Abstract. In the paper the differential inequality
\[ \Delta_p u + B(x, u) \leq 0, \]
where \( \Delta_p u = \text{div}(\|\nabla u\|^{p-2}\nabla u) \), \( p > 1 \), \( B(x, u) \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \) is studied. Sufficient conditions on the function \( B(x, u) \) are established, which guarantee nonexistence of an eventually positive solution. The generalized Riccati transformation is the main tool.

Keywords: \( p \)-Laplacian, oscillation criteria

MSC 2000: 35B05

1. Introduction

In the paper we study positive solutions of the partial differential inequality
\[ \Delta_p u + B(x, u) \leq 0, \]
where \( \Delta_p u = \text{div}(\|\nabla u\|^{p-2}\nabla u) \) is the \( p \)-Laplace operator, \( p > 1 \), \( B(x, u): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is a continuous function, \( \| \cdot \| \) is the usual Euclidean norm in \( \mathbb{R}^n \). Inequality (1) covers several equations and inequalities studied in literature. If \( p = 2 \) then (1) reduces to the semilinear Schrödinger inequality
\[ \Delta u + B(x, u) \leq 0, \]

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studied in [6], [7]. Another important special case of (1) is the half-linear differential equation

\[ \Delta_p u + c(x)|u|^{p-1} \text{sgn} u = 0, \]

studied in [2], [3]. For important applications of equations with \( p \)-Laplacian see [1].

The aim of this paper is to introduce sufficient conditions for nonexistence of a solution which would be eventually positive (i.e., positive outside of some ball in \( \mathbb{R}^n \)). Remark that in a similar way one can study also negative solutions of the inequality

\[ \Delta_p u + B(x, u) \geq 0, \]

and a combination of these results produces criteria for nonexistence of a solution of the inequality

\[ u[\Delta_p u + B(x, u)] \leq 0 \]

which would have no zero outside of some ball in \( \mathbb{R}^n \), the so-called weak oscillation criteria. A simple version of this procedure is used in Corollary 6. A more elaborated version of this procedure can be found in [6].

The following notation is used throughout the paper: \( \langle \cdot, \cdot \rangle \) denotes the scalar product, \( q = \frac{2}{p-1} \) is the conjugate number to the number \( p \),

\[
\begin{align*}
\Omega(a, b) &= \{ x \in \mathbb{R}^n : a \leq ||x|| \leq b \}, \\
\Omega_a &= \Omega(a, \infty) = \{ x \in \mathbb{R}^n : a \leq ||x|| \}, \\
S_a &= \partial \Omega_a = \{ x \in \mathbb{R}^n : ||x|| = a \},
\end{align*}
\]

and \( \omega_1 = \int_{S_1} 1 \, d\sigma \) is the measure of the \( n \)-dimensional unit sphere in \( \mathbb{R}^n \).

2. Riccati transformation

The main tool used for the study of positive solutions is the generalized Riccati transformation. The special case of this transformation has been used in [6], where inequality (2) is studied. A simple version of this transformation, convenient for the half-linear equation, has been introduced in [2].

Our approach combines both these methods. We use the transformation

\[ \tilde{w}(x) = -\alpha(||x||) \frac{\|\nabla u(x)\|^{p-2} \nabla u(x)}{\varphi(u(x))} \]

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\( \alpha \in C^1([a_0, \infty), \mathbb{R}^+), \varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+) \) which maps a positive \( C^1 \) function \( u(x) \) into an \( n \)-vector function \( \vec{w}(x) \).

**Lemma 1.** Let \( u \) be a positive solution of (1) on \( \Omega_{a_0} \). The \( n \)-vector function \( \vec{w}(x) \) is well-defined by (5) and satisfies the Riccati-type inequality

\[
\text{div } \vec{w}(x) \geq \frac{\alpha(||x||)B(x, u(x))}{\varphi(u(x))} + \frac{\alpha'(||x||)}{\alpha(||x||)} (\vec{\nu}(x), \vec{w}(x)) + \alpha^{1-q}(||x||)\varphi^{q-2}(u(x))\varphi'(u(x))||\vec{w}(x)||^q
\]

on \( \Omega_{a_0} \), where \( \vec{\nu}(x) = \frac{x}{||x||} \) is the outward unit normal vector to the sphere \( S||x|| \).

