MEAN VALUES AND ASSOCIATED MEASURES OF 
$\delta$-SUBHARMONIC FUNCTIONS

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Abstract. Let $u$ be a $\delta$-subharmonic function with associated measure $\mu$, and let $v$ be a superharmonic function with associated measure $\nu$, on an open set $E$. For any closed ball $B(x, r)$, of centre $x$ and radius $r$, contained in $E$, let $M(u, x, r)$ denote the mean value of $u$ over the surface of the ball. We prove that the upper and lower limits as $s, t \to 0$ with $0 < s < t$ of the quotient $(M(u, x, s) - M(u, x, t))/(M(v, x, s) - M(v, x, t))$, lie between the upper and lower limits as $r \to 0+$ of the quotient $\mu(B(x, r))/\nu(B(x, r))$. This enables us to use some well-known measure-theoretic results to prove new variants and generalizations of several theorems about $\delta$-subharmonic functions.

Keywords: superharmonic, $\delta$-subharmonic, Riesz measure, spherical mean values

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1. Introduction

Let $E$ be an open subset of $\mathbb{R}^n$, let $u$ be $\delta$-subharmonic on $E$, and let $v$ be superharmonic on $E$. Let $\mu$ and $\nu$ be the Borel measures associated with $u$ and $v$ by the Riesz Decomposition Theorem, so that $\mu$ is signed and $\nu$ is positive. Let $B(x, r)$ denote the closed ball with centre $x$ and radius $r$ contained in $E$, and let $M(u, x, r)$ denote the spherical mean value of $u$ over $\partial B(x, r)$. We shall prove that the upper and lower limits as $s, t \to 0$ with $0 < s < t$ of

\begin{equation}
\frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)}
\end{equation}

lie between the upper and lower limits as $r \to 0+$ of

\begin{equation}
\frac{\mu(B(x, r))}{\nu(B(x, r))}.
\end{equation}
This enables us to use the measure-theoretic results of Besicovitch [3], [4] to study the behaviour of δ-subharmonic functions.

This work was inspired by a recent paper of Sodin [12]. However, the techniques we devised have much wider ramifications, so that Sodin’s results appear only as fairly minor details. We generalize not only Sodin’s results, but also some due to Armitage [2] and Watson [14]. We also present new analogues of some theorems about Poisson integrals which appeared in [1], [5], and [16], a new form of the Domination Principle, and variants of recent results of Fuglede [10].

Our starting point is the well-known formula

$$M(u, x, s) = M(u, x, t) + p_n \int_s^t r^{1-n} \mu(B(x, r)) \, dr,$$

in which $0 < s < t$, $B(x, t) \subseteq E$, and $p_n = \max\{1, n - 2\}$. See, for example, [2] Lemma 3. We shall put

$$I_{\mu}(x; s, t) = p_n \int_s^t r^{1-n} \mu(B(x, r)) \, dr.$$

Then the quotient (1) can be written as either

$$\frac{I_{\mu}(x; s, t)}{I_{\nu}(x; s, t)}$$

or

$$\frac{M(u, x, s) - M(u, x, t)}{I_{\mu}(x; s, t)}.$$

From (3) it is easy to see the connection with (2).

If we take $\nu$ to be the Lebesgue measure $\lambda$, we have

$$I_{\lambda}(x; s, t) = p_n v_n (t^2 - s^2)/2,$$

where $v_n = \lambda(B(0, 1))$. Then, up to a multiplicative constant, (4) becomes

$$\frac{M(u, x, s) - M(u, x, t)}{t^2 - s^2}.$$

Theorem 3 below gives conditions on this quotient which ensure that $\mu$ can be written in the form

$$\mu = \omega - \sum_j c_j \delta_j$$
where $\omega$ is a positive measure, each $c_j$ is a specific positive constant, each $\delta_j$ is a unit mass at a given point $x_j$, and there are countably many indices. This is analogous to decomposition formulas for the boundary measures of Poisson integrals given in [5], [1], and [16].

Theorem 4 shows how the quotient (4) can be used to determine which sets are positive for $\mu$. Roughly, if

$$\limsup_{0<s<t\to0} \frac{M(u,x,s) - M(u,x,t)}{I_\nu(x;s,t)} \geq 0$$

for all $x \in S$, then $S$ is positive for $\mu$. The condition can be weakened on a $\nu$-null subset of $S$. This result contains as special cases those due to Sodin [12], which include the one known as Grishin’s lemma [11].

Theorems 5 and 6 generalize results of Armitage [2] by extending them to points where his conditions that an infinity occur no longer hold. Theorems 10 and 11 similarly extend results of Watson [14].

By analogy with results on half-space Poisson integrals given in [1] and [16], Theorem 7 gives conditions on the quotient (1) which ensure that $u - Av$ is superharmonic for some real number $A$. For example, the condition

$$\limsup_{0<s<t\to0} \frac{M(u,x,s) - M(u,x,t)}{M(v,x,s) - M(v,x,t)} \geq A$$

for all $x \in E$, is sufficient. A minor modification of the proof, in the special case where $u$ is a positive superharmonic function, $E$ is Greenian, and $v = G_E\nu$ is a Green potential, yields the following domination principle as Theorem 8: If

$$\limsup_{0<s<t\to0} \frac{M(u,x,s) - M(u,x,t)}{M(v,x,s) - M(v,x,t)}$$

is never $-\infty$, and is greater than or equal to 1 for $\nu$-almost all $x$, then $u \geq v$.

