Abstract. This paper deals with the relations between graph automorphisms and direct factors of a semimodular lattice of locally finite length.

Keywords: semimodular lattice, graph automorphism, direct factor

MSC 2000: 06C10

1. Introduction

Each lattice dealt with in the present paper is assumed to be of locally finite length (i.e., all its bounded chains are finite).

For a lattice $L$ let $G(L)$ be the corresponding unoriented graph.

An automorphism of the graph $G(L)$ is called also a graph automorphism of the lattice $L$. The graph isomorphism of lattices is defined analogously.

We denote by $C$ the class of all finite lattices $L$ such that each automorphism of $G(L)$ turns out to be a lattice automorphism.

In connection with Birkhoff’s problem 6 from [1], the following result has been proved in [5] (by using the results of [2] and [6]):

(∗) Let $L$ be a finite modular lattice. Then the following conditions are equivalent:

(i) $L$ belongs to $C$.

(ii) No direct factor of $L$ having more than one element is self-dual.

The natural question arises whether in (∗) the assumption of modularity can be replaced by the assumption that $L$ is semimodular.

In Section 3 we show by an example that the answer is “No”.

We define the notions of an interval of type (C) in $L$ and of a graph automorphism of type (C) (cf. Definitions 2.1 and 2.2).
Let $A$ be a direct factor of a lattice $L$ and $\emptyset \neq X \subseteq L$. We say that $A$ is orthogonal to $X$ if for any $x_1, x_2 \in X$, the components of $x_1$ and $x_2$ in the direct factor $A$ are equal.

Let $C_1$ be the class of all lattices $L$ such that each graph automorphism of type (C) of $L$ is a lattice automorphism.

We prove (by applying the results and the methods of [3], [5] and [6]):

(*$_1$) Let $L$ be a semimodular lattice. Then the following conditions are equivalent:

(i) $L$ belongs to $C_1$.
(ii) If $A$ is a direct factor of $L$ such that $A$ is self-dual and orthogonal to each interval of type (C) in $L$, then $A$ is trivial (i.e., $\text{card } A = 1$).

2. Preliminaries

In what follows, $L$ is a lattice. For the notion of the unoriented graph $G(L)$ of $L$ cf., e.g. [1], [2].

If $x, y \in L$, $x < y$ and if the interval $[x, y]$ of $L$ is a two-element set, then we write $x \prec y$ or $y \succ x$.

Hence a graph automorphism of $L$ is a one-to-one mapping $\varphi$ of $L$ onto $L$ such that, whenever $x, y \in L$ and $x \prec y$, then

(i) either $\varphi(x) \prec \varphi(y)$ or $\varphi(y) \prec \varphi(x)$,
(ii) either $\varphi^{-1}(x) \prec \varphi^{-1}(y)$ or $\varphi^{-1}(y) \prec \varphi^{-1}(x)$.

2.1. Definition. Let $L_0$ be a sublattice of $L$ such that $L_0$ is isomorphic to the lattice in Fig. 1; then the convex closure $\overline{L_0}$ of $L_0$ in $L$ is said to be an interval of type (C) in $L$.

Fig. 1

2.2. Definition. A graph automorphism $\varphi$ of $L$ is said to be of type (C) if, whenever $L_1$ is an interval of type (C) in $L$ and $x, y \in L_1$, $x \prec y$, then $\varphi(x) \prec \varphi(y)$ and $\varphi^{-1}(x) \prec \varphi^{-1}(y)$. 460
It is easy to verify that if $L$ is modular, then it has no sublattice of type (C); consequently, in this case each graph automorphism of $L$ is of type (C). Therefore (*1) is a corollary of (*1).

We denote by $L^\sim$ the lattice dual to $L$. If $L$ and $L^\sim$ are isomorphic, then $L$ is said to be self-dual.

3. An example

Let us recall that if $L$ can be expressed as a direct product $L_1 \times L_2$ and if $x = (x_1, x_2) \in L$, $y = (y_1, y_2) \in L$, then $x \prec y$ if and only if either $x_1 \prec y_1$ and $x_2 = y_2$, or $x_1 = x_2$ and $y_1 \prec y_2$.