**Proof.** Let \( u(x) \geq 0 \) be a solution of (1) on \( \Omega_{a_0} \) and let \( \vec{w}(x) \) be defined by (5). From (5) it follows that

\[
\text{div } \vec{w} = \frac{\alpha}{\varphi(u)} \Delta u - ||\nabla u||^{p-2} \left( \nabla u, \nabla \left( \frac{\alpha}{\varphi(u)} \right) \right)
\]

and in view of (1)

\[
\text{div } \vec{w} \geq \frac{\alpha B(x, u)}{\varphi(u)} - \frac{\alpha' ||\nabla u||^{p-2}}{\varphi(u)} (\nabla u, \vec{\nu}) + \frac{\alpha \varphi'(u)}{\varphi^2(u)} ||\nabla u||^p
\]

holds (the dependence on \( x \in \Omega_{a_0} \) is suppressed in the notation). In view of (5) this inequality is equivalent to (6). \( \square \)

3. **Nonexistence of Positive Solution**

The main result of the paper is the following

**Theorem 1.** Let \( a_0 \geq 0 \). Suppose that there exist functions

\( \alpha \in C^1([a_0, \infty), \mathbb{R}^+), \varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+), \ c \in C(\mathbb{R}^n, \mathbb{R}), \)

and numbers \( k, l, k > 0, l > 1, \) such that

(i) \( B(x, u) \geq c(x)\varphi(u) \) for \( x \in \mathbb{R}^n, u > 0, \)

(ii) \( \varphi'(u)\varphi^{q-2}(u) \geq k \) for \( u > 0, \)

(iii) \( \lim_{r \to \infty} \int_{\Omega(a_0, r)} \left[ \alpha(||x||)c(x) - \frac{1}{p} \left( \frac{l}{m} \right)^{p-1} \alpha'(||x||) \alpha^{-p} ||x|| \right] \ dx = +\infty, \)

(iv) \( \lim_{r \to \infty} \int_{a_0}^{r} \alpha^{-\frac{1}{q-r}}(r) \frac{dx}{r} = +\infty. \)

Then (1) has no positive solution on \( \Omega_a \) for arbitrary \( a > 0. \)
Proof. Suppose, by contradiction, that \( u \) is a solution of (1) positive on \( \Omega_a \) for some \( a > a_0 \). Lemma 1 and the assumptions (i), (ii) imply

\[
\text{div} \, \bar{w} \geq \alpha c + \alpha' \langle \bar{v}, \bar{w} \rangle + \alpha^{1-q} k \| \bar{w} \|^q
\]

\[
= \alpha c + \alpha^{1-q} k \frac{q}{l} \left[ \frac{\| \bar{w} \|^q}{q} + \langle \bar{w}^{\alpha q -2} \alpha' \rangle \right] + \alpha^{1-q} k \frac{1}{l^*} \| \bar{w} \|^q,
\]

where \( l^* = \frac{l}{l^{1+}} \) is the conjugate number to the number \( l \). The Young inequality implies

\[
\frac{\| \bar{w} \|^q}{q} + \langle \bar{w}^{\alpha q -2} \alpha' \rangle + \frac{1}{p} \left( \frac{\alpha q -2 |\alpha'|}{qk} \right)^p \geq 0.
\]

Combining both these inequalities we obtain

\[
\text{div} \, \bar{w} \geq \alpha c - \alpha^{1-q} k \frac{q}{l^p} \left( \frac{\alpha q -2 |\alpha'|}{qk} \right)^p + \alpha^{1-q} k \frac{1}{l^*} \| \bar{w} \|^q
\]

\[
= \alpha c - \frac{1}{p} \left( \frac{l}{qk} \right)^{p-1} |\alpha'|^p \alpha^{1-p} + \alpha^{1-q} k \frac{1}{l^*} \| \bar{w} \|^q.
\]

Integration of the last inequality over \( \Omega(a, r) \) and the Gauss-Ostrogradski divergence theorem gives

\[
\int_{S_r} \langle \bar{w}, \bar{v} \rangle \, d\sigma - \int_{S_a} \langle \bar{w}, \bar{v} \rangle \, d\sigma \geq \frac{k}{l^*} \int_{\Omega(a, r)} \alpha^{1-q} \| \bar{w} \|^q \, dx
\]

\[
+ \int_{\Omega(a, r)} \left[ \alpha c - \frac{1}{p} \left( \frac{l}{qk} \right)^{p-1} |\alpha'|^p \alpha^{1-p} \right] \, dx.
\]

By assumption (iii) there exists \( r_0, r_0 > a \), such that

\[
\int_{\Omega(a, r)} \left[ \alpha c - \frac{1}{p} \left( \frac{l}{qk} \right)^{p-1} |\alpha'|^p \alpha^{1-p} \right] \, dx + \int_{S_a} \langle \bar{w}, \bar{v} \rangle \, d\sigma \geq 0 \quad \text{for } r > r_0.
\]

