ON $k$-STRONG DISTANCE IN STRONG DIGRAPHS

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(Received January 15, 2001)

Abstract. For a nonempty set $S$ of vertices in a strong digraph $D$, the strong distance $d(S)$ is the minimum size of a strong subdigraph of $D$ containing the vertices of $S$. If $S$ contains $k$ vertices, then $d(S)$ is referred to as the $k$-strong distance of $S$. For an integer $k \geq 2$ and a vertex $v$ of a strong digraph $D$, the $k$-strong eccentricity $se_k(v)$ of $v$ is the maximum $k$-strong distance $d(S)$ among all sets $S$ of $k$ vertices in $D$ containing $v$. The minimum $k$-strong eccentricity among the vertices of $D$ is its $k$-strong radius $srad_k(D)$ and the maximum $k$-strong eccentricity is its $k$-strong diameter $siam_k(D)$. The $k$-strong center ($k$-strong periphery) of $D$ is the subdigraph of $D$ induced by those vertices of $k$-strong eccentricity $srad_k(D)$ ($siam_k(D)$). It is shown that, for each integer $k \geq 2$, every oriented graph is the $k$-strong center of some strong oriented graph. A strong oriented graph $D$ is called strongly $k$-self-centered if $D$ is its own $k$-strong center. For every integer $r \geq 6$, there exist infinitely many strongly 3-self-centered oriented graphs of 3-strong radius $r$. The problem of determining those oriented graphs that are $k$-strong peripheries of strong oriented graphs is studied.

Keywords: strong distance, strong eccentricity, strong center, strong periphery

MSC 2000: 05C12, 05C20

1. Introduction

The familiar distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph is the length of a shortest $u - v$ path in $G$. Equivalently, this distance is the minimum size of a connected subgraph of $G$ containing $u$ and $v$. This concept was extended in [2] to connected digraphs, in particular to strongly connected (strong) oriented graphs. We refer to [4] for graph theory notation and terminology not described here.

Research supported in part by the Western Michigan University Arts and Sciences Teaching and Research Award Program.
A digraph $D$ is strong if for every pair $u, v$ of distinct vertices of $D$, there is both a directed $u - v$ path and a directed $v - u$ path in $D$. A digraph $D$ is an oriented graph if $D$ is obtained by assigning a direction to each edge of a graph $G$. The graph $G$ is referred to as the underlying graph of $D$. In this paper we will be interested in strong oriented graphs. The underlying graph of a strong oriented graph is necessarily 2-edge-connected. Let $D$ be a strong oriented graph of order $n \geq 3$ and size $m$. For two vertices $u$ and $v$ of $D$, the strong distance $sd(u,v)$ between $u$ and $v$ is defined in [2] as the minimum size of a strong subdigraph of $D$ containing $u$ and $v$. If $u \neq v$, then $3 \leq sd(u,v) \leq m$. In the strong oriented graph $D$ of Figure 1, $sd(v,w) = 3$, $sd(u,y) = 4$, and $sd(u,x) = 5$.

![Figure 1. A strong oriented graph](image)

A generalization of distance in graphs was introduced in [5]. For a nonempty set $S$ of vertices in a connected graph $G$, the Steiner distance $d(S)$ of $S$ is the minimum size of a connected subgraph of $G$ containing $S$. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to $S$. We now extend this concept to connected strong digraphs. For a nonempty set $S$ of vertices in a strong digraph $D$, the strong Steiner distance $d(S)$ is the minimum size of a strong subdigraph of $D$ containing $S$. We will refer to such a subgraph as a Steiner subdigraph with respect to $S$, or, simply, $S$-subdigraph. Since $D$ itself is strong, $d(S)$ is defined for every nonempty set $S$ of vertices of $D$. We denote the size of a digraph $D$ by $m(D)$. If $|S| = k$, then $d(S)$ is referred to as the $k$-strong Steiner distance (or simply $k$-strong distance) of $S$. Thus $3 \leq d(S) \leq m(D)$ for each set $S$ of vertices in a strong digraph $D$ with $|S| \geq 2$. Then the 2-strong distance is the strong distance studied in [2], [3]. For example, in the strong oriented graph $D$ of Figure 1, let $S_1 = \{u,v,x\}$, $S_2 = \{u,v,y\}$, and $S_3 = \{v,w,y\}$. Then the 3-strong distances of $S_1$, $S_2$, and $S_3$ are $d(S_1) = 5$, $d(S_2) = 4$, and $d(S_3) = 3$.