In Theorem 9, we use (1) to determine the $\mu$-null subsets of $E$. One of its corollaries is an extension to $\delta$-subharmonic functions of the fact that polar sets are null for the restriction of $\nu$ to the set where $v$ is finite. A different such extension was established by Fuglede [10].

Given a Borel subset $B$ of $E$, we denote by $\mu_B$ the restriction of $\mu$ to $B$. 

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Theorem 1 contains the necessary measure theory. It is implicit in [15], but may not have been stated explicitly before. References are given to the original papers of Besicovitch; an alternative source is [9].

**Theorem 1.** Let \( \mu \) be a signed measure and \( \nu \) a positive measure on \( E \). Let

\[
f(x) = \lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}
\]

whenever the limit exists, let \( Z^+ = \{ x \in E : f(x) = \infty \} \), and let \( Z^- = \{ x \in E : f(x) = -\infty \} \). Then \( f \) is defined and finite \( \nu \)-a.e. on \( E \), and there are positive \( \nu \)-singular measures \( \sigma^+ \) and \( \sigma^- \), concentrated on \( Z^+ \) and \( Z^- \) respectively, such that

\[
d\mu = f\,d\nu + d\sigma^+ - d\sigma^-.
\]

**Proof.** By [3] Theorem 2, \( f \) is defined and finite \( \nu \)-a.e. By [4] Theorem 6, \( f \) is the Radon-Nikodým derivative of \( \mu \) with respect to \( \nu \), so that (5) holds with \( \sigma^+ \) and \( \sigma^- \) the positive and negative variations of the \( \nu \)-singular part of \( \mu \).

To show that \( \sigma^+ \) is concentrated on \( Z^+ \), we put \( d\omega = f\,d\nu - d\sigma^- \). Then both \( \nu \) and \( \omega \) are \( \sigma^+ \)-singular, so that by [3] Theorem 3,

\[
\lim_{r \to 0} \frac{\nu(B(x, r))}{\sigma^+(B(x, r))} = 0
\]

and

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{\sigma^+(B(x, r))} = \lim_{r \to 0} \frac{\omega(B(x, r))}{\sigma^+(B(x, r))} + 1 = 1,
\]

for \( \sigma^+ \)-almost all \( x \). Hence \( f(x) = \infty \) for \( \sigma^+ \)-almost all \( x \), so that \( \sigma^+ \) is concentrated on \( Z^+ \). Similarly, \( \sigma^- \) is concentrated on \( Z^- \).

**Corollary 1.** Let \( \mu \) be a signed measure and \( \nu \) a positive measure on \( E \), and let \( S \) be a Borel subset of \( E \). If

\[
\limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} > -\infty
\]

for all \( x \in S \) at which the upper limit is defined, and

\[
\limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} \geq A
\]

for \( \nu \)-almost all \( x \in S \), then \( (\mu - A\nu)_S \geq 0 \).

**Proof.** By Theorem 1, \( d\mu_S = f\,d\nu_S + d\sigma^+_S - d\sigma^-_S \) with \( f(x) \) equal to the upper limit in (7) and \( \sigma^-_S \) concentrated on \( \{ x \in S : f(x) = -\infty \} \). By (6) this set is empty, and by (7) \( f \geq A \). Hence \( d\mu_S - A\,d\nu_S \geq d\sigma^+_S \geq 0 \). \( \Box \)
Corollary 2. Let $\mu$ be a signed measure and $\nu$ a positive measure on $E$. Let $S$ be a Borel subset of $E$ such that, for each $x \in S$, either

$$\lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = 0$$

or the limit does not exist. Then $\mu_S = 0$.

Proof. By Theorem 1, $\{x \in E: (8) \text{ holds}\}$ is $\mu$-null, and the set of points where $f$ is undefined is also $\mu$-null. \qed

We include for completeness the definition of

$$\limsup_{0 < s < t \to 0} f(s, t),$$

although it is the natural one. Those of the corresponding lim inf and lim are then obvious.

Definition. Suppose that $f(s, t)$ is defined as an extended-real number whenever $0 < s < t < a$, and that $\ell \in \mathbb{R}$. We write

$$\limsup_{0 < s < t \to 0} f(s, t) = \ell$$

if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $f(s, t) < \ell + \varepsilon$ whenever $0 < s < t < \delta$, and there is a sequence $\{(s_k, t_k)\}$ such that $0 < s_k < t_k \to 0$ and $f(s_k, t_k) \to \ell$ as $k \to \infty$. We also write

$$\limsup_{0 < s < t \to 0} f(s, t) = \infty$$

if there is a sequence $\{(s_k, t_k)\}$ such that $0 < s_k < t_k \to 0$ and $f(s_k, t_k) \to \infty$. Finally, we write

$$\limsup_{0 < s < t \to 0} f(s, t) = -\infty$$

if to each $A \in \mathbb{R}$ there corresponds $\delta > 0$ such that $f(s, t) < A$ whenever $0 < s < t < \delta$.