From this we immediately obtain

3.1. Lemma. Let $L_1, L_2$ be lattices and let $\varphi$ be a graph isomorphism of $L_1$ onto $L_2$. Put $L = L_1 \times L_2$. For each $x = (x_1, x_2) \in L$ we set

$$\varphi(x) = (\varphi^{-1}(x_2), \varphi(x_1)).$$

Then $\psi$ is a graph automorphism of $L$.

Consider the lattices $L_1$ and $L_2$ in Fig. 2 or Fig. 3, respectively. Both $L_1$ and $L_2$ are semimodular.

![Fig. 2](image)

![Fig. 3](image)

3.2. Lemma. Both $L_1$ and $L_2$ are directly indecomposable.

Proof. The assertion for $L_1$ was proved in [5], pp. 164–165. The proof for $L_2$ is similar. □

3.3. Lemma. Let $i \in \{1, 2\}$. Then the lattice $L_i^\sim$ fails to be self-dual.

Proof. It is easy to verify that $L_i^\sim$ fails to be semimodular. Therefore $L_i^\sim$ is not isomorphic to $L_i$. □
Put \( L = L_1 \times L_2 \).

Since any two direct product decompositions of \( L \) have a common refinement and since \( L_1, L_2 \) are directly indecomposable by 3.3, we conclude

3.4. Lemma. Let \( A \) be a direct factor of \( L \) having more than one element. Then the lattice \( A \) is isomorphic to some of the lattices \( L, L_1, L_2 \).

By the same argument as in 3.3 we obtain

3.5. Lemma. The lattice \( L \) is not self-dual.

Now, 3.3, 3.4 and 3.5 yield

3.6. Corollary. The lattice \( L \) satisfies the condition \((ii)\) from (*)

It is easy to verify that there exists a graph isomorphism \( \varphi \) of \( L_1 \) onto \( L_2 \) such that \( \varphi \) fails to be a lattice isomorphism. Hence there are \( x_1, y_1 \) in \( L_1 \) such that \( x_1 \prec y_1 \) and \( \varphi(x_1) \succ \varphi(y_1) \). Consequently, if \( \psi \) is defined as above, then \( \psi \) is not a lattice automorphism of \( L \).

In view of 3.1 we conclude that in (*)&, the assumption of modularity cannot be replaced by the assumption of semimodularity of the lattice \( L \).

We also remark that \( \psi \) is an example of a graph automorphism on a semimodular lattice such that \( \psi \) is not of type (C).

4. Proof of (*)&

In this section we assume that the lattice \( L \) is semimodular.

4.1. Lemma. Suppose that \( B \) is a direct factor of \( L \) such that

(i) \( B \) is self-dual;
(ii) \( B \) is orthogonal to each interval of type (C) in \( L \);
(iii) \( \text{card } B > 1 \).

Then \( L \) does not belong to \( \mathcal{C}_1 \).

Proof. There is a lattice \( A \) such that there exists an isomorphism \( \psi \) of \( L \) onto \( A \times B \). Further, in view of (i), there is an isomorphism \( \chi \) of the lattice \( B \) onto \( B^\sim \).

For each \( x \in L \) we put \( \varphi(x) = y \), where

\[
\psi(x) = (a, b), \quad y = \psi^{-1}((a, \chi(b))).
\]

Then \( \varphi \) is a graph automorphism of the lattice \( L \) (cf. [5], Lemma 1.1). Moreover, (ii) yields that \( \varphi \) is of type (C). By applying Lemma 1.2 of [5] we conclude that \( \varphi \) fails to be a lattice automorphism. Therefore \( L \) does not belong to \( \mathcal{C}_1 \). \( \square \)
Let \( L_1 \) and \( L_2 \) be semimodular lattices. Suppose that \( \varphi \) is a graph isomorphism of \( L_1 \) onto \( L_2 \) such that

(a) if \( X \) is an interval of type (C) in \( L_1 \) and \( x_1, x_2 \in X, \ x_1 \prec x_2 \), then \( \varphi(x_1) \prec \varphi(x_2) \);

(b) if \( Y \) is an interval of type (C) in \( L_2 \) and \( y_1, y_2 \in Y, \ y_1 \prec y_2 \), then \( \varphi^{-1}(y_1) \prec \varphi^{-1}(y_2) \).