It was shown in [2] that strong distance is a metric on the vertex set of a strong oriented graph $D$. As such, certain properties are satisfied. Among these are: (1) $sd(u,v) \geq 0$ for vertices $u$ and $v$ of $D$ and $sd(u,v) = 0$ if and only if $u = v$ and (2) $sd(u,w) \leq sd(u,v) + sd(v,w)$ for vertices $u, v, w$ of $D$. These two properties can be considered in a different setting. Let $D$ be a strong oriented graph and let $S \subseteq V(D)$, where $S \neq \emptyset$. Then $d(S) \geq 0$ and $d(S) = 0$ if and only if $|S| = 1$, which is property (1). Let $S_1 = \{u, w\}$, $S_2 = \{u, v\}$, and $S_3 = \{v, w\}$. Then the triangle inequality $sd(u,w) \leq sd(u,v) + sd(v,w)$ given in (2) can be restated as $d(S_1) \leq d(S_2) + d(S_3)$.

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where, of course, $|S_i| = 2$ for $1 \leq i \leq 3$, $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$. We now describe an extension of (2).

**Proposition 1.1.** For an integer $k \geq 2$, let $S_1, S_2, S_3$ be sets of $k$ vertices in a strong oriented graph with $|S_i| = k$ for $1 \leq i \leq 3$. If $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$, then

$$d(S_1) \leq d(S_2) + d(S_3).$$

**Proof.** Let $D_i$ be an $S_i$-digraph of size $d(S_i)$ for $i = 1, 2, 3$. Define a digraph $D'$ to be the subdigraph of $D$ with vertex set $V(D_2) \cup V(D_3)$ and arc set $E(D_2) \cup E(D_3)$. Since $S_2 \cap S_3 \neq \emptyset$ and $D_2$ and $D_3$ are strong subdigraphs of $D$, it follows that $D'$ is also a strong subdigraph of $D$ with $S_1 \subseteq V(D')$. Thus $m(D_1) \leq m(D')$. Therefore,

$$d(S_1) = m(D_1) \leq m(D') \leq m(D_2) + m(D_3) = d(S_2) + d(S_3),$$

as desired. 

As an example, consider the strong oriented graph $D$ of Figure 2. Let $S_1 = \{s, v, x\}$, $S_2 = \{v, x, z\}$, and $S_3 = \{s, x, y\}$. Then $|S_i| = 3$ for $1 \leq i \leq 3$, where $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$. For each $i$ with $1 \leq i \leq 3$, let $D_i$ be an $S_i$-subdigraph of size $d(S_i)$ in $D$, which is also shown in Figure 2. Hence $d(S_1) = 3$, $d(S_2) = 4$, and $d(S_3) = 5$. Note that the subdigraph $D'$ of $D$ described in the proof of Proposition 1.1 has size 6. Thus $d(S_1) \leq m(D') \leq d(S_2) + d(S_3)$.

![Diagram](image.png)

Figure 2. An example of an extension of (2)

The extended triangle inequality $d(S_1) \leq d(S_2) + d(S_3)$ stated in Proposition 1.1 suggests a generalization of strong distance in strong oriented graphs, which we introduce in this paper.
2. On $k$-strong eccentricity, radius, and diameter

Let $v$ be a vertex of a strong oriented graph $D$ of order $n \geq 3$ and let $k$ be an integer with $2 \leq k \leq n$. The $k$-strong eccentricity $s_e(v)$ is defined by

$$s_e(v) = \max\{d(S); \ S \subseteq V(D), v \in S, |S| = k\}.$$ 

The $k$-strong diameter $s_{diam}(D)$ is

$$s_{diam}(D) = \max\{s_e(v); \ v \in V(D)\};$$

while the $k$-strong radius $s_{rad}(D)$ is defined by

$$s_{rad}(D) = \min\{s_e(v); \ v \in V(D)\}.$$ 

To illustrate these concepts, consider the strong oriented graph $D$ of Figure 3. The 3-strong eccentricity of each vertex of $D$ is shown in Figure 3. Thus $s_{rad}(D) = 8$ and $s_{diam}(D) = 12$.

