ON GENERAL SOLVABILITY PROPERTIES OF 
p-LAPALACIAN-LIKE EQUATIONS

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Abstract. We discuss how the choice of the functional setting and the definition of the weak solution affect the existence and uniqueness of the solution to the equation

\[-\Delta_p u = f \text{ in } \Omega,\]

where \(\Omega\) is a very general domain in \(\mathbb{R}^N\), including the case \(\Omega = \mathbb{R}^N\).

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1. Introduction

The object of our study is the second order quasilinear elliptic differential operator \(\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)\), where \(p > 1\) is a real number. Note that we define \(\Delta_p u = 0\) for \(\nabla u = 0\) and \(1 < p < 2\). We concentrate on the following basic question: “How the choice of an appropriate function space affects the existence and uniqueness of the weak solution to the equation

\[(1.1) \quad -\Delta_p u = f \text{ in } \Omega,\]

where \(\Omega \subset \mathbb{R}^N\)?” Let us point out that \(\Omega\) is considered to be a bounded, an (unbounded) exterior domain or, possibly, \(\Omega = \mathbb{R}^N\). The choice of an appropriate

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function space and the relation between $p$ and the dimension $N$ then play the essential role in the questions of existence, nonexistence or uniqueness of the weak solution to Eq. (1.1). While for $\Omega$ a bounded domain the situation seems to be more or less clear and often treated in literature, for $\Omega = \mathbb{R}^N$ or $\Omega$ an exterior domain in $\mathbb{R}^N$ we can observe some phenomena which may seem to be surprising without deeper insight of the problem and a careful definition of the notion of a weak solution (cf. [7]). We start our exposition with very general existence and uniqueness results in abstract Banach spaces. Then we consider the typical situations: $\Omega$ a bounded domain, an (unbounded) exterior domain and the whole of $\mathbb{R}^N$, and point out some differences between these cases. Let us remind the reader that problems of this type were treated e.g. in [1], [2], [3] or [5].

2. Some general existence and uniqueness results

Let $\Omega \subset \mathbb{R}^N$ be a domain and let $L^{1,p}(\Omega) := \{ u \in L^1_{\text{loc}}(\Omega); \nabla u \in [L^p(\Omega)]^N \}$. Here $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$, where $\partial_i u := \partial u/\partial x_i$ ($i = 1, \ldots, N$) is the weak (distributional) derivative of $u$.

Let $X$ be a linear function space with the following properties:

(X1) $X \subset L^{1,p}(\Omega)$.

(X2) By $\|u\|_X := \|\nabla u\|_{p;\Omega}$ for $u \in X$ a norm is defined on $X$ so that $X$ equipped with this norm is a reflexive Banach space where $\| \cdot \|_{p;\Omega}$ is the usual $L^p$-norm of $|\nabla u| := \left( \sum_{i=1}^{N} |\partial_i u|^2 \right)^{1/2}$.

Let us denote by $X^*$ the dual space, by $\| \cdot \|_{X^*}$ the norm on $X^*$ and by $\langle \cdot, \cdot \rangle_X$ the duality pairing between $X^*$ and $X$. We define the operator $J: X \to X^*$ by

$$\langle J(u), v \rangle_X = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v$$

for any $u, v \in X$. Then the operator $J$ has the following properties:

(J1) $\langle J(u), u \rangle_X = \|u\|_X^p$ for any $u \in X$;

(J2) $\langle J(u) - J(v), u - v \rangle_X > 0$ for any $u, v \in X, u \neq v$;

(J3) $J$ and $J^{-1}$ are continuous operators.

Indeed, the properties (J1) and (J2) as well as the continuity of $J$ are obvious. It then follows from the theory of monotone operators (see e.g. [4]) that $J$ is surjective.
To prove the continuity of $J^{-1}$ we use the inequality

\begin{equation}
\langle J(u) - J(v), u - v \rangle_X \geq (\|u\|_X^{p-1} - \|v\|_X^{p-1})(\|u\|_X - \|v\|_X)
\end{equation}

which is an immediate consequence of the Hölder inequality. Let us suppose that $J^{-1}: X^* \to X$ is not continuous. Then there exists a sequence $(f_n) \subset X^*$, $f_n \to f$, i.e. strongly, in $X^*$ and

\[ \|J^{-1}(f_n) - J^{-1}(f)\|_X \geq \delta \]

for some $\delta > 0$. Denote $u_n = J^{-1}(f_n)$, $u = J^{-1}(f)$. It follows from (J1) that

\[ \|f_n\|_{\mathcal{X}}\|u_n\|_X \geq \langle f_n, u_n \rangle_X = \langle J(u_n), u_n \rangle_X = \|u_n\|_{\mathcal{X}}, \]

i.e. $(u_n) \subset X$ is a bounded sequence. Due to (X2) we can assume (after passing to a subsequence, if necessary) that there exists $\tilde{u} \in X$ such that $u_n \rightharpoonup \tilde{u}$, i.e. weakly, in $X$. Hence we have

\begin{equation}
\langle J(u_n) - J(\tilde{u}), u_n - \tilde{u} \rangle_X = \langle J(u_n) - J(u), u_n - \tilde{u} \rangle_X + \langle J(u) - J(\tilde{u}), u_n - \tilde{u} \rangle_X \to 0
\end{equation}

since $J(u_n) = f_n \to f = J(u)$ in $X^*$. If we set $u = u_n$ and $v = \tilde{u}$ in (2.1) then (2.2) implies $\|u_n\|_X \to \|\tilde{u}\|_X$. Then (X2) yields $u_n \to \tilde{u}$ in $X$ and so by (J2) we get $u = \tilde{u}$, a contradiction. Actually, we have proved

**Theorem 2.1.** The operator $J$ is a homeomorphism between $X$ and $X^*$. In particular, given $f \in X^*$, the equation $J(u) = f$ has a unique solution $u_f \in X$ and $\|u_f\|_X \leq \|f\|_{X^*}^{1/(p-1)}$.

Note that the equation $J(u) = f$ can be interpreted also as an Euler equation of the functional

\[ \Phi_f(u) = \frac{1}{p}\|u\|_X^p - \langle f, u \rangle_X, \quad u \in X, \]

and its solution as a minimizer of $\Phi_f$. Indeed, it is easy to verify that $\Phi_f: X \to \mathbb{R}$ is a coercive, strictly convex and weakly lower semicontinuous functional. So for arbitrary $f \in X^*$, there exists a unique minimizer $u_f \in X$ of $\Phi_f$ which is also its unique critical point.
3. THE CASE OF A BOUNDED DOMAIN

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the Dirichlet problem

\[
\begin{aligned}
-\Delta_p u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Define $X := C^\infty_0(\Omega) = W^{1,p}_0(\Omega)$ and let $f \in X^*$. It is well known that the space $X$ equipped with the norm $\|\nabla \cdot \|_{p,\Omega}$ satisfies (X1) and (X2). We then define a weak solution of (3.1) as a function $u \in X$ for which the identity

\[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \langle f, v \rangle_X
\]

holds for every $v \in X$. It follows from Theorem 2.1 that (3.2) is uniquely solvable for any $f \in X^*$.