We apply similar steps as in Section 2 of [5]. For the sake of completeness, we recall the corresponding notation.

Let \( A_1 \) be the set of all intervals \([x, y]\) of \( L_1 \) such that \( x \prec y \) and \( \varphi(x) \prec \varphi(y) \).

Further, let \( B_1 \) be the set of all intervals \([u, v]\) of \( L_1 \) such that \( u \prec v \) and \( \varphi(u) \succ \varphi(v) \).

Similarly we define the sets \( A_2 \) and \( B_2 \) of intervals of \( L_2 \) (with \( \varphi^{-1} \) instead of \( \varphi \)).

Choose \( x_1^0 \in L_1, \ x_2^0 \in L_2 \). We denote by \( A_1^0 \) the set of all elements \( x \in L_1 \) such that either \( x = x_1^0 \), or there exist \( y_1, y_2, \ldots, y_n \in L_1 \) such that

(i) \( y_1 = x_1^0, \ y_n = x \),

(ii) for each \( i \in \{1, 2, \ldots, n - 1\} \), the elements \( y_i, \ y_{i+1} \) are comparable and the corresponding interval belongs to \( A_1 \).

Similarly we define the set \( B_1^0 \) (taking \( B_1 \) instead of \( A_1 \)). The subsets \( A_2^0 \) and \( B_2^0 \) are defined analogously (taking \( x_2^0 \) and \( \varphi^{-1} \) instead of \( x_1^0 \) and \( \varphi \)).

We apply the notion of the internal direct product decomposition of a lattice \( L \) with the central element \( x^0 \) in the same sense as in [5] (cf. also [6]). By using this notion and by applying the assumption given above we conclude that the results of [3] (cf. Theorem 2 in [3] and the lemmas applied for proving this Theorem) yield

**4.2. Proposition.** Under the assumptions as above, there exist internal direct product decompositions

\[
\psi_1 : \ L_1 \to A_1^0 \times B_1^0 \quad \text{(with the central element } x_1^0) \]

\[
\psi_2 : \ L_2 \to A_2^0 \times B_2^0 \quad \text{(with the central element } x_2^0) \]

such that

(i) the lattices \( A_1^0 \) and \( A_2^0 \) are isomorphic,

(ii) the lattice \( B_1^0 \) is isomorphic to \((B_2^0)^\sim\).

Now suppose that the lattice \( L \) satisfies the condition (ii) of (\( \ast_1 \)). Let \( \varphi \) be a graph automorphism of type (C) of the lattice \( L \).
Choose \( x^0 \in L \). We put \( L = L_1 = L_2 \) and \( x^0 = x^0_1 = x^0_2 \). The fact that \( \varphi \) is of type (C) yields that the conditions (a) and (b) are satisfied. Hence we can apply Proposition 4.2.

The further steps are the same as in Part 3 of [5]. By using them we obtain

**4.3. Lemma.** Let \( L \) be a semimodular lattice satisfying the condition (ii) of \((\ast_1)\). Then the condition (i) of \((\ast_1)\) is valid.

In view of 4.1 and 4.3, we infer that \((\ast_1)\) holds.

If \( L_1 \) is a sublattice of \( L \) and \( a, b \in L_1, a < b \), then we denote by \([a, b]_1\) the corresponding interval of \( L_1 \). We put \( a \prec_1 b \) if \([a, b]_1\) is a two-element set.

We say that \( L_1 \) is a \( c \)-sublattice of \( L \) if, whenever \( a, b \in L_1 \) and \( a \prec_1 b \), then \( a < b \).

We remark that Theorem 2 in the paper [7] by Ratanaprasert and Davey (this theorem solved a problem proposed in [4]) implies that in Definition 2.1 above it suffices to consider only those sublattices \( L_0 \) of \( L \) which are \( c \)-sublattices of \( L \).

**References**


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