![Figure 3. A strong oriented graph $D$ with $s_{rad}(D) = 8$ and $s_{diam}(D) = 12$](image)

For a nontrivial strong oriented graph $D$ of order $n$, the radius sequence $S_r(D)$ of $D$ is defined as

$$S_r(D): s_{rad}(D), s_{rad}(D), s_{rad}(D), \ldots, s_{rad}(D)$$

and the diameter sequence $S_d(D)$ of $D$ is defined as

$$S_d(D): s_{diam}(D), s_{diam}(D), s_{diam}(D), \ldots, s_{diam}(D).$$

For example, the strong oriented graph $D$ in Figure 4 has order 9. Since $s_{rad}(D) = 6$, $s_{rad}(D) = 9$, and $s_{rad}(D) = 12$ for $4 \leq k \leq 9$, it follows that $S_r(D): 6, 9, 12, 12, \ldots, 12$. Moreover, $s_{diam}(D) = 9$ and $s_{diam}(D) = 12$ for $3 \leq k \leq 9$. 

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Thus $S_d(D) = 9, 12, 12, \ldots, 12$. Note that both $S_r(D)$ and $S_d(D)$ are nondecreasing sequences. This is no coincidence, as we now see.

**Proposition 2.1.** For a nontrivial strong oriented graph $D$ of order $n$ and every integer $k$ with $2 \leq k \leq n - 1$,

- (a) $srad_k(D) \leq srad_{k+1}(D)$ and
- (b) $sdiam_k(D) \leq sdiam_{k+1}(D)$.

**Proof.** To verify (a), let $u$ and $v$ be two vertices of $D$ with $se_k(u) = srad_k(D)$ and $se_{k+1}(v) = srad_{k+1}(D)$. Let $S$ be a set of $k$ vertices of $D$ such that $se_k(u) = d(S) = srad_k(D)$. Now let $x$ be a vertex of $D$ such that $x = v$ if $v \notin S$ and $x \in V(D) - S$ if $v \in S$. Let $S' = \{x\} \cup S$. Since $S \subseteq S'$, it follows that $d(S) \leq d(S')$. Moreover, $S'$ is a set of $k + 1$ vertices of $D$ containing $v$ and so $d(S') \leq se_{k+1}(v)$. Thus

$$srad_k(D) = d(S) \leq d(S') \leq se_{k+1}(v) = srad_{k+1}(D)$$

and so (a) holds. To verify (b), let $S$ be a set of $k$ vertices of $D$ with $d(S) = sdiam_k(D)$. If $S'$ is any set of $k + 1$ vertices of $D$ with $S \subseteq S'$, then

$$sdiam_k(D) = d(S) \leq d(S') \leq sdiam_{k+1}(D)$$

and so (b) holds.

Equalities in (a) and (b) of Proposition 2.1 hold for certain strong oriented graphs, for example, the directed $n$-cycle $C_n^*$ for $n \geq 3$. In fact, $srad_k(C_n^*) = sdiam_k(C_n^*) = n$ for all $k$ with $2 \leq k \leq n$. As another example, let $D$ be the strong oriented graph of order $n \geq 3$ with $V(D) = \{v_1, v_2, \ldots, v_n\}$ such that for $1 \leq i < j \leq n$, $(v_i, v_j) \in E(D)$, except when $i = 1$ and $j = n$, and $(v_n, v_1) \in E(D)$ (see Figure 5). Then $srad_k(D) = sdiam_k(D) = n$ for all $k$ with $2 \leq k \leq n$. In fact, there are many other strong oriented graphs $D$ with the property that $srad_k(D) = sdiam_k(D)$.