We can now establish the connection on which all our results are based.

Theorem 2. If $u$ is $\delta$-subharmonic on $E$ with associated measure $\mu$, and $\nu$ is a positive measure on $E$, then

$$\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{I_\nu(x; s, t)} \leq \limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$
whenever the latter exists. The reverse inequality holds for lower limits, and

\[ \lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)} = \lim_{r\to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} \]

for \( \nu \)-almost all \( x \in E \).

**Proof.** Given \( x \) for which the upper limit on the right-hand side of (9) exists, denote that upper limit by \( \ell \). If \( \ell = \infty \) there is nothing to prove. Otherwise, given a real number \( A > \ell \) we can find \( \delta > 0 \) such that

\[ \frac{\mu(B(x, r))}{\nu(B(x, r))} < A \quad \text{whenever } 0 < r < \delta. \]

If \( \nu(B(x, r)) = 0 \) for all \( r < \eta (\leq \delta) \), then the above inequality can hold only if \( \mu(B(x, r)) < 0 \) for all such \( r \). Then \( I_\nu(x; s, t) = 0 \) whenever \( t < \eta \), and

\[ \mathcal{M}(u, x, s) - \mathcal{M}(u, x, t) = I_\mu(x; s, t) < 0, \]

so that (9) holds with both sides \(-\infty\). On the other hand, if \( \nu(B(x, r)) > 0 \) for all \( r \), then

\[ \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)} = \frac{p_n}{I_\nu(x; s, t)} \int_s^t r^{1-n} \nu(B(x, r)) \frac{\mu(B(x, r))}{\nu(B(x, r))} dr < A \]

whenever \( 0 < s < t < \delta \), and again (9) holds.

Obviously (9) implies the reverse inequality for lower limits. Now (10) follows from \([3]\) Theorem 2.

The particular cases of Theorem 2, in which \( \nu \) is the Lebesgue measure \( \lambda \) or the unit mass \( \delta_x \) at \( x \), are of special importance.

**Corollary 1.** If \( u \) is \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \), then

\[ \lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^2 - s^2} = \frac{p_n}{2} \lim_{r\to 0} \frac{\mu(B(x, r))}{r^n} \]

whenever the latter exists.

**Proof.** Whenever \( B(x, t) \subseteq E \), we have

\[ I_\lambda(x; s, t) = p_n \int_s^t r^{1-n} (\nu_n r^n) dr = p_n \nu_n (t^2 - s^2)/2, \]

so that the result follows from Theorem 2.
Corollary 2. If \( u \) is \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \), then for each \( x \in E \) we have

\[
\mu(\{x\}) = \lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\log(t/s)} \quad \text{if } n = 2,
\]

and

\[
\mu(\{x\}) = \lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{s^{2-n} - t^{2-n}} \quad \text{if } n \geq 3.
\]

Proof. Writing \( \delta = \delta_x \), we have

\[
I_\delta(x; s, t) = pn \int_s^t r^{1-n} \, dr = \begin{cases} 
\log(t/s) & \text{if } n = 2, \\
\frac{s^{2-n} - t^{2-n}}{2-n} & \text{if } n \geq 3,
\end{cases}
\]

so that Theorem 2 gives the result. \( \square \)

3. A Representation Theorem

Theorem 2 and its corollaries enable us to prove a new representation theorem for \( \delta \)-subharmonic functions, which is analogous to known results about Poisson integrals on a ball due to Bruckner, Lohwater and Ryan [5], and on a half-space due to Armitage [1] and Watson [16].

**Theorem 3.** Let \( u \) be \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \). If

\[
\lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^2 - s^2} \geq 0
\]

for \( \lambda \)-almost all \( x \in E \), and

\[
\lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x_j, s) - \mathcal{M}(u, x_j, t)}{t^2 - s^2} = -\infty
\]

for only the points \( x_j \) in a countable set \( C \), then \( \mu \) can be written in the form

\[
\mu = \omega + \sum_j \left( \lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x_j, s) - \mathcal{M}(u, x_j, t)}{\log(t/s)} \right) \delta_j
\]

if \( n = 2 \),

\[
\mu = \omega + \sum_j \left( \lim_{0<s<t\to 0} \frac{\mathcal{M}(u, x_j, s) - \mathcal{M}(u, x_j, t)}{s^{2-n} - t^{2-n}} \right) \delta_j
\]

if \( n \geq 3 \).
if \( n \geq 3 \), where \( \omega \) is a positive measure such that \( \omega(C) = 0 \), and \( \delta_j \) is the unit mass at \( x_j \).

**Proof.** In view of Theorem 2 Corollary 1, condition (11) implies that

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} \geq 0
\]

for \( \lambda \)-almost all \( x \in E \), and condition (12) implies that

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} = -\infty
\]

only if \( x \in C \). Therefore, by Theorem 1,

\[
d\mu = f\,d\lambda + d\sigma^+ - d\sigma^-
\]

with \( f \geq 0 \) and \( \sigma^- \) concentrated on \( C \). Furthermore, for each \( j \), Theorem 2 Corollary 2 shows that the limits in (13) and (14) are equal to \( \mu(\{x_j\}) \). Thus

\[
d\mu = (f\,d\lambda + d\sigma^+) + \sum_j \mu(\{x_j\})\delta_j
\]

yields the required representation. \( \square \)

In particular, Theorem 3 allows the following characterization of a point mass.