In what follows, for $1 < p < N$ we set

\[
p^* = \frac{Np}{N-p} \quad \text{(the critical Sobolev exponent)}, \quad p^{*'} = \frac{Np}{p - 1} = \frac{Np}{Np - N + p}.
\]

In the case $p > N$ we set $p^* = \infty$, $p^{*'} = 1$, and finally for $p = N$ we put $p^* = q$, $p^{*'} = \frac{q}{q-1}$, where $q \in (1, \infty)$ is an arbitrarily chosen number. It follows from the Sobolev imbedding theorem that any $f \in L^{p''}(\Omega)$ can be identified with an $f \in X^*$ and $\langle f, v \rangle_X = \int_\Omega fv$ for any $v \in X$. The above considerations immediately imply

**Theorem 3.1.** Let $f \in L^{p''}(\Omega)$. Then the Dirichlet problem (3.1) has a unique weak solution $u_f \in X$, i.e.

\[
\int_\Omega |\nabla u_f|^{p-2} \nabla u_f \cdot \nabla v = \int_\Omega fv
\]

for any $v \in X$ (or equivalently for any $v \in C^\infty_0(\Omega)$).

For the Neumann problem

\[
\begin{aligned}
-\Delta_p u &= f \quad \text{in } \Omega, \\
|\nabla u|^{p-2} \partial u / \partial \nu &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(here $\partial u / \partial \nu$ denotes the derivative with respect to the exterior normal) the situation is different. A weak solution of (3.3) is usually defined by the same integral identity as (3.2) but now with the test space $X$ replaced by $\bar{X} := W^{1,p}(\Omega)$, where $W^{1,p}(\Omega) =$
\{ u \in L^p(\Omega); \nabla u \in [L^p(\Omega)]^N \}. Since \( \| \nabla \cdot \|_{p;\Omega} \) is only a seminorm on \( \tilde{X} \), we cannot apply Theorem 2.1 as in the case of the Dirichlet problem. Roughly speaking, we have to rule out the constants from \( \tilde{X} \). One possibility is to restrict ourselves (since \( \Omega \) is bounded) to the subspace \( X := \{ u \in \tilde{X}; \int_{\Omega} u = 0 \} \).

Now, \( \| \nabla \cdot \|_{p;\Omega} \) defines a norm on \( X \) but additional information about \( \Omega \) is needed in order to guarantee that \( (X, \| \nabla \cdot \|_{p;\Omega}) \) is complete. It is proved in [11] that this is the case if and only if the Poincaré inequality

\begin{equation}
\| u \|_{p;\Omega} \leq c \| \nabla u \|_{p;\Omega} \quad \forall u \in X
\end{equation}

holds. One of the sufficient conditions for (3.4) to hold is \( \partial \Omega \in C^0 \) (i.e. for any \( x_0 \in \partial \Omega \) there is a neighbourhood \( U(x_0) \subset \mathbb{R}^N \) such that \( U(x_0) \cap \partial \Omega \) is a \( C^0 \) manifold in \( \mathbb{R}^N \)—see [11]). So, assuming \( \partial \Omega \in C^0 \), we verify (X1), (X2), and for any \( f \in X^* \) there exists a unique \( u_f \in X \) satisfying (3.2) with this choice of \( X \).

In order to apply Sobolev’s imbedding theorems for \( X \) we need now \( \partial \Omega \in C^{0,1} \) (the boundary is locally Lipschitzian—this property is defined analogously as \( \partial \Omega \in C^0 \)). Remark also that the norm \( \| \nabla \cdot \|_{p;\Omega} \) on \( X \) is equivalent to the usual Sobolev norm \( \| \cdot \|_{W^{1,p}(\Omega)} \) in this case. If this is the case, any \( f \in L^{p'}(\Omega) \) defines \( f \in X^* \) satisfying \( (f, v)_X = \int_{\Omega} fv \) for any \( v \in X \). But now any constant function on \( \Omega \) is identified with the zero element of \( X(X^*) \) and by the same argument any \( u \in X \) (\( f \in L^{p'}(\Omega) \)) is identified with \( \tilde{u} = u - \int_{\Omega} u \) (\( \tilde{f} = f - \int_{\Omega} f \)). Thus we have

**Theorem 3.2.** Let \( \partial \Omega \in C^{0,1} \), \( f \in L^{p'}(\Omega) \). Then the Neumann problem (3.3) has a unique family of weak solutions \( u_{f,c} = u_f + c \), \( c \in \mathbb{R} \), where \( \int_{\Omega} u_f = 0 \) (i.e.

\[ \int_{\Omega} |\nabla u_{f,c}|^{p-2} \nabla u_{f,c} \cdot \nabla v = \int_{\Omega} fv \]

for any \( v \in W^{1,p}(\Omega) \)) if and only if

\[ \int_{\Omega} f = 0. \]
4. The case $\Omega = \mathbb{R}^N$

In this section we discuss the existence of a weak solution of the equation

\begin{equation}
- \Delta_p u = f \text{ in } \mathbb{R}^N.
\end{equation}

For $1 < p < N$ set $\mathfrak{H}^1_0(\mathbb{R}^N) := \{ u \in L^1(\mathbb{R}^N); u \in L^{p'}(\mathbb{R}^N) \}$ where $p^* := \frac{Np}{N-p}$. Let us recall some facts from [3], [9], [11] and [12]. In the sense of a direct decomposition we have

\begin{equation}
\left\{ \begin{array}{l}
\mathbb{R}^N = H^1_0(\mathbb{R}^N) \oplus \mathbb{R}, \\
u = (u - c_u) + c_u, \\
\text{where } (u - c_u) \in \mathfrak{H}^1_0(\mathbb{R}^N) \text{ and } \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} u, \text{ where } B_R := \{ x \in \mathbb{R}^N; |x| < R \}.
\end{array} \right.
\end{equation}

Here, $|B_R|$ denotes the Lebesgue measure of $B_R$. Moreover, we have

\begin{equation}
\mathfrak{H}^1_0(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)} ||\nabla||_{p;\mathbb{R}^N}
\end{equation}

by the Sobolev imbedding, and $||\nabla||_{p;\mathbb{R}^N}$ is a norm on $X := \mathfrak{H}^1_0(\mathbb{R}^N)$ so that $X$ is complete. Thus (X1) and (X2) are verified and we can apply Theorem 2.1. In particular, we have

**Theorem 4.1.** Let $f \in L^{p^*}(\mathbb{R}^N)$. Then there is a unique $u_f \in X$ such that the integral identity

\begin{equation}
\int_{\mathbb{R}^N} |\nabla u_f|^{p-2} \nabla u_f \cdot \nabla v = \int_{\mathbb{R}^N} fv
\end{equation}

holds for any $v \in X$ (or equivalently for any $v \in C_0^\infty(\mathbb{R}^N)$).