**Figure 4. A strong oriented graph**

**Figure 5. A strong oriented graph $D$ of order $n$ with $srad_k(D) = sdiam_k(D)$ for $2 \leq k \leq n$**
On the other hand, for a strong oriented graph $D$, the difference between $\text{srad}_{k+1}(D)$ and $\text{srad}_k(D)$ (or $\text{sdiam}_{k+1}(D)$ and $\text{sdiam}_k(D)$) can be arbitrarily large for some $k$.

**Proposition 2.2.** For every integer $N \geq 3$, there exist a strong oriented graph $D$ and an integer $k$ such that

$$\text{srad}_{k+1}(D) - \text{srad}_k(D) \geq N \text{ and } \text{sdiam}_{k+1}(D) - \text{sdiam}_k(D) \geq N.$$  

**Proof.** Let $\ell \geq 3$ be an integer. For each $i$ with $1 \leq i \leq \ell$, let $D_i$ be a copy of the directed $N$-cycle $C_N$ and let $v_i \in V(D_i)$. Now let $D$ be the strong oriented graph obtained from the digraphs $D_i$ (1 $\leq i \leq \ell$) by identifying the $\ell$ vertices $v_1, v_2, \ldots, v_\ell$. It can be verified that $\text{srad}_{k+1}(D) - \text{srad}_k(D) = N$ and $\text{sdiam}_{k+1}(D) - \text{sdiam}_k(D) = N$ for all $k$ with $2 \leq k \leq \ell - 1$. 

For an integer $k \geq 2$, the $k$-strong radius and $k$-strong diameter of a strong oriented graph satisfy familiar inequalities, which are verified with familiar arguments.

**Proposition 2.3.** Let $k \geq 2$ be an integer. For every strong oriented graph $D$,

$$\text{srad}_k(D) \leq \text{sdiam}_k(D) \leq 2\text{srad}_k(D).$$

**Proof.** The inequality $\text{srad}_k(D) \leq \text{sdiam}_k(D)$ follows directly from the definitions. It was shown in [2] that result is true for $k = 2$. So we may assume that $k \geq 3$. Let $S_1 = \{w_1, w_2, \ldots, w_k\}$ be a set of vertices of $D$ with $d(S) = \text{sdiam}_k(D)$ and let $v$ be a vertex of $D$ with $\text{se}_k(v) = \text{srad}_k(D)$. Define $S_2 = \{v, w_1, w_2, \ldots, w_{k-1}\}$ and $S_3 = \{v, w_2, w_3, \ldots, w_k\}$. Thus $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$. It then follows from Proposition 1.1 that

$$\text{sdiam}_k(D) = d(S_1) \leq d(S_2) + d(S_3) \leq 2\text{srad}_k(D),$$

producing the desired result. 

3. **On $k$-strong centers and peripherals**

A vertex $v$ in a strong digraph $D$ is a $k$-strong central vertex if $\text{se}_k(v) = \text{srad}_k(G)$, while the $k$-strong center $\text{SC}_k(D)$ of $D$ is the subgraph induced by the $k$-strong central vertices of $D$. These concepts were first introduced in [3] for $k = 2$. For example, consider the strong digraph $D$ of Figure 4, which is also shown in Figure 6. Each vertex of $D$ is labeled with its 3-strong eccentricity. Thus the vertices $z$, $y$, $z$ are the 3-strong central vertices of $D$. The 3-strong center $\text{SC}_3(D)$ of $D$ is a 3-cycle as shown in Figure 6.

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It was shown in [3] that every 2-strong center of every strong oriented graph \( D \) lies in a block of the underlying graph of \( D \). However, it is not true in general for \( k \geq 3 \). For example, although the 3-strong center of the strong oriented graph \( D \) in Figure 6 lies in a block of the underlying graph of \( D \), the 4-strong center of \( D \) is \( D \) itself and \( D \) is not a block. On the other hand, as Hedetniemi (see [1]) showed that every graph is the center of some connected graph, it was also shown in [3] that every oriented graph is the 2-strong center of some strong digraph. We now extend this result by showing that, for each integer \( k \geq 2 \), every oriented graph is the \( k \)-strong center of some strong digraph.

**Theorem 3.1.** Let \( k \geq 2 \) be an integer. Then every oriented graph is the \( k \)-strong center of some strong digraph.