**Corollary.** Let \( u \) be \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \). If

\[
\lim_{0 < s < t \to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^2 - s^2}
\]

is 0 for \( \lambda \)-almost all \( x \in E \), is finite except at \( x_0 \), and is \( \infty \) at \( x_0 \), then \( \mu \) is a positive constant multiple of the unit mass at \( x_0 \).

**Proof.** Applying Theorem 3 to \(-u\), we obtain

\[
-\mu = \omega - \left( \lim_{0 < s < t \to 0} \frac{\mathcal{M}(u, x_0, s) - \mathcal{M}(u, x_0, t)}{\log(t/s)} \right) \delta_0
\]

if \( n = 2 \),

\[
-\mu = \omega - \left( \lim_{0 < s < t \to 0} \frac{\mathcal{M}(u, x_0, s) - \mathcal{M}(u, x_0, t)}{s^{2-n} - t^{2-n}} \right) \delta_0
\]

if \( n \geq 3 \), where \( \omega \) is a positive measure such that \( \omega(\{x_0\}) = 0 \). By Theorem 2 Corollary 2, \(-\mu = \omega - \mu(\{x_0\})\delta_0\) in either case. Applying Theorem 3 to \( u \) itself, we find that \( \mu \) is positive, so that \( \omega \) is null. \( \square \)
4. Positive sets for associated measures

The proof of its corollary illustrates how Theorem 3 can sometimes be used to show that the measure associated with a $\delta$-subharmonic function is positive. Theorem 4 below is a refinement that allows us to determine which are the positive sets for the measure. It is similar in essence to the case $Y = \emptyset$ of [15] Theorem 6.

Recall that $\mu_S$ denotes the restriction of $\mu$ to the set $S$.

**Theorem 4.** Let $\mu$ be $\delta$-subharmonic with associated measure $\mu$ on $E$, let $S$ be a Borel subset of $E$, and let $\nu$ be a positive measure on $E$. If

$$
\limsup_{0<s<t \to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{I_\nu(x;s,t)}
$$

is not $-\infty$ for any $x \in S$, and is nonnegative for $\nu$-almost all $x \in S$, then $\mu_S \geq 0$.

**Proof.** By Theorem 2,

$$
\limsup_{r \to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}
$$

is not $-\infty$ for any $x \in S$, and is nonnegative for $\nu$-almost all $x \in S$. Therefore $\mu_S \geq 0$, by Theorem 1 Corollary 1. \hfill \Box

Theorem 4 contains the results of Sodin [12] which, in turn, are extensions of Grishin’s lemma [11]. Other extensions of Grishin’s lemma were obtained by Fuglede [10]. Our next corollary extends Sodin’s theorem to $n$-dimensions.

**Corollary 1.** Let $\mu$ be $\delta$-subharmonic with associated measure $\mu$ on $E$, and let $S$ be the set of points in $E$ with the following property: There are sequences $\{s_k\}$ and $\{t_k\}$, which depend on the point $x$, such that $0 < s_k < t_k \to 0$ and

$$
\mathcal{M}(u,x,s_k) \leq \mathcal{M}(u,x,t_k)
$$

(15)

for all $k$. Then $\mu_S \leq 0$.

**Proof.** Sodin proved that $S$ is a Borel set. If $x \in S$, then

$$
\liminf_{0<s<t \to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{I_\lambda(x;s,t)} < 0.
$$

Therefore $\mu_S \leq 0$, by Theorem 4. \hfill \Box

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Theorem 4 is much stronger than its first corollary. To see this, consider the case where \( d\mu(y) = f(y)\,dy \) with \( f \) continuous and nonnegative, and with the zero set \( Z \) of \( f \) nonempty but with empty interior. Then, whenever \( B(x, t) \subseteq E \) and \( 0 < s < t \), we have \( M(u, x, s) > M(u, x, t) \), so that the corollary can only be applied to \(-u\) and not to \( u \), and it yields only the inequality \( \mu \geq 0 \). However, for any \( x \in Z \) we have

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} = 0,
\]

so that

\[
\lim_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{I_\lambda(x; s, t)} = 0
\]

by Theorem 2. Now Theorem 4 can be applied to both \( u \) and \(-u\) (with \( S = Z \)), and confirms that \( \mu_Z \) is null.

The next corollary generalizes both of Sodin’s “remarks” to \( n \)-dimensions, with weaker hypotheses.

**Corollary 2.** Let \( u \) be \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \), and let \( S \) be a Borel subset of \( E \) on which there is defined a positive measure \( \nu \) such that for some constant \( \beta \geq 0 \)

\[
\nu(B(x, r)) \geq \kappa r^\beta
\]

whenever \( x \in S \) and \( 0 < r < r_x \), where \( \kappa = \kappa_x > 0 \). Let \( \alpha > 0 \), and let \( h \) be an absolutely continuous function on \([0, \alpha]\) such that \( h'(r) = o(r^{\beta - n + 1}) \) as \( r \to 0 \). If, to each \( x \in S \), there correspond sequences \( \{s_k\} \) and \( \{t_k\} \) such that \( 0 < s_k < t_k \to 0 \) and

\[
M(u, x, s_k) - M(u, x, t_k) \geq h(s_k) - h(t_k) \quad \forall k,
\]

then \( \mu_S \geq 0 \).