Let us now consider the case $p \geq N \geq 2$. As is shown in [7], if $f \in L^1(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} f \neq 0$ then there is no $u \in L^1(\mathbb{R}^N)$ satisfying

\begin{equation}
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_{\mathbb{R}^N} fv
\end{equation}

for arbitrary $v \in C_0^\infty(\mathbb{R}^N)$.

A natural question arises: “Does this result contradict Theorem 2.1?” The answer is NO and in the remaining part of this section we will justify it.

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Let us recall again some facts from [3], [9], [11] and [12]. For $\emptyset \neq M \subset \subset \mathbb{R}^N$ (i.e. $M$ is an open nonempty and bounded set) define

$$L_M^{1,p}(\mathbb{R}^N) := \left\{ u \in L^{1,p}(\mathbb{R}^N) ; \int_M u = 0 \right\}.$$ 

Then in the sense of a direct decomposition

$$L^{1,p}(\mathbb{R}^N) = L_M^{1,p}(\mathbb{R}^N) \oplus \mathbb{R}, \quad u = (u - m_u) + m_u,$$

where $m_u := \frac{1}{|M|} \int_M u$.

Moreover, in the case $N \leq p < \infty$ we have

$$L_M^{1,p}(\mathbb{R}^N) = C_{0,M}^\infty(\mathbb{R}^N) \|^{\| \nabla \cdot \|_{p;\mathbb{R}^N}}, \quad \text{if } p > N,$$

where $C_{0,M}^\infty(\mathbb{R}^N) := \{ u \in C_0^\infty(\mathbb{R}^N) ; \int_M u = 0 \}$.

Let $X := L_M^{1,p}(\mathbb{R}^N)$ and let $R > 0$ be such that $M \subset B_{2R}$ and $f \in L^{p'}(\mathbb{R}^N)$ satisfy

$$\int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^{p'} |x|^{p'} \, dx < \infty \quad \text{if } p > N,$$

and

$$\int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^{\frac{N}{N-1} \frac{N-1}{N} (\ln \frac{|x|}{R})} \, dx < \infty \quad \text{if } p = N.$$

**Lemma 4.1.** The assumptions of Theorem 2.1 are satisfied with $X$ and $f$ given above.

**Proof.** Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, supp $\varphi \subset \mathbb{R}^N \setminus \overline{B_{2R}}$. The following auxiliary estimates were proved in [12], Lemma II.9.2, p. 95:

$$\left( \int_{\mathbb{R}^N} \frac{|\varphi(x)|^p}{|x|^p} \, dx \right)^{\frac{1}{p}} \leq \frac{p}{|N-1|} \| \nabla \varphi \|_{p;\mathbb{R}^N} \quad \text{if } p > 1, p \neq N$$

and

$$\left( \int_{\mathbb{R}^N} \frac{|\varphi(x)|^N}{|x|^N (\ln \frac{|x|}{R})^N} \, dx \right)^{\frac{1}{N}} \leq \frac{N}{|N-1|} \| \nabla \varphi \|_{N;\mathbb{R}^N}.$$

Let us also recall the (extended) Poincaré inequality (see [3], estimate (2.12)):

$$\| u \|_{p,B_{R'}} \leq c(R, M) \| \nabla u \|_{p,B_{R'}}.$$
for all $u \in L_M^{1,p}(\mathbb{R}^N)$, valid even for $1 \leq p < \infty$ and all $R'$ such that $M \subset B_{R'}$.

We prove that $f$ defines a continuous linear functional on $X$. Indeed, let $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \eta(x) \leq 1$,

$$\eta(x) = \begin{cases} 
1 & \text{for } |x| \leq 2R, \\
0 & \text{for } |x| \geq 4R.
\end{cases}$$

For $\varphi \in X$ we consider

$$\langle f, \varphi \rangle_X = \langle f, \eta \varphi \rangle_X + \langle f, (1 - \eta) \varphi \rangle_X.$$ 

Set $\varphi_1 := \eta \varphi$, $\varphi_2 := (1 - \eta) \varphi$. Then

$$|\langle f, \varphi_1 \rangle_X| \leq \|f\|_{p', \mathbb{R}^N} \|\varphi_1\|_{p, \mathbb{R}^N}$$

and since $\int_M \varphi_1 = 0$, we have

$$\|\varphi_1\|_{p, \mathbb{R}^N} \leq c(R, M)\|\nabla \varphi_1\|_{p, \mathbb{R}^N} \leq c(R, M)(\|\eta \nabla \varphi\|_{p, \mathbb{R}^N} + \|\varphi \nabla \eta\|_{p, \mathbb{R}^N}).$$

On the other hand, since $\text{supp } \eta \subset B_{4R}$, $|\nabla \eta| \leq C_R$, we get by (4.11)

$$\|\varphi \nabla \eta\|_{p, \mathbb{R}^N} \leq C_R \|\varphi\|_{p, B_{4R}} \leq C_R c(R, M)\|\nabla \varphi\|_{p, B_{4R}}$$

and

$$\|\varphi_1\|_{p, \mathbb{R}^N} \leq c(R, M)(1 + C_R c(R, M))\|\nabla \varphi\|_{p, B_{4R}}.$$

For $\varphi_2$ we get

$$|\langle f, \varphi_2 \rangle_X| \leq \int_{\mathbb{R}^N} |f(x)| |\varphi_2(x)| \, dx \leq \int_{\mathbb{R}^N} (|f(x)| |x|^{-1} |\varphi_2(x)|) \, dx$$

$$\leq \left( \int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^p |x|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} |\varphi_2(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \left( \int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^p |x|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} |\varphi_2(x)|^p \, dx \right)^{\frac{1}{p}} \frac{p}{|N - p|} \|\nabla \varphi_2\|_{p, \mathbb{R}^N}$$

by (4.9) and (4.7) if $p > N$.