**Proof.** For an oriented graph \( D \), we construct a strong oriented graph \( D^* \) from \( D \) by adding the \( 3k \) new vertices \( u_i, v_i, w_i \) \((1 \leq i \leq k)\) and arcs (1) \((u_i, v_i), (v_i, w_i)\), and \((u_i, w_i)\) for all \( i \) with \( 1 \leq i \leq k \) and (2) \((u_i, x)\) and \((x, v_i)\) for all \( x \in V(D) \) and for all \( i \) with \( 1 \leq i \leq k \). The oriented graph \( D^* \) is shown in Figure 7. Certainly, \( D^* \) is strong. Next, we show that \( D \) is the \( k \)-strong center of \( D^* \).

Let \( U = \{u_1, u_2, \ldots, u_k\} \), \( V = \{v_1, v_2, \ldots, v_k\} \), and \( W = \{w_1, w_2, \ldots, w_k\} \). For each \( x \in V(D) \), let \( S(x) = \{x\} \cup (W - \{w_k\}) \). Then \( sc_k(x) = d(S) = 6(k - 1) \). For each \( u_i \in U \), where \( 1 \leq i \leq k \), let \( S(u_i) = \{u_i\} \cup (W - \{w_i\}) \). Then \( sc_k(u_i) = 6(k - 1) \).
For each \( v_i \in V \), \( 1 \leq i \leq k \), let \( S(v_i) = \{ v_i \} \cup (W - \{ w_i \}) \). Then \( \se_k(v_i) = d(S) = 6(k - 1) + 3 \) for \( 1 \leq i \leq k \).

For each \( w_i \in W \), where \( 1 \leq i \leq k \), let \( S = W \). Then \( \se_k(w_i) = d(S) = 6k \) for \( 1 \leq i \leq k \). Since \( \se_k(x) = 6(k - 1) \) for all \( x \in V(D) \) and \( \se_k(v) > 6(k - 1) \) for all \( v \in V(D^*) - V(D) \), it follows that \( D \) is the \( k \)-strong center of \( D^* \), as desired. \( \square \)

Independently, V. Castellana and M. Raines also discovered Theorem 3.1 (personal communication). A vertex \( v \) in a strong digraph \( D \) is called a \( k \)-strong peripheral vertex if \( \se_k(v) = \text{sdiam}_k(D) \), while the subgraph induced by the \( k \)-strong peripheral vertices of \( D \) is the \( k \)-strong periphery \( SP_k(D) \) of \( D \). Also, these concepts were first introduced in [3] for \( k = 2 \). A strong digraph \( D \) and its 3-strong periphery are shown in Figure 8. The following result appeared in [3].

**Theorem A.** If \( D \) is an oriented graph with \( \text{srad}_2(D) = 3 \) and \( \text{sdiam}_2(D) > 3 \), then \( D \) is not the 2-strong periphery of any oriented graph.

We now extend Theorem A to the \( k \)-strong periphery of a strong oriented graph for \( k \geq 3 \) and show that not all oriented graphs are the \( k \)-strong peripheries of strong oriented graphs.

**Theorem 3.2.** Let \( k \geq 3 \) be an integer. If \( D \) is an oriented graph with \( \text{sdiam}_k(D) > \text{srad}_k(D) \), then \( D \) is not the \( k \)-strong periphery of any oriented graph.