**Proof.** Given \( x \in S \) and \( \varepsilon > 0 \), for all sufficiently large \( k \) we have

\[
M(u, x, s_k) - M(u, x, t_k) \geq -\int_{s_k}^{t_k} h'(r)\,dr \geq -\varepsilon \kappa p_n \int_{s_k}^{t_k} r^{\beta - n + 1} \,dr
\]

\[
\geq -\varepsilon p_n \int_{s_k}^{t_k} r^{1 - n} \nu(B(x, r))\,dr = -\varepsilon I_\nu(x; s_k, t_k),
\]

so that

\[
\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{I_\nu(x; s_k, t_k)} \geq 0.
\]

By Theorem 4, \( \mu_S \geq 0 \). \( \square \)
In the above corollary, the case \( n = \beta = 2 \) is [12] Remark 1, which does not mention a measure \( \nu \). The choice \( \nu = \lambda \) gives the result. The case \( n = 2, \beta > 0 \), is [12] Remark 2. With regard to the existence of \( \nu \), Sodin mentioned only the work of Tricot [13]. However, there are many other results in this direction. For example, if \( S \) is a \( q \)-set for some \( q \in [0, n] \) (as, for example, in [8]), then the \( q \)-dimensional Hausdorff measure \( \nu \) on \( S \) satisfies \( \nu(B(x, r)) \sim (2r)^q \) as \( r \to 0 \), at every regular point of \( S \).

5. Specific rates

The next theorem generalizes one due to Armitage [2], which we deduce as a corollary.

**Theorem 5.** Let \( u \) be \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \). Let \( \alpha > 0 \), let \( f \) be a positive, increasing, absolutely continuous function on \([0, \alpha]\), and let

\[
\hat{f}(s, t) = p_n \int_s^t r^{1-n} f(r) \, dr
\]

whenever \( 0 \leq s < t \leq \alpha \). Then

\[
\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{f(s, t)} \leq \limsup_{r \to 0} \frac{\mu(B(x, r))}{f(r)}
\]

for every \( x \in E \).

**Proof.** Given \( x \), define a positive measure \( \nu \) on \( B(x, \alpha) \) by putting

\[
d\nu(y) = ||x - y||^{1-n} f'(||x - y||) \, dy + \sigma_n f(0) \, d\delta_x(y),
\]

where \( \sigma_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \). Then

\[
\nu(B(x, r)) = \sigma_n \int_0^r f'(s) \, ds + \sigma_n f(0) = \sigma_n f(r)
\]

if \( 0 < r \leq \alpha \), so that

\[
I_\nu(x; s, t) = p_n \int_s^t r^{1-n} \sigma_n f(r) \, dr = \sigma_n \hat{f}(s, t)
\]

whenever \( 0 < s < t \leq \alpha \). The result now follows from Theorem 2. \(\square\)
Armitage’s result did not involve differences of spherical mean values, and so required an additional hypothesis on \( \hat{f} \), as follows.

**Corollary 1.** Let \( u \) be \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \). Let \( \alpha > 0 \), let \( f \) be a positive, increasing, absolutely continuous function on \([0, \alpha]\), and let

\[
\hat{f}(s, t) = n \int_s^t r^{1-n} f(r) \, dr
\]

whenever \( 0 \leq s < t \leq \alpha \). If \( \hat{f}(0, \alpha) = \infty \), then

\[
\limsup_{s \to 0} \frac{M(u, x, s)}{f(s, \alpha)} \leq \limsup_{r \to 0} \frac{\mu(B(x, r))}{f(r)}
\]

for all \( x \in E \).

**Proof.** Given \( x \in E \), let \( \ell \) denote the left-hand side of (16). In view of (16), it suffices to prove that

\[
\limsup_{s \to 0} \frac{M(u, x, s)}{f(s, \alpha)} \leq \ell.
\]

We may assume that \( \ell < \infty \). Given a real number \( A > \ell \), choose \( \delta > 0 \) such that

\[
\frac{M(u, x, s) - M(u, x, t)}{f(s, t)} < A \quad \text{whenever } 0 < s < t < \delta.
\]

Fix \( t < \delta \). Given \( \varepsilon > 0 \), choose \( \eta < t \) such that both

\[
\frac{M(u, x, t)}{f(s, \alpha)} < \varepsilon \quad \text{and} \quad \frac{\hat{f}(t, \alpha)}{f(s, \alpha)} < \varepsilon
\]

whenever \( 0 < s < \eta \). Then

\[
\frac{M(u, x, s)}{f(s, \alpha)} = \frac{M(u, x, s) - M(u, x, t)}{f(s, t)} \cdot \frac{\hat{f}(s, t)}{f(s, \alpha)} + \frac{M(u, x, t)}{f(s, \alpha)} < A \left(1 - \frac{\hat{f}(t, \alpha)}{f(s, \alpha)}\right) + \varepsilon < \max\{A, (1 - \varepsilon)A\} + \varepsilon
\]

if \( 0 < s < \eta \), and (17) follows. \( \square \)
The extra generality of Theorem 5 over Corollary 1 allows us to generalize the corollary of Armitage’s theorem and remove its restrictions on \( q \).