Similarly, we get

$$|\langle f, \varphi_2 \rangle_X| \leq \int_{\mathbb{R}^N} \left( \frac{|f(x)|}{|x|^R} \right) \left( \frac{|\nabla \varphi_2(x)|}{|x|^{N-1}} \right) \, dx$$

$$\leq \left( \int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^{\frac{N}{N-1}} |x|^{\frac{N}{N-1}} \ln \left( \frac{|x|^R}{R} \right) \, dx \right)^{\frac{N-1}{N}} \left( \int_{\mathbb{R}^N} \frac{|\varphi_2(x)|^N}{|x|^N \ln \left( \frac{|x|^R}{R} \right)} \, dx \right)^{\frac{1}{N}}$$

$$\leq \left( \int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^{\frac{N}{N-1}} |x|^{\frac{N}{N-1}} \ln \left( \frac{|x|^R}{R} \right) \, dx \right)^{\frac{N-1}{N}} \frac{N}{N-1} \|\nabla \varphi_2\|_{N; \mathbb{R}^N}.$$
by (4.10) and (4.8). Now, by (4.11) again
\[
\|\nabla \varphi\|_{p,R^N} \leq \|\nabla \varphi\|_{p,R^N \setminus B_{2R}} + C_R \|\varphi\|_{p,B_{4R}} \\
\leq \|\nabla \varphi\|_{p,R^N \setminus B_{2R}} + C_R c(R,M) \|\nabla \varphi\|_{p,B_{4R}} \leq C_1(R,M) \|\nabla \varphi\|_{p,R^N}.
\]
Thus we have an estimate
\[
|\langle f, \varphi \rangle_X| \leq c \|\nabla \varphi\|_{p,R^N}
\]
for any \( \varphi \in X \), where the constant depends only on \( R > 0 \), i.e. \( f \in X^* \). Since \((X1)\) and \((X2)\) are satisfied, the proof of the lemma is complete. \( \square \)

**Remark 4.1.** It follows from Lemma 4.1 and Theorem 2.1 that for any \( f \in L^p(R^N) \) satisfying (4.7) (if \( p > N \)) and (4.8) (if \( p = N \)) there exists a unique \( u_f \in X \) such that
\[
\int_{\mathbb{R}^N} |\nabla u_f|^p \nabla u_f \cdot \nabla \varphi = \int_{\mathbb{R}^N} f \varphi
\]
holds for any \( \varphi \in X \).

**Theorem 4.2.** Let \( X \) and \( f \) be as in Lemma 4.1. Then \( f \in L^1(R^N) \) and moreover, there is a unique family \( u_{f,c} = u_f + c, \ c \in \mathbb{R} \), \( u_{f,c} \in L^{1,p}(R^N) \) satisfying
\[
(4.12) \int_{\mathbb{R}^N} |\nabla u_{f,c}|^{p-2} \nabla u_{f,c} \cdot \nabla \varphi = \int_{\mathbb{R}^N} f \varphi
\]
for any \( \varphi \in C_0^\infty(R^N) \) if and only if
\[
\int_{\mathbb{R}^N} f = 0.
\]

**Proof.** Let \( p > N \). Then it follows from Hölder’s inequality that for any \( T > 2R \) we have
\[
\int_{\{2R \leq |x| \leq T \}} |f(x)| \, dx = \int_{\{2R \leq |x| \leq T \}} |f(x)| |x|^{-1} \, dx \\
\leq \left( \int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^p |x|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N \setminus B_{2R}} |x|^{-p} \, dx \right)^{\frac{1}{p}},
\]
\[
\int_{\{2R \leq |x| \leq T \}} |x|^{-p} \, dx = \omega_N \int_2^T r^{N-1-p} \, dr \leq \frac{\omega_N}{p-N} (2R)^{N-p}.
\]
(Here \( \omega_N \) is the measure of the unit sphere in \( \mathbb{R}^N \).)
Let $p = N$. Then from H"older’s inequality we have for any $T > 2R$

$$\int_{\{2R \leq |x| \leq T\}} |f(x)| \, dx \leq \left( \int_{\mathbb{R}^N \setminus B_{2R}} |f(x)|^\frac{N}{N-2} |x|^\frac{N-N}{N-2} \left( \ln \frac{|x|}{R} \right)^\frac{N-2}{N-1} \, dx \right)^\frac{N-2}{N-1} \times \left( \int_{\mathbb{R}^N \setminus B_{2R}} |x|^{-N} \left( \ln \frac{|x|}{R} \right)^{-N} \, dx \right)^\frac{1}{N}.$$ 

$$\int_{\{2R \leq |x| \leq T\}} |x|^{-N} \left( \ln \frac{|x|}{R} \right)^{-N} \, dx = \omega_N \int_{2R}^T r^{-1} \left( \ln \frac{r}{R} \right)^{-N} \, dr \leq \frac{\omega_N}{R(N-1)} (\ln 2)^{1-N}.$$ 

Hence from $f \in L^p'(\mathbb{R}^N)$, (4.7) (if $p > N$) and (4.8) (if $p = N$) we get that $f \in L^1(\mathbb{R}^N)$.

Assume now $\int_{\mathbb{R}^N} f = 0$. As mentioned above any $\varphi \in L^{1,p}(\mathbb{R}^N)$ splits as

$$\varphi = (\varphi - m_\varphi) + m_\varphi,$$

where $m_\varphi = \frac{1}{|M|} \int_M \varphi$. Then

$$\int_{\mathbb{R}^N} fm_\varphi = 0 = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla m_\varphi,$$

which together with the fact that (4.11) holds for any $\varphi \in X$ (cf. Remark 4.1) yields

(4.13) \[ \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^N} f \varphi \]

for any $\varphi \in L^{1,p}(\mathbb{R}^N)$ and, in particular, for any $\varphi \in C^\infty_0(\mathbb{R}^N)$.

If conversely, (4.13) holds for all $\varphi \in C^\infty_0(\mathbb{R}^N)$ then we can choose $\varphi = g_k$, where $g_k \in C^\infty_0(\mathbb{R}^N), 0 \leq g_k \leq 1, g_k(x) = 1$ for $|x| \leq k$ and $\|\nabla g_k\|_{p;\mathbb{R}^N} \to 0$ as $k \to \infty$ (cf. [3]). Then

(4.14) \[ \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla g_k \to 0 \]

and since $fg_k \to f$ a.e. in $\mathbb{R}^N$, $|fg_k| \leq |f|$, by Lebesgue’s theorem we conclude

$$\int_{\mathbb{R}^N} fg_k \to \int_{\mathbb{R}^N} f.$$

On the other hand, by (4.13), (4.14) $\int_{\mathbb{R}^N} fg_k \to 0$, i.e. $\int_{\mathbb{R}^N} f = 0$. \qed

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Let us assume that \( p > N \). Then due to the Morrey estimate (see [6], Theorem 7.17) the space \( L^{1,p}_M(\mathbb{R}^N) \) is isometrically isomorphic to

\[
\hat{H}^{1,p}_M(\mathbb{R}^N) := \{ u \in L^{1,p}(\mathbb{R}^N) : |u(x) - u(y)| \\
\leq C(N,p)\|\nabla u\|_{p;\mathbb{R}^N}|x - y|^{1 - \frac{N}{p}} \quad \forall x, y \in \mathbb{R}^N, u(0) = 0 \}.
\]

The corresponding isometric isomorphism \( J_p : L^{1,p}_M(\mathbb{R}^N) \to \hat{H}^{1,p}_M(\mathbb{R}^N) \) is defined by

\[
(J_p \tilde{u})(x) := \tilde{u}(x) - \tilde{u}(0),
\]
where \( \tilde{u} \) denotes the unique continuous representative belonging to the equivalence class \( u \in L^{1,p}_M(\mathbb{R}^N) \).