**Proof.** Let \( D \) satisfy the conditions of the theorem. Assume, to the contrary, that \( D \) is the \( k \)-strong periphery of some oriented graph \( D' \). Assume that \( \text{srad}_k(D) = r \) and \( \text{sdiam}_k(D) = d \). So \( d > r \geq 3 \). Let \( u \) be a \( k \)-strong central vertex of \( D \). Since \( \text{sdiam}_k(D) = d > r \), we have \( \text{sdiam}_k(D') = d' \geq d > r \). Moreover, since \( D \) is the \( k \)-strong periphery of \( D' \) and \( u \in V(D) \), it follows that \( D' \) contains a set \( S = \{ u, v_1, v_2, \ldots, v_{k-1} \} \) such that \( d(S) = \text{sdiam}_k(D') = d' \). Because \( u \) is a \( k \)-strong central vertex of \( D \), that is, \( u \) has \( k \)-strong eccentricity \( r \) in \( D \), and \( r < d' \), at least one vertex from \( \{ v_1, v_2, \ldots, v_{k-1} \} \) does not belong to \( V(D) \). Assume, without loss of generality, that \( v_1 \notin V(D) \). Then the \( k \)-strong eccentricity \( \se_k(v_1) \) of \( v_1 \) in \( D' \) is
at least $d(S)$ and so $s_k(v_1) \geq d(S) = d'$. Thus $s_k(v_1) = d'$, which implies that $v_1$ is a $k$-strong peripheral vertex of $D'$. Since $v_1 \notin V(D)$, it follows that $D$ is not the $k$-strong periphery of $D'$, which is a contradiction. \hfill \square

In [3], a sufficient condition was established for an oriented graph $D$ to be the 2-strong periphery of some oriented graph $D'$, which we state next.

**Theorem B.** Let $D$ be an oriented graph of order $n$ with strong diameter at least 4. If $id v + od v < n - 1$ for every vertex $v$ of $D$, then $D$ is the 2-strong periphery of some oriented graph $D'$.

Observe that if $v$ is a vertex of an oriented graph $D$ of order $n$ such that $id v + od v < n - 1$, then there is a vertex $u \in V(D)$ such that $v$ and $u$ are nonadjacent vertices of $D$, that is, $v$ belongs to an independent set, namely $\{u, v\}$, of cardinality 2 in $D$. Thus the sufficient condition given in Theorem B is equivalent to that every vertex in $D$ belongs to an independent set of cardinality 2 in $D$. We now extend Theorem B to obtain a sufficient condition for an oriented digraph $D$ to be the $k$-strong periphery of some oriented graph $D'$ for all integers $k \geq 2$.

**Theorem 3.3.** Let $k \geq 2$ be an integer and let $D$ be a connected oriented graph. If every vertex of $D$ belongs to an independent set of cardinality $k$ in $D$, then $D$ is the $k$-strong periphery of some oriented graph $D'$.

**Proof.** By Theorem B the result holds for $k = 2$. So we assume that $k \geq 3$. Let $D$ be an oriented graph of order $n$ which satisfies the conditions of the theorem and let $V(D) = \{u_1, u_2, \ldots, u_n\}$. We construct a new oriented graph $D'$ of order $2n + 2$ with $V(D') = V(D) \cup \{v_1, v_2, \ldots, v_n, x, y\}$ such that the arc set of $D'$ consists of $E(D)$ together with arcs (1) $(u_i, v_i)$ and $(v_i, u_j)$ for $1 \leq i \leq n$ and $1 \leq j \leq n$, (2) $(v_i, v_j)$ for $1 \leq i < j \leq n$, and (3) $(y, x), (v_i, x), (x, u_i), (u_i, y), (y, v_i)$ for $1 \leq i \leq n$. The oriented graph $D'$ is shown in Figure 9. We claim that $D$ is the $k$-strong periphery of $D'$. We will show it only for $k = 3$ since the argument for $k \geq 4$ is similar.

![Figure 9. An oriented graph $D'$ containing $D$ as its $k$-strong periphery](image)
We first show that $se_3(u_i) = 6$ in $D'$ for all $i$ with $1 \leq i \leq n$. Without loss of generality, we consider only $u_1 \in V(D)$ and show that $se_3(u_1) = 6$. Let $S_1 = \{u_1, u_p, u_q\}$ be an independent set of three vertices in $D'$, where $2 \leq p < q \leq n$. Then the size of a strong subdigraph containing $S_0$ is at least 6. On the other hand, the directed 6-cycle $C$ shown in Figure 10 contains $S_0$. Thus $d(S_0) = 6$ and so $se_3(u_1) \geq 6$.