Hence

\[
\int_{S_r} \langle \bar{w}, \bar{v} \rangle \, d\sigma \geq \frac{k}{l^*} g(r)
\]

holds for \( r > r_0 \), where

\[
g(r) = \int_{\Omega(a, r)} \alpha^{1-q}(\| \bar{w} \|) \| \bar{w}(x) \|^q \, dx.
\]

The Hölder inequality gives

\[
\int_{S_r} \langle \bar{w}, \bar{v} \rangle \, d\sigma \leq \left( \int_{S_r} \| \bar{w} \|^q \, d\sigma \right)^{\frac{1}{q}} \left( \int_{S_r} 1 \, d\sigma \right)^{\frac{1}{p}} = \alpha^{\frac{1}{p}}(r)(g'(r))^\frac{1}{q} \omega_r^\frac{1}{q} r^{-\frac{n}{p}}.
\]

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From (7) and (8) we obtain
\[(g'(r))^\frac{1}{q} \alpha^\frac{1}{q}(r) \omega^\frac{1}{q} r^{\frac{n-1}{p}} \geq \frac{k}{r} g(r) \quad \text{for } r \geq r_0\]
and equivalently
\[\frac{g'(r)}{g(r)} \geq \left( \frac{k}{r} \right)^q \alpha^{-\frac{1}{q}}(r) r^{(1-n)\frac{1}{p}} = \left( \frac{k}{r} \right)^q \alpha^{-\frac{1}{q}}(r) r^{\frac{n-1}{p}} \quad \text{for } r \geq r_0.
\]
Integration of this inequality over the interval \((r_0, \infty)\) gives a convergent integral on the left-hand side and a divergent integral on the right-hand side of this inequality, by virtue of the assumption (iv). This contradiction completes the proof.

Remark 1. For \(\varphi(u) = u^{p-1}\) we have \(\varphi'(u)\varphi^{q-2}(u) = p - 1\) and the assumption (ii) holds with \(k = p - 1\). Conversely, \(\varphi(u) \geq \left( \frac{k}{p-1} \right)^{p-1} u^{p-1}\) is necessary for (ii) to be satisfied. Remark also that neither sign restrictions, nor radial symmetry, are supposed for the function \(c(x)\) in (i).

**Corollary 2** (Leighton type criterion). Let \(p \geq n\). Suppose that there exists a continuous function \(c(x)\) such that
\[B(x, u) \geq c(x) u^{p-1} \quad \text{for } u > 0\]
and
\[
\lim_{r \to \infty} \int_{\Omega(1, r)} c(x) \, dx = +\infty.
\]
Then (1) has no positive solution on \(\Omega_a\) for arbitrary \(a > 0\).

**Proof.** Follows from Theorem 1 for \(\alpha(r) \equiv 1\) and \(\varphi(u) = u^{p-1}\).

Remark 2. Remark that (10) is known to be a sufficient condition for oscillation of (3) provided \(p \geq n\), see [2]. It is also known that the condition \(p \geq n\) in this criterion cannot be omitted.

**Corollary 3.** Suppose that (9) holds and there exists \(m > 1\) such that
\[
\lim_{r \to \infty} \int_{\Omega(1, r)} \left[ \|x\|^{p-n} c(x) - m \left| \frac{p-n}{p} \|x\|^{\frac{p-1}{p}} \right| \right] \, dx = +\infty.
\]
Then (1) has no positive solution on \(\Omega_a\) for arbitrary \(a > 0\).

**Proof.** Follows from Theorem 1 for \(\alpha(r) = r^{p-n}\) and \(\varphi(u) = u^{p-1}, \, m = l^{p-1}\).

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Remark 3. If the limit \( \lim_{r \to \infty} \frac{1}{\ln r} \int_{B(1,r)} \|x\|^{p-n} c(x) \, dx \) exists, or if this limit equals \(+\infty\), then (11) is equivalent to the condition

\[
\lim_{r \to \infty} \frac{1}{\ln r} \int_{B(1,r)} \|x\|^{p-n} c(x) \, dx > \omega_1 \left| \frac{n-n}{p} \right|^p.
\]

This condition is very close to the criterion for oscillation of the half-linear equation [5, Corollary 2.1], which contains “\( \lim sup \)” instead of “\( \lim \)” and one additional condition

\[
\lim_{r \to \infty} \left[ r^{p-1} \left( C_0 - \int_{B(1,r)} \|x\|^{1-n} c(x) \, dx \right) \right] > -\infty,
\]

where

\[
C_0 = \lim_{r \to \infty} \frac{p-1}{r^{p-1}} \int_1^r t^{p-2} \int_{B(1,t)} \|x\|^{1-n} c(x) \, dx \, dt.
\]

Among other, the constant \( \left| \frac{n-n}{p} \right|^p \) is here shown to be optimal.