Hence for \( p > N \) we can alternatively set \( X = \hat{H}^{1,p}_M(\mathbb{R}^N) \) and \( (X, \| \nabla \cdot \|_{p;\mathbb{R}^N}) \) satisfies (X1) and (X2).

Let \( \mathbb{R}^N_+ := \{ x \in \mathbb{R}^N ; |x| > 0 \} \) and

\[
D_{N,p}(\mathbb{R}^N) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^N_+) ; \int_{\mathbb{R}^N} |f(x)| |x|^{1 - \frac{N}{p}} \, dx < \infty \right\}.
\]

Then by

\[
\| f \|_{D_{N,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |f(x)| |x|^{1 - \frac{N}{p}} \, dx
\]

a norm is defined and \( (D_{N,p}(\mathbb{R}^N), \| \cdot \|_{D_{N,p}(\mathbb{R}^N)}) \) is a Banach space.

Let \( u \in X \) and \( f \in D_{N,p}(\mathbb{R}^N) \). It follows from (4.15) and (4.16) that

\[
\left| \int_{\mathbb{R}^N} f(x) u(x) \, dx \right| \leq C(N,p)\|\nabla u\|_{p;\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x)| |x|^{1 - \frac{N}{p}} \, dx \\
= C(N,p)\|f\|_{D_{N,p}(\mathbb{R}^N)}\|\nabla u\|_{p;\mathbb{R}^N},
\]

i.e. \( D_{N,p}(\mathbb{R}^N) \subset X^* \).

**Theorem 4.3.** Let \( p > N \) and \( X \) be as above. Let \( f \in L^1_{\text{loc}}(\mathbb{R}^N) \) and assume that for some \( q > p \) the inequality

\[
\int_{\mathbb{R}^N \setminus B_1} |f(x)|^q |x|^q \, dx < \infty
\]

holds. Then there exists a unique family \( u_{f,c} = u_f + c, c \in \mathbb{R}, u_f \in X, X = \hat{H}^{1,p}_M(\mathbb{R}^N), u_{f,c} \in L^{1,p}(\mathbb{R}^N) \), satisfying

\[
\int_{\mathbb{R}^N} |\nabla u_{f,c}|^{p-2} \nabla u_{f,c} \cdot \nabla \varphi = \int_{\mathbb{R}^N} f \varphi
\]

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for any \( \varphi \in C_0^\infty(\mathbb{R}^N) \) if and only if
\[
\int_{\mathbb{R}^N} f = 0.
\]

**Proof.** We prove that \( f \in D_{N,p}(\mathbb{R}^N) \). Indeed, by Hölder’s inequality we obtain
\[
\int_{\mathbb{R}^N} |f(x)||x|^{1-\frac{2}{p}} \, dx \leq \int_{B_1} |f(x)| \, dx + \int_{\mathbb{R}^N \setminus B_1} |f(x)||x|^{1-\frac{2}{p}} \, dx
\]
\[
\leq \|f\|_{1,B_1} + \left( \int_{\mathbb{R}^N \setminus B_1} |f(x)|^q |x|^q \, dx \right)^\frac{1}{q} \left( \int_{\mathbb{R}^N \setminus B_1} |x|^{-N\frac{2}{p}} \, dx \right)^\frac{1}{q}.
\]

The rest of the proof follows the lines of the proof of Theorem 4.2. \( \square \)

**Remark 4.2.** Our Theorems 4.2 and 4.3 generalize a necessary condition given in [7]. In particular, we get from here that any constant is a weak solution of
\[
-\Delta_p u = 0 \quad \text{in} \quad \mathbb{R}^N.
\]

5. **The case of an exterior domain**

Let \( G := \mathbb{R}^N \setminus K \), where \( \emptyset \neq K \subset \subset \mathbb{R}^N \), \( 0 \in K \). Let us consider the Dirichlet problem
\[
\begin{cases}
-\Delta_p u = f \quad \text{in} \ G, \\
u = 0 \quad \text{on} \ \partial G.
\end{cases}
\]

We want to prove existence and uniqueness of a weak solution of (5.1). Define the space
\[
\tilde{H}_0^{1,p}(G) := \frac{C_0^\infty(G) \cap \mathcal{W}}{\|\cdot\|_{p,G}}.
\]

Let \( 1 < p < N \). Then due to the Sobolev imbedding we have \( \tilde{H}_0^{1,p}(G) \hookrightarrow L^{p'}(G) \) and therefore \( X := \tilde{H}_0^{1,p}(G) \) verifies (X1) and (X2). We can apply the abstract Theorem 2.1 and, in particular, we have the following result.

**Theorem 5.1.** Let \( f \in L^{p'}(G) \) be given. Then there is a unique \( u_f \in X \) such that
\[
\int_G |\nabla u_f|^{p-2} \nabla u_f : \nabla \varphi = \int_G f \varphi
\]
holds for any \( \varphi \in X \) (or equivalently, for any \( \varphi \in C_0^\infty(G) \)).
Let \( p \geq N \). Then \( \tilde{H}^{1,p}_{0}(G) \) coincides with the space

\[
\tilde{H}^{1,p}_{0}(G) := \{ u \in L^{1,p}(G); \ u \in L^{p}(G_{R}) \text{ for every } R > 0 \text{ and } \eta u \in W^{1,p}_{0}(G) \text{ for any } \eta \in C_{\infty}^{0}(\mathbb{R}^{N}) \},
\]

where \( G_{R} = G \cap B_{R} \) (see [12], Theorems I. 2.7, I. 2.16). Now, we can literally follow the approach from Section 4, case \( p \geq N \), to get the following result.

**Theorem 5.2.** Let \( f \in L^{p'}(G) \), let \( f \) satisfy (4.7) for \( p > N \) and (4.8) for \( p = N \). Then there exists a unique \( u_{f} \in X \) such that (5.2) holds for any \( \varphi \in X \) (or equivalently, for any \( \varphi \in C_{\infty}^{0}(G) \)).

**Remark 5.1.** Let us point out that contrary to the case of the whole of \( \mathbb{R}^{N} \) we do not need any additional condition of the type "\( \int f = 0 \)" because the constants are ruled out due to the homogeneous Dirichlet boundary conditions.