![Figure 10. A directed 6-cycle $C$ in $D'$ containing $S_0$.](image)

To show that $se_3(u_1) \leq 6$. Let $S$ be a set of three vertices of $D$ containing $u_1$. Then the only possible choices for $S$ are $S_1 = \{u_1, u_i, u_j\}$, where $2 \leq i < j \leq n$, $S_2 = \{u_1, v_i, v_j\}$, where $1 \leq i < j \leq n$, $S_3 = \{u_1, u_i, v_j\}$, where $i \geq 2$ and $1 \leq j \leq n$, $S_4 = \{u_1, x, y\}$, $S_5 = \{u_1, u_i, y\}$, where $2 \leq i \leq n$, $S_6 = \{u_1, u_i, x\}$, where $2 \leq i \leq n$, $S_7 = \{u_1, v_i, y\}$, and $S_8 = \{u_1, v_i, x\}$, where $1 \leq i \leq n$. If $S = S_1$, then the directed 6-cycle $u_1, v_i, v_j, u_1$ is a strong subdigraph of $D'$ containing $S$ and so $d(S) \leq 6$. Let $S = S_2 = \{u_1, v_i, v_j\}$, where $1 \leq i < j \leq n$. If $i = 1$, then the directed 4-cycle $u_1, v_i, u_j, u_1$ is a strong subdigraph of $D'$ containing $S$ and so $d(S) \leq 4$. If $i \geq 2$, then the directed 4-cycle $u_1, y, v_i, v_j, u_1$ is a strong subdigraph of $D'$ containing $S$ and so $d(S) \leq 4$. Let $S = S_3 = \{u_1, u_i, v_j\}$, where $i \geq 2$ and $1 \leq j \leq n$. If $j = 1$ or $j = i$, say $j = 1$, then the directed 4-cycle $u_1, v_1, u_1, v_i, u_1$ is a strong subdigraph of $D'$ containing $S$ and so $d(S) \leq 4$; Otherwise, the directed 5-cycle $u_1, v_1, u_i, v_i, v_j, u_1$ is a strong subdigraph of $D'$ containing $S$ and so $d(S) \leq 5$. If $S = S_4$, then the directed 3-cycle $u_1, y, x, u_1$ is a strong subdigraph of $D'$ containing $S$ and so $d(S) \leq 3$. If $S = S_5$ (or $S = S_6$), then the directed 5-cycle $u_1, v_1, u_i, y, v_i, u_1$ contains $S$ (or the directed 5-cycle $u_1, v_1, x, u_i, v_i, u_1$ contains $S$). Thus $d(S) \leq 5$. Let $S = S_7 = \{u_1, v_i, y\}$ or $S = S_8 = \{u_1, v_i, x\}$, where $1 \leq i \leq n$. If $i = 1$, then directed 4-cycle $u_1, y, v_1, x, u_1$ contains $S$ and $d(S) \leq 4$. If $i \geq 2$, then either the directed 5-cycle $u_1, v_1, u_i, y, v_i, u_1$ contains $S$ or the directed 5-cycle $u_1, v_1, x, u_i, v_i, u_1$ contains $S$. Thus $d(S) \leq 5$. Hence $d(S) \leq 6$ for all possible choices for $S$ and so $se_3(u_1) \leq 6$. Therefore, $se_3(u_1) = 6$. Similarly, $se_3(u_i) = 6$ for all $i$ with $2 \leq i \leq n$.

Next we show that $se(x) \leq 5$ and $se(y) \leq 5$ in $D'$. Let $S$ be a set of three vertices in $D'$ containing $x$. Then the only possible choices for $S$ are $S_1 = \{x, u_i, u_j\}$, where
Let $D$ be a nontrivial strong digraph of order $n$ and let $k$ be an integer with $2 \leq k \leq n$. Then $D$ is called strongly $k$-self-centered if $srad_k D = sdiam_k D$, that is, if $D$ is its own $k$-strong center. For example, the directed $n$-cycle $C_n^+$ and the strong digraph $D$ in Figure 5 are $k$-self-centered for all $k$ with $2 \leq k \leq n$. The 2-self-centered digraph was studied in [3]. The following result was established in [3].

**Theorem C.** For every integer $r \geq 3$, there exist infinitely many strongly 2-self-centered oriented graphs of strong radius $r$.

We now extend Theorem C to strongly 3-self-centered oriented graphs.