**Corollary 2.** Let \( u \) be \( \delta \)-subharmonic with associated measure \( \mu \) on \( E \), and let \( x \in E \). Then
\[
\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{s^{q+2-n} - t^{q+2-n}} \leq \left( \frac{n-2}{n-q-2} \right) \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^q}
\]
if \( 0 \leq q < n - 2 \),
\[
\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{\log(t/s)} \leq p_n \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^{n-2}},
\]
and
\[
\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{s^{q+2-n} - t^{q+2-n}} \leq \left( \frac{p_n}{q+2-n} \right) \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^q}
\]
if \( q > n - 2 \).

**Proof.** If we take \( f(r) = r^q \) \( (q \geq 0) \) in Theorem 5, so that \( \hat{f}(s, t) = p_n \int_s^t r^{q+1-n} \, dr \), then \( \hat{f}(s, t) \) is equal to \( p_n \) times
\[
\frac{s^{q+2-n} - t^{q+2-n}}{n-q-2} \quad \text{if } q < n - 2,
\]
\[
\frac{\log(t/s)}{t^{q+2-n} - s^{q+2-n}} \quad \text{if } q = n - 2,
\]
\[
\frac{q+2-n}{q+2-n} \quad \text{if } q > n - 2,
\]
which gives the result. \( \square \)

If \( S \) is a regular \( q \)-set \( [8] \) contained in \( E \), and \( \mu \) is the \( q \)-dimensional Hausdorff measure on \( S \), then
\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{r^q} = 2^q
\]
for \( \mu \)-almost all \( x \in S \). Therefore, for such \( x \),
\[
\lim_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{s^{q+2-n} - t^{q+2-n}} = \frac{p_n 2^q}{n-q-2}
\]
if \( q \neq n - 2 \), and
\[
\lim_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{\log(t/s)} = p_n 2^q
\]
if \( q = n - 2 \), for any superharmonic function \( u \) whose associated measure is \( \mu \). These identities follow easily from Theorem 5 Corollary 2.
6. CONDITIONS FOR SUPERHARMONICITY

Theorem 2 can easily be re-written in a form that generalizes [2] Theorem 1, which we deduce as a corollary. This formulation is then used to provide conditions under which \( u - Av \) is superharmonic for some real number \( A \), as well as a new version of the domination principle.

**Theorem 6.** Let \( u \) be \( \delta \)-subharmonic and \( v \) superharmonic on \( E \), with associated measures \( \mu \) and \( \nu \) respectively. Then

\[
\limsup_{0<s<t\to 0} \frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)} \leq \limsup_{r\to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}
\]

whenever the latter exists. The reverse inequality holds for lower limits, and

\[
\lim_{0<s<t\to 0} \frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)} = \lim_{r\to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}
\]

for \( \nu \)-almost all \( x \in E \).

**Proof.** The result follows from Theorem 2, because

\[ M(v, x, s) - M(v, x, t) = I_v(x; s, t) \]

by [2] Lemma 3. \( \square \)

**Corollary.** Let \( u \) be \( \delta \)-subharmonic and \( v \) superharmonic on \( E \), with associated measures \( \mu \) and \( \nu \) respectively. If \( x \in E \) and \( v(x) = \infty \), then

\[
\limsup_{s \to 0} \frac{M(u, x, s)}{M(v, x, s)} \leq \limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}
\]

and the reverse inequality holds for lower limits.

**Proof.** Given \( x \in E \) such that \( v(x) = \infty \), let \( \ell \) denote the left-hand side of (18). In view of (18), it suffices to prove that

\[
\limsup_{s \to 0} \frac{M(u, x, s)}{M(v, x, s)} \leq \ell.
\]

We may assume that \( \ell < \infty \). Given a real number \( A > \ell \), choose \( \delta > 0 \) such that

\[
\frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)} < A \quad \text{whenever } 0 < s < t < \delta.
\]
Since $M(v, x, r) \to \infty$ as $r \to 0$, we may suppose that $M(v, x, r) > 0$ for all $r < \delta$. Fix $t < \delta$. Given $\varepsilon > 0$, choose $\eta < t$ such that both
\[
\frac{M(v, x, t)}{M(v, x, s)} < \varepsilon \quad \text{and} \quad \frac{M(u, x, t)}{M(v, x, s)} < \varepsilon
\]
whenever $0 < s < \eta$. Then
\[
\frac{M(u, x, s)}{M(v, x, s)} = \frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)} \left(1 - \frac{M(v, x, t)}{M(v, x, s)}\right) + \frac{M(u, x, t)}{M(v, x, s)} < \max\{A, (1 - \varepsilon)A\} + \varepsilon
\]
if $0 < s < \eta$. This proves (19). \[\square\]

We now use Theorem 6 to prove analogues of a domination theorem and a uniqueness theorem about Poisson integrals on half-spaces given in [1] and [16]. Conditions for the measure to be positive or null in that context translate into conditions for superharmonicity or harmonicity here.