Let us consider the Neumann problem

\[
\begin{cases}
-\Delta_{p} u = f \text{ in } G, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial G.
\end{cases}
\]

Choose \( M \) such that \( \emptyset \neq M \subset \subset G \). Then a subspace of \( L^{1,p}(G) \) is given by

\[
L^{1,p}_{M}(G) := \{ u_{0} \in L^{1,p}(G); \int_{M} u_{0} = 0 \}
\]

and in the sense of a direct sum

\[
L^{1,p}(G) = L^{1,p}_{M}(G) \oplus \mathbb{R},
\]

where

\[
m_{u} := |M|^{-1} \int_{M} u, \quad u_{0} := u - m_{u}.
\]

By

\[
|u|_{1,p;G;M} := \|\nabla u\|_{p;G} + \left| \int_{M} u \right|
\]

a norm is defined on \( L^{1,p}(G) \) (see [9], Lemma 4.1) such that \( L^{1,p}(G) \) equipped with this norm is a reflexive Banach space (see [9], Theorem 4.5).
Clearly, for \( u_0 \in L^{1,p}_M(G) \) we have

\[
|u_0|_{1,p;G;M} = \|\nabla u_0\|_{p;G}.
\]

We assume now that \( \partial G \in C^0 \) and choose \( R_0 = R_0(M,K) > 0 \) so that \( \overline{M} \subset B_{R_0} \) and \( \overline{K} \subset B_{R_0} \) and we write \( G_{R_0} := G \cap B_{R_0} \). By [11], Lemma 4.2, for \( u \in L^{1,p}(G) \), we see that \( u|_{G_{R_0}} \in L^p(G_{R_0}) \) and there exist \( G' \subset G \) and a constant \( C_{R_0} > 0 \) such that

\[
\|u\|_{p;G_{R_0}} \leq C_{R_0} (\|\nabla u\|_{p;G} + \|u\|_{p;G'}) \quad \forall u \in L^{1,p}(G).
\]

Because of the Poincaré-type inequality

\[
\|u\|_{p;G'} \leq C_{G'} \|\nabla u\|_{p;G} \quad \forall u \in L^{1,p}(G)
\]

(with \( C_{G'} = C(G',G,p) > 0 \), see [9], Theorem 5.1), by (5.8), (5.9) we get

\[
\|u\|_{W^{1,p}(G_{R_0})} \leq (1 + C_{G'}) \frac{1}{\gamma} \|\nabla u_0\|_{p;G} \quad \forall u \in L^{1,p}_M(G).
\]

**Lemma 5.1.** Assume that \( \partial G \in C^{0,1} \) (e.g. \( \partial G = \partial K \) is a Lipschitz manifold). Then there exists a linear extension

\[
E: L^{1,p}_M(G) \rightarrow L^{1,p}_M(\mathbb{R}^N)
\]

such that \( Eu_0|_G = u_0 \) \( \forall u_0 \in L^{1,p}_M(G) \). In addition, there is a constant \( C_E > 0 \) such that

\[
\|\nabla Eu_0\|_{p;\mathbb{R}^N} \leq C_E \|\nabla u_0\|_{p;G} \quad \forall u_0 \in L^{1,p}_M(G).
\]

**Proof.** a) Because of \( \partial G \in C^{0,1} \), there exists a linear extension

\[
\tilde{E}: W^{1,p}(G_{R_0}) \rightarrow W^{1,p}_0(\mathbb{R}^N), \quad \tilde{E}v|_{G_{R_0}} = v \quad \forall v \in W^{1,p}(G_{R_0})
\]

and a constant \( \tilde{C} = \tilde{C}(G_{R_0},p) > 0 \) such that

\[
\|\tilde{E}v\|_{W^{1,p}(\mathbb{R}^N)} \leq \tilde{C} \|v\|_{W^{1,p}(G_{R_0})} \quad \forall v \in W^{1,p}(G_{R_0})
\]

(see e.g. [10], Théoréme 3.9).
b) As we mentioned above, \( u_0 \in L^{1,p}_M(G) \) implies \( u_0|_{G_{R_0}} \in W^{1,p}(G_{R_0}) \). With help of \( \tilde{E} \) we define

\[
(Eu_0)(x) := \begin{cases} 
  u_0(x) & \text{for } x \in G, \\
  \tilde{E}(u_0|_{G_{R_0}})(x) & \text{for } x \in \mathbb{R}^N \setminus \overline{G} = K.
\end{cases}
\]

Since \( M \subset \subset G \) it is clear that \( Eu_0 \in L^{1,p}_M(\mathbb{R}^N) \) for \( u_0 \in L^{1,p}_M(G) \) and \( Eu_0|_G = u_0 \).

By (5.11) and (5.13) we see

\[
\| \nabla Eu_0 \|_{\mathbb{R}^N} \leq \| \nabla u_0 \|_{G} + \| \nabla \tilde{E}(u_0|_{G_{R_0}}) \|_{K} \\
\leq \| \nabla u_0 \|_{G} + \tilde{C} \| u_0 \|_{W^{1,p}(G_{R_0})} \leq CE \| \nabla u_0 \|_{G},
\]

with \( CE := 1 + \tilde{C}(1 + C_1^p)^\frac{1}{p} \).

□

Obviously we get

**Corollary 5.1.** Let \( \partial G \in C^{0,1} \). Then

\[
L^{1,p}_M(G) = \{ v|_G ; \ v \in L^{1,p}_M(\mathbb{R}^N) \}.
\]

Let \( \partial G \in C^{0,1} \). Due to (5.4) any \( u \in L^{1,p}(G) \) can be written as \( u = u_0 + m_u \).

Define a linear map \( E_1 : L^{1,p}(G) \rightarrow L^{1,p}(\mathbb{R}^N) \) by

\[
E_1 u := Eu_0 + m_u.
\]

Then \( E_1 u|_G = u \ \forall u \in L^{1,p}(G) \).

This extension enables us to apply the result found for the whole space \( \mathbb{R}^N \) to the underlying case. But the price we have to pay is the assumption \( \partial G \in C^{0,1} \). On the other hand, without any regularity assumptions on \( \partial G \) we never may expect any imbedding theorems for \( G \).

Let \( 1 < p < N \). We recall the decomposition (4.2) and the density property (4.3).

**Lemma 5.2.** Let \( \partial G \in C^{0,1} \) and

\[
\tilde{H}^{1,p}(G) := \{ u^* \in L^{1,p}(G) ; \ u^* \in L^p(G) \}.
\]

Then in the sense of a direct decomposition

\[
L^{1,p}(G) = \tilde{H}^{1,p}(G) \oplus \mathbb{R}, \quad u = u^* + c_u
\]

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where \((G_R := G \cap B_R)\),

\[
(5.19) \quad c_u := \lim_{R \to \infty} \frac{1}{|G_R|} \int_{G_R} u.
\]

Further, the map \(J : L^1_M(G) \to \tilde{H}^{1,p}(G)\), \(J u := u^*\), is an isometric isomorphism and \((\tilde{H}^{1,p}(G), \|\nabla \cdot \|_p)\) is a reflexive Banach space.

With \(C_{SOB} > 0\) (the constant for the Sobolev imbedding) and \(C_E > 0\) from (5.12), we have

\[
(5.20) \quad \|u^*\|_{p', G} \leq C_{SOB} C_E \|\nabla u^*\|_{p; G} \quad \forall u \in \tilde{H}^{1,p}(G).
\]

Further, \(\tilde{H}^{1,p}(G) = \{v^*|_G; v^* \in \tilde{H}_0^{1,p}(\mathbb{R}^N)\}\).