**Theorem 4.1.** For every integer $r \geq 6$, there exist infinitely many strongly 3-self-centered oriented graphs of strong radius $r$.

**Proof.** For each integer $r \geq 6$, we construct an infinite sequence $\{D_n\}$ of strongly 3-self-centered oriented graphs of strong radius $r$. We consider two cases, according to whether $r$ is even or $r$ is odd.

**Case 1.** $r$ is even. Let $r = 2p$, where $p \geq 3$. Let $D_1$ be the digraph obtained from the directed $p$-cycle $C_p : u_1, u_2, \ldots, u_p$ by adding the $2(p-1)$ new vertices $u_1, u_2, \ldots, u_p$,
with $w, u, v,$ and $v_p$ and the new arcs $(1) \ (u_i, u_{i+1}), (v_i, v_{i+1})$ for $1 \leq i \leq p-2$ and $(2) \ (v, u_1), (u_{p-1}, v), (v, v_1),$ and $(v_{p-1}, v)$ for all $v \in V(C_p)$. The digraph $D_1$ is shown in Figure 11 for $r = 6$. Let $U = \{u_1, u_2, \ldots, u_{p-1}\}, V = \{v_1, v_2, \ldots, v_{p-1}\}$, and $W = \{w_1, w_2, \ldots, w_p\}$. We show that $D_1$ is a strongly 3-self-centered digraph with 3-strong radius $r$.

![Figure 11. The digraph $D_1$ in Case 1 for $r = 6$](image)

First, we make an observation. If $S = \{u, v, w\}$, where $u \in U$, $v \in V$, and $w \in W$, then $d(S) \geq r$ by the construction of $D_1$. On the other hand, let $D_S$ be the strong subdigraph in $D_1$ consisting of two $p$-cycles $w, v_1, v_2, \ldots, v_{p-1}, w$ and $w, u_1, u_2, \ldots, u_{p-1}, w$. Since $D_S$ contains $S$ and has size $2p = r$, it follows that $d(S) = r$. Therefore, for every vertex $x$ of $V(D_1)$, there is a set $S$ of three vertices of $D_1$ such that $S$ contains $x$ and $d(S) = r$. This implies that $se_3(x) \geq r$ for all $x \in V(D_1)$. So it remains to show that $se_3(x) \leq r$ for all $x \in V(D_1)$. There are two subcases.

**Subcase 1.1.** $x \in U$ or $x \in V$. Without loss of generality, assume that $x \in U$. We will only consider $x = u_1 \in U$ since the proofs for other vertices are similar. Let $S$ be a set of three vertices in $D_1$ containing $u_1$. If $S \cap V \neq \emptyset$ and $S \cap W \neq \emptyset$, then $d(S) = r$ by the observation above. So we may assume that $S$ is one of the following sets: $S_1 = \{u_1, u_i, u_j\}$, where $2 \leq i < j \leq p-1$, $S_2 = \{u_1, u_i, v_j\}$, where $2 \leq i \leq p-1$ and $1 \leq j \leq p$, $S_3 = \{u_1, u_i, v_j\}$, where $2 \leq i \leq p-1$ and $1 \leq j \leq p-1$, $S_4 = \{u_1, u_i, v_j\}$, where $1 \leq i < j \leq p-1$, and $S_5 = \{u_1, w_i, v_j\}$, where $1 \leq i \leq j \leq p$. If $S = S_1, S_2$, then the directed $p$-cycle $w_j, u_1, u_2, \ldots, u_{p-1}, w_j$ is a strong subdigraph in $D_1$ containing $S$ and so $d(S) \leq p$. If $S = S_3, S_4$, then the strong subdigraph $D_S$ in $D_1$ consisting of two $p$-cycles $w, v_1, v_2, \ldots, v_{p-1}, w$ and $w, u_1, u_2, \ldots, u_{p-1}, w$ contains $S$ and so $d(S) \leq 2p = r$. If $S = S_5$, then the strong subdigraph consisting of two $p$-cycles $w, v_1, v_2, \ldots, v_{p-1}, w$ and $w, u_1, u_2, \ldots, u_{p-1}, w$ contains $S$ and so $d(S) \leq 2