**Theorem 7.** Let $u$ be $\delta$-subharmonic on $E$, and let $v$ be superharmonic on $E$ with associated measure $\nu$. If
\[
\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)}
\]
is never $-\infty$, and is greater than or equal to $A$ for $\nu$-almost all $x$, then $u - Av$ is superharmonic on $E$.

**Proof.** Let $\mu$ be the measure associated to $u$. By Theorem 6, our hypotheses imply that
\[
\limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}
\]
is never $-\infty$, and is greater than or equal to $A$ for $\nu$-almost all $x$. Therefore, we can use Theorem 1 Corollary 1 to show that $\mu - A\nu \geq 0$. Hence $u - Av$ is superharmonic on $E$. \[\square\]

Note that the case $A = 0$ of Theorem 7 gives a condition for $u$ itself to be superharmonic. Theorem 7 is analogous to both [16] Theorem 2 and an earlier result about Poisson integrals on a disc, [5] Theorem 2. It also implies the following condition for $u$ to be harmonic; compare [1] Theorem 4 and the comment on that result in [16] (p. 470).
**Corollary.** Let \( u \) be \( \delta \)-subharmonic on \( E \), and let \( v \) be superharmonic on \( E \) with associated measure \( \nu \). If

\[
\lim_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)}
\]

is finite whenever it exists, and is 0 for \( \nu \)-almost all \( x \), then \( u \) is harmonic on \( E \).

**Proof.** Applying Theorem 7 to both \( u \) and \(-u\), we see that both functions are superharmonic on \( E \). \( \square \)

A minor variation in the proof of Theorem 7 yields a new form of the Domination Principle ([7], pp. 67, 194).

**Theorem 8.** Let \( E \) be Greenian, let \( v = G_E \nu \) be the Green potential of a positive measure \( \nu \) on \( E \), and let \( u \) be a positive superharmonic function on \( E \). If

\[
\limsup_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)}
\]

is never \(-\infty\), and is greater than or equal to 1 for \( \nu \)-almost all \( x \), then \( u \geq G_E \nu \).

**Proof.** Let \( \mu \) be the measure associated to \( u \). As in the proof of Theorem 7, our hypotheses imply that \( \mu \geq \nu \), so that \( G_E \mu \geq G_E \nu \). Since \( u \) is positive, its greatest harmonic minorant is nonnegative, and so \( u \geq G_E \nu \). \( \square \)

7. Null sets for associated measures

Theorem 4 obviously implies a condition for a set to be null for the associated measure \( \mu \). In this section we state the result explicitly and relate it to known theorems. For example, if \( u \) is superharmonic on \( E \) and \( S = \{ x \in E : u(x) < \infty \} \), it is well-known that any polar subset of \( E \) is \( \mu_S \)-null ([7], p. 68). That result was generalized to \( \delta \)-subharmonic functions, with \( S \) replaced by \( \{ x \in E : \liminf_{y \to x} |u(y)| < \infty \} \), by Fuglede [10] Theorem 2.1. Theorem 9 Corollary 2 gives a different generalization.

**Theorem 9.** Let \( u \) be \( \delta \)-subharmonic and \( v \) superharmonic on \( E \), with associated measures \( \mu \) and \( \nu \) respectively, and let \( S \) be a Borel subset of \( E \). If

\[
\lim_{0 < s < t \to 0} \frac{M(u, x, s) - M(u, x, t)}{M(v, x, s) - M(v, x, t)}
\]

is not infinite for any \( x \in S \), and is zero for \( \nu \)-almost all \( x \in S \), then \( \mu_S \) is null.

**Proof.** Write \( M(v, x, s) - M(v, x, t) \) as \( I_\nu(x; s, t) \), and apply Theorem 4 to both \( u \) and \( -u \). \( \square \)
The first corollary gives a restricted version of the theorem which involves quotients of the form $\mathcal{M}(u, x, s)/\mathcal{M}(v, x, s)$, and thus parallels Theorem 6 Corollary.

**Corollary 1.** Let $u$ be $\delta$-subharmonic with associated measure $\mu$ on $E$, let $v$ be superharmonic on $E$, and let $S$ be a Borel subset of $E$. If, for each $x \in S$, $v(x) = \infty$ and there is a null sequence $\{r_k\}$ such that

$$\lim_{k \to \infty} \frac{\mathcal{M}(u, x, r_k)}{\mathcal{M}(v, x, r_k)} = 0,$$

then $\mu_S$ is null.

**Proof.** For any $x \in S$ we have $\mathcal{M}(v, x, r) \to \infty$ as $r \to 0$. Therefore, for any fixed $t$ such that $B(x, t) \subseteq E$,

$$\lim_{k \to \infty} \frac{\mathcal{M}(u, x, r_k) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, r_k) - \mathcal{M}(v, x, t)} = 0$$

in view of (20). Therefore

$$\lim_{0 < s < t \to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

is zero if it exists, and it exists for $\nu$-almost all $x$ (where $\nu$ is the measure associated to $v$) by Theorem 6. Now Theorem 9 shows that $\mu_S$ is null. \qed

**Corollary 2.** Let $u$ be $\delta$-subharmonic with associated measure $\mu$ on $E$. If $S$ is a Borel subset of $E$ such that for each $x \in S$

$$\liminf_{r \to 0} |\mathcal{M}(u, x, r)| < \infty,$$

then any polar subset of $E$ is $\mu_S$-null.