Let

\[
(5.21) \quad C_0^\infty(G) := \{\Phi \in C^\infty(G); \exists R_\Phi \geq R_0: \Phi(x) = 0 \text{ for } |x| \geq R_\Phi\}.
\]

Then

\[
(5.22) \quad \{\psi|_G; \psi \in C_0^\infty(\mathbb{R}^N)\} \subset C_0^\infty(G) \subset \tilde{H}^{1,p}(G)
\]

and

\[
(5.23) \quad \tilde{H}^{1,p}(G) = \overline{C_0^\infty(G)}^{\|\nabla \cdot \|_p G}.
\]

**Proof.**  a) If \(u \in L^1(G), u = u_0 + m_u\), then by virtue of (5.4), with \(u_0 \in L^1_M(G)\) and \(m_u \in \mathbb{R}\), we have \(v := E_1 u = E u_0 + m_u \in L^1_M(\mathbb{R}^N)\). By (4.2), \(v = v^* + c_v\) with \(v^* \in \tilde{H}_0^{1,p}(\mathbb{R}^N)\) and \(c_v \in \mathbb{R}\). Let \(u^* := v^*|_G = (v - c_v)|_G = u - c_v = u_0 + m_u - c_v\).

Therefore \(u = u^* + c_v\). Since \(u^* \in L^p(G)\), we get

\[
\left|G_R\right|^{-1} \int_{G_R} u - c_v = \left|G_R\right|^{-1} \int_{G_R} (u(y) - c_v) \, dy \leq \left|G_R\right|^{-1} \|u^*\|_{p; G_R} \left|G_R\right|^{\frac{p-1}{p}} \quad \left|G_R\right| \to \infty.
\]

Thus \(c_v = c_u = \lim_{R \to \infty} \left|G_R\right|^{-1} \int_{G_R} u\).

If \(u^* \in \tilde{H}^{1,p}(G) \cap \mathbb{R}\) then because of \(|G| = \infty\) we have \(u^* = 0\), proving (5.18), (5.19).

If conversely \(u^* \in \tilde{H}^{1,p}(G) \subset L^1(G)\) is given then \(u^* = u_0 + m_u\), \(u_0 \in L^1(G)\), \(m_u \in \mathbb{R}\). Then \(E_1 u^* = E_1 u_0 + m_u =: v\). Then \(v = v^* + c_v\), \(v^* \in \tilde{H}_0^{1,p}(\mathbb{R}^N)\), \(c_v \in \mathbb{R}\). Further \(u^* = v|_G = v^*|_G + c_v\). Then \(c_v = (u^* - v^*|_G) \in L^p(G) \cap \mathbb{R}\) and again by
\(|G| = \infty\) we see that \(c_v = 0\), that is \(u^* = v^*|G\), proving \(\tilde{H}^{1,p}(G) = \{v^*|G; \ v^* \in \tilde{H}_0^{1,p}(\mathbb{R}^N)\}\).

Moreover, we derive (5.20) from
\[
\|u^*\|_{p^*;G} \leq \|v^*\|_{p^*;\mathbb{R}^N} \leq C_{\text{SOB}} \|\nabla v^*\|_{p;\mathbb{R}^N} = C_{\text{SOB}} \|\nabla u_0\|_{p;\mathbb{R}^N} \leq C_{\text{SOB}} C_E \|\nabla u_0\|_{p;G} = C_{\text{SOB}} C_E \|\nabla u^*\|_{p;G}
\]
and therefore completeness of \(\tilde{H}^{1,p}(G)\) follows. If \(u^* \in \tilde{H}^{1,p}(G)\), \(u^* = v^*|G\) with \(v^* \in \tilde{H}^{1,p}(\mathbb{R}^N)\), then by (4.3) there exists a sequence \((v_k) \subset C_0^\infty(\mathbb{R}^N)\) with \(\|\nabla v^* - \nabla v_k\|_{p;\mathbb{R}^N} \to 0\). Then \(\Phi_k := v_k|G \in \mathcal{C}_G^\infty(G)\) and
\[
\|\nabla u^* - \nabla \Phi_k\|_{p;G} \leq \|\nabla u^* - \nabla v_k\|_{p;\mathbb{R}^N} \to 0,
\]
which proves (5.23). Finally, the properties of the map \(J: L_{\text{H}}^{1,p}(G) \to \tilde{H}^{1,p}(G)\) are obvious.

\textbf{Lemma 5.3.} Let \(G \subset \mathbb{R}^N\) be a domain with \(|G| = \infty\) and let \(1 < p < N\). Let us suppose conversely that \(\tilde{H}^{1,p}(G)\) defined by (5.17) is complete with respect to the \(\|\nabla \cdot\|_{p;G}\)-norm. Then there is a constant \(C > 0\) such that
\[
(5.24) \quad \|u\|_{p^*;G} \leq C \|\nabla u\|_{p;G} \quad \forall u \in \tilde{H}^{1,p}(G).
\]

\textbf{Proof.} Let \(T: \tilde{H}^{1,p}(G) \to L^{p^*}(G)\) be defined by \(Tu^* := u^* \forall u^* \in \tilde{H}^{1,p}(G)\). Suppose that \((u_j^*) \subset \tilde{H}^{1,p}(G)\) and \(u^* \in \tilde{H}^{1,p}(G)\) with \(\|\nabla u^* - \nabla u_j^*\|_{p;G} \to 0\). Suppose in addition that there is \(v \in L^{p^*}(G)\) with
\[
\|v - Tu_j^*\|_{p^*;G} = \|v - u_j^*\|_{p^*;G} \to 0.
\]
Then for \(\Phi \in C_0^\infty(G)\) and \(i = 1, \ldots, N\) we have
\[
\int_G v \partial_i \Phi = \lim \int_G u_j^* \partial_i \Phi = - \lim \int_G \Phi \partial_j u_j^* = - \int_G \Phi \partial_i u^*,
\]
proving that \(v\) has the weak derivatives \(\partial_i u^*\). Then \(\nabla v = \nabla u^*\) and therefore, since \(G\) is a domain, \(u^* = v + c\). Since \(u^*, v \in L^{p^*}(G)\) and \(|G| = \infty\) we see that \(c = 0\) and \(v = u^*\). This proves closedness of \(T\) and since \(D(T) = \tilde{H}^{1,p}(G)\) by Banach’s closed graph theorem the boundedness of \(T\) and therefore (5.24) follow.
Theorem 5.3. Let $G \subset \mathbb{R}^N$ be an exterior domain with $\partial G \in C^{0,1}$ and $X := \tilde{H}^{1,p}(G)$. Given $f \in L^{p'}(\mathbb{R}^N)$ there exists a unique $u_f \in X$ such that

$$\int_G |\nabla u_f|^{p-2}\nabla u_f \cdot \nabla v = \int_G f v \quad \forall v \in X.$$ 

Proof. By (5.20), for $v \in X$ we have

$$\left| \int_G f v \right| \leq \|f\|_{p'; G} C_{SOB} C_E \|\nabla v\|_{p;G}.$$ 

□

Let $p \geq N$. We recall (4.6). Then the following assertion holds.