**Corollary 4.** Let \( p \geq n, p > 2, (9) \) and

\[
\lim_{r \to \infty} \int_{B(1,r)} \ln(\|x\|) c(x) \, dx = +\infty.
\]

Then (1) has no positive solution on \( B(\alpha, \rho) \) for arbitrary \( \alpha > 0 \).

**Proof.** Let \( \alpha > e, p \geq n, p > 2, \alpha(r) = \ln r \). Since

\[
\lim_{r \to \infty} \frac{\alpha^r}{r} = \lim_{r \to \infty} \frac{r^{p-1} \ln^{\frac{p}{p-1}} r}{r^{p-1}} = 1,
\]

the condition (iv) of Theorem 1 holds. Further,

\[
\int_{B(1,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|) \, dx = \omega_1 \int_e^r \xi^{n-1-p} \ln^{1-p} \xi \, d\xi
\]

\[
\leq \omega_1 \int_e^r \xi \ln \xi \xi \, d\xi = \omega_1 \frac{1}{p-2} [1 - \ln^{2-p} r].
\]

Hence \( \lim_{r \to \infty} \int_{B(1,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|) \, dx \) exists and (12) is equivalent to the condition (iii) of Theorem 1. Now Theorem 1 implies the conclusion. \( \square \)

The choice \( \alpha(r) = \ln^{\beta} r \) leads to

**Corollary 5.** Let \( p \geq n, \) let (9) hold and suppose that there exists \( \beta, \beta \in (0, p-1) \) such that

\[
\lim_{r \to \infty} \int_{B(1,r)} \ln^{\beta}(\|x\|) c(x) \, dx = +\infty.
\]

Then (1) has no positive solution on \( B(\alpha, \rho) \) for arbitrary \( \alpha > 0 \).

**Proof.** The proof is a complete analogue of the proof of Corollary 4. \( \square \)
Following terminology in [6], a function \( f : \Omega \to \mathbb{R} \) is called \textit{weakly oscillatory} if and only if \( f(x) \) has a zero in \( \Omega \cap \Omega_a \) for every \( a > 0 \). The inequality (4) is called \textit{weakly oscillatory} in \( \Omega \) whenever every solution \( u \) of the inequality is oscillatory in \( \Omega \).

\textbf{Corollary 6.} Let \( B(x, u) : \mathbb{R}^{n+1} \to \mathbb{R} \) be a continuous function which is odd with respect to the variable \( u \), i.e. let \( B(x, -u) = -B(x, u) \). Let the assumptions of Theorem 1 be satisfied. Then inequality (4) is weakly oscillatory in \( \mathbb{R}^n \).

\textbf{Proof.} Suppose that there exists \( a > 0 \) such that inequality (4) has a solution \( u \) without zeros on \( \Omega_a \). If \( u \) is a positive function, then Theorem 1 yields a contradiction. Further, if \( u \) is a negative solution on \( \Omega_a \), then \( v(x) := -u(x) \) is a positive solution of (4) on \( \Omega_a \) and the same argument as in the first part of this proof leads to a contradiction. \( \square \)

3.1. Perturbed half-linear differential inequality. Let us consider a perturbed half-linear differential inequality

\begin{equation}
\Delta_p u + c(x)|u|^{p-1}\text{sgn } u + \sum_{i=1}^{m} q_i(x)\psi_i(u) \leq 0, \tag{13}
\end{equation}

where \( c(x), q_i(x) \) are continuous functions, \( \psi_i(u) \) are continuously differentiable, positive and nondecreasing for \( u > 0 \). Define

\[ q(x) = \min\{c(x), q_1(x), q_2(x), \ldots, q_m(x)\} \]

and

\[ \varphi(u) = u^{p-1} + \sum_{i=1}^{m} \psi_i(u). \]

Then

\[ c(x)|u|^{p-1}\text{sgn } u + \sum_{i=1}^{m} q_i(x)\psi_i(u) \geq q(x)\varphi(u), \quad \varphi'(u)\varphi'^{-2}(u) \geq p - 1 \]

and hence Theorem 1 can be applied. Remark that since \( q_i \) may change sign, a standard argument based on the Sturmian majorant and a comparison with half-linear differential equation (3) cannot be applied (as has been explained for \( p = 2 \) already in [6]).
References


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