**Proof.** Let $N$ be a polar subset of $E$, and let $v$ be a superharmonic function on $E$ such that $v(x) = \infty$ for every $x \in N$. Then, for any $x \in S \cap N$, the condition (21) implies the existence of a null sequence $\{r_k\}$ such that (20) holds. By Corollary 1, $\mu_{S \cap N}$ is null. \qed

Corollary 1 is considerably stronger than Corollary 2. To illustrate this, we consider an open ball $B$ with a $G_\delta$ polar subset $N$. We construct two positive superharmonic functions $u, v$ on $B$, with $u(x) = v(x) = \infty$ for all $x \in N$, such that $\mu(N) = 0$.
(see [2], p. 61) and $\nu(B \setminus N) = 0$ (see [6]), where $\mu, \nu$ are the measures associated to $u, v$ respectively. Since $\mu$ and $\nu$ are mutually singular, we have

$$\lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = 0$$

for $\nu$-almost all $x$ [3], so that

$$\lim_{r \to 0} \frac{\mathcal{M}(u, x, r)}{\mathcal{M}(v, x, r)} = 0$$

by [2] Theorem 1. So Corollary 1 confirms that there is a $\nu$-null set $M$ such that $\mu_{N \setminus M}$ is null, but Corollary 2 is inapplicable because (21) fails to hold for any $x \in N$.

8. More extensions of known results

We conclude with two extensions of results in [14].

**Theorem 10.** Let $u$ be $\delta$-subharmonic and $v$ superharmonic on $E$, with associated measures $\mu$ and $\nu$ respectively. Let

$$f(x) = \lim_{0 < s < t \to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

whenever the limit exists, let $Z^+ = \{ x \in E : f(x) = \infty \}$, and let $Z^- = \{ x \in E : f(x) = -\infty \}$. Then $f$ is defined and finite $\nu$-a.e. on $E$, and there are positive $\nu$-singular measures $\sigma^+$ and $\sigma^-$, concentrated on $Z^+$ and $Z^-$ respectively, such that $d\mu = f d\nu + d\sigma^+ - d\sigma^-$. 

**Proof.** By Theorem 6,

$$f(x) = \lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

whenever this limit exists. The result now follows from Theorem 1. \qed

Note that, if $f(x)$ is finite whenever it exists, then $\mu$ is absolutely continuous with respect to $\nu$.

Theorem 10 generalizes [14] Theorem 6, which we now deduce as a corollary.

**Corollary.** Let $u$ be $\delta$-subharmonic and $v$ superharmonic on $E$, with associated measures $\mu$ and $\nu$ respectively, let $X = \{ x \in E : v(x) = \infty \}$, let

$$g(x) = \lim_{r \to 0} \frac{\mathcal{M}(u, x, r)}{\mathcal{M}(v, x, r)}$$

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whenever the limit exists, let \( Z^+ = \{ x \in X : g(x) = \infty \} \), and let \( Z^- = \{ x \in X : g(x) = -\infty \} \). Then \( g \) is defined and finite \( \nu \)-a.e. on \( X \), and there are positive \( \nu \)-singular measures \( \sigma^+ \) and \( \sigma^- \), concentrated on \( Z^+ \) and \( Z^- \) respectively, such that \( d\mu_X = g d\nu_X + d\sigma^+ - d\sigma^- \).

**Proof.** If \( x \in X \), then

\[
\limsup_{r \to 0} \frac{\mathcal{M}(u, x, r)}{\mathcal{M}(v, x, r)} \leq \limsup_{0 < s < t \to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}
\]

by (19), and the reverse inequality holds for lower limits. Therefore \( g(x) \) is equal to the \( f(x) \) in Theorem 10, whenever \( f(x) \) exists. The result follows. \( \Box \)

Theorem 10 enables us to prove a corresponding generalization of [14] Theorem 8, as follows. This generalization provides conditions under which a Borel set is a positive set for the Riesz measure of a \( \delta \)-subharmonic function, whereas [14] Theorem 8 applied only to a Borel polar set.

**Theorem 11.** Let \( u \) be \( \delta \)-subharmonic and \( v \) superharmonic on \( E \), with associated measures \( \mu \) and \( \nu \) respectively. Let \( q \in [0, n-2] \), and let \( S \) be a Borel subset of \( E \). If

\[
\limsup_{0 < s < t \to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} > -\infty
\]

for all \( x \in S \setminus Y \), where \( Y \) is an \( m_q \)-null Borel set, if

\[
\limsup_{0 < s < t \to 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} \geq 0
\]

for \( \nu \)-almost all \( x \in S \setminus Y \), and if

\[
(22) \quad \liminf_{r \to 0} r^{n-q-2} \mathcal{M}(u, x, r) > -\infty
\]

for \( |\mu| \)-almost all \( x \in Y \), then \( \mu_S \geq 0 \). If (22) is replaced by

\[
\liminf_{r \to 0} r^{n-q-2} \mathcal{M}(u, x, r) \geq 0,
\]

then the result remains valid if \( 0 < m_q(Y) < \infty \).

**Proof.** Follow the proof of [14] Theorem 8, but use Theorem 10 above instead of [14] Theorem 6. \( \Box \)
References


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