Lemma 5.4. Let $G \subset \mathbb{R}^N$ be an exterior domain with $\partial G \in C^{0,1}$. Let $\emptyset \neq M \subset G$ and

$$(5.25) \quad C_{0,M}^\infty(\overline{G}) := \left\{ \Phi \in C^\infty(\overline{G}); \int_M \Phi \, dy = 0 \text{ and } \exists R_\Phi > 0: \Phi(x) = 0 \text{ for } |x| \geq R_\Phi \right\}.$$ 

Then $\{\Phi|_G; \Phi \in C_{0,M}^\infty(\mathbb{R}^N)\} \subset C_{0,M}^\infty(\overline{G})$ and for $p \geq N$ we have

$$(5.26) \quad L_M^{1,p}(G) = \{\Phi|_G; \Phi \in C_{0,M}^\infty(\mathbb{R}^N)\}^{\|\nabla \cdot \|_{p;G}}$$

and

$$(5.27) \quad L_M^{1,p}(G) = \{v|_G; v \in L_M^{1,p}(\mathbb{R}^N)\}.$$ 

Proof. If $u \in L_M^{1,p}(G)$ then $Eu \in L_M^{1,p}(\mathbb{R}^N)$ and by (4.6) there exists a sequence $(\Phi_k) \subset C_0^\infty(\mathbb{R}^N)$ with $\|\nabla E u - \nabla \Phi_k\|_{p;\mathbb{R}^N} \to 0$. □

Theorem 5.4. Let $X := L_M^{1,p}(G)$. Let $R \geq R_0(G)$ and suppose that $f \in L^{p'}(G)$ satisfies (4.7) if $p > N$ or (4.8) if $p = N$. Then there exists a unique $u \in L_M^{1,p}(G)$ with

$$(5.28) \quad \int_G |\nabla u|^{p-2}\nabla u \cdot \nabla v = \int_G f v \quad \forall v \in L_M^{1,p}(G).$$

Further, (5.28) holds even for all $v \in C_0^\infty(\overline{G})$ if and only if $\int_G f = 0$. 120
P r o o f.  a) Existence is clear.

b) If \( f = 0 \), then \( v \in C_0^\infty(\overline{\Omega}) \) may be decomposed into \( v = v_0 + m_v \), \( v_0 \in L_1^{1,p}(\Omega) \), \( m_v \in \mathbb{R} \). Since \( \int_\Omega f m_v = 0 \) and \( \nabla m_v = 0 \), (5.28) holds for \( v \in C_0^\infty(\overline{\Omega}) \), too. Conversely, consider again the sequence \( (\eta_k) \subset C_0^\infty(\mathbb{R}^N) \) with \( \eta_k|_{B_R} \to 1 \) (\( k \to \infty \)) uniformly for every fixed \( R > 0 \) and \( \|\nabla \eta_k\|_{p,\mathbb{R}^N} \to \infty \). Then with \( v := \eta_k \) we conclude from (5.28) for \( k \to \infty \): \( \int_\Omega f = 0 \).  

In the case \( N < p < \infty \) we have an additional “realization” of \( L_1^{1,p}(\Omega) \) corresponding to the case \( \Gamma = \mathbb{R}^N \).

Lemma 5.5. Let \( \Omega \subset \mathbb{R}^N \) be an exterior domain with \( \partial \Omega \subset C^{0,1} \) and let \( N < p < \infty \). Let \( x_0 \in \Omega \) be fixed and let

\[
\tilde{H}_1^{1,p}(\Omega) := \{ \tilde{u} \in L_1^{1,p}(\Omega) : |\tilde{u}(x) - \tilde{u}(y)| \\
\leq C(N,p)|x - y|^{1 - \frac{2}{p}} \|\nabla \tilde{u}\|_{p,\Omega} \quad \forall x, y \in \overline{\Omega}, \text{ and } \tilde{u}(x_0) = 0 \}.
\]

Then \( \tilde{H}_1^{1,p}(\Omega) \) equipped with the norm \( \|\nabla \tilde{u}\|_{p,\Omega} \) is a reflexive Banach space,

\[
\tilde{H}_1^{1,p}(\Omega) = \{ (\hat{v} - \hat{v}(x_0))|_{\overline{\Omega}} : \hat{v} \in \tilde{H}_1^{1,p}(\mathbb{R}^N) \}
\]

(with \( \tilde{H}_1^{1,p}(\mathbb{R}^N) \) by (4.15)), and there is an isometrically isomorphic map \( I_p : L_1^{1,p}(\Omega) \to \tilde{H}_1^{1,p}(\Omega) \).

P r o o f. If \( u \in L_1^{1,p}(\Omega) \) then \( v := Eu \in L_1^{1,p}(\mathbb{R}^N) \). Denote by \( \hat{v} \) the unique Hölder continuous representative of \( v \). Then \( \hat{v} := (\hat{v} - \hat{v}(0)) \in \tilde{H}_1^{1,p}(\mathbb{R}^N) \) and \( \tilde{u} := (\hat{v} - \hat{v}(x_0)) \in \tilde{H}_1^{1,p}(\Omega) \). Clearly, if \( \tilde{u} \in \tilde{H}_1^{1,p}(\Omega) \) then \( E \tilde{u} \in L_1^{1,p}(\mathbb{R}^N) \) and \( \tilde{v} := E \tilde{u} - (E \tilde{u})(0) \in \tilde{H}_1^{1,p}(\mathbb{R}^N), \quad \tilde{u} = (\tilde{v} - \tilde{v}(x_0))|_{\Omega} \).

Further, the map \( I_p u := (E \tilde{u} - E \tilde{u}(x_0)), \quad I_p : L_1^{1,p}(\Omega) \to \tilde{H}_1^{1,p}(\Omega) \) is an isometric isomorphism.  

Theorem 5.5. Let \( \Omega \subset \mathbb{R}^N \) be an exterior domain with \( \partial \Omega \subset C^{0,1} \) and \( 0 \in \mathbb{R}^N \setminus \overline{\Omega} \) and let \( N < p < \infty \). Let \( f \in L_1^{1,1}(\Omega) \) and assume that for some \( q > p \),

\[
\int_\Omega |f(x)|^q |x|^q \, dx < \infty.
\]

Then there exists a unique family \( u_{f,c} = u_f + c \) with \( u_f \in X := \tilde{H}_1^{1,p}(\Omega) \) and \( c \in \mathbb{R} \) satisfying

\[
\int_\Omega |\nabla u_{f,c}|^{p-2} \nabla u_{f,c} \cdot \nabla \varphi = \int_\Omega f \varphi \quad \forall \varphi \in C_0^\infty(\overline{\Omega})
\]

(see (5.21)) if and only if \( \int_\Omega f = 0 \).

P r o o f. The proof is performed analogously to that of Theorem 4.3.  

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References


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