this kind see e.g. [1].

A solution of (3.6) is understood to be a solution of the integral equation

\[
x(t) = x_0 + \int_a^t a(s)x(s)\, ds + \int_a^t \varphi(s)\, ds
\]
where \(d \in [0, 1]\) and \(x_0 = x(d)\).

More generally we can consider the integral equation of the form

\[
x(t) = \int_a^t a(s)x(s)\, ds + g(t)
\]
with \(g \in G(X)\).

Let us set

\[
A(t) = \int_a^t a(s)\, ds \quad \text{and} \quad f(t) = \int_a^t \varphi(s)\, ds, \quad t \in [0, 1].
\]
Assume that \(D: 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1\) is an arbitrary partition of \([0, 1]\). Then using the properties of the Bochner integral we get

\[
\sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\| = \sum_{j=1}^{k} \left\| \int_{\alpha_{j-1}}^{\alpha_j} a(s)\, ds \right\|
\leq \sum_{j=1}^{k} \int_{\alpha_{j-1}}^{\alpha_j} \|a(s)\|\, ds = \int_0^1 \|a(s)\|\, ds < \infty
\]
and therefore \(A \in BV(L(X))\). Since the function \(\|a\|\) is Lebesgue integrable over \([0, 1]\) we have

\[
\|A(t) - A(r)\| \leq \left| \int_r^t \|a(s)\|\, ds \right|
\]
for \(t, r \in [0, 1]\) and this yields the continuity of \(A\) on \([0, 1]\). Hence \(\lim_{t \rightarrow r^+} A(t) = A(r)\) for \(r \in [0, 1]\) and \(\lim_{t \rightarrow r^-} A(t) = A(r)\) for \(r \in (0, 1]\) and consequently we have \(\Delta^+ A(r) = 0\) for \(r \in [0, 1]\) and \(\Delta^- A(r) = 0\) for \(r \in (0, 1]\) and the function \(A: [0, 1] \rightarrow L(X)\) satisfies the condition (U) given in Theorem 1.1. Similarly the function \(f: [0, 1] \rightarrow X\) is also continuous and belongs trivially to \(G(X)\).
It is a matter of routine to show that if \( x \in G(X) \) then the integrals \( \int_0^1 d[A(s)]x(s) \) and \( \int_0^1 a(s)x(s) \, ds \) both exist and
\[
\int_0^1 d[A(s)]x(s) = \int_0^1 a(s)x(s) \, ds.
\]

Since \( g \) is assumed to belong to \( G(X) \), every solution of (3.8) also belongs to \( G(X) \) and therefore the equation (3.8) is equivalent to
\[
x(t) = \int_t^d d[A(s)]x(s) + g(t) = g(d) + \int_t^d d[A(s)]x(s) + g(t) - g(d).
\]

Hence by Theorem 2.10 in [9] there exists a unique solution \( x: [0, 1] \to X, x \in G(X) \) of (3.8) and by Theorem 3.4 we get after a straightforward calculation
\[
x(t) = \Phi(t)\Phi^{-1}(t_0)g(d) + g(t) - g(d) - \Phi(t)\int_t^d d[\Phi^{-1}(s)](g(s) - g(d))
\]
\[
= g(t) - \Phi(t)\int_t^d d[\Phi^{-1}(s)]g(s)
\]
where the function \( \Phi: [0, 1] \to L(X) \) is a solution of (1.5) with \( A \) given by \( A(t) = \int_t^1 a(s) \, ds \) for \( t \in [0, 1] \).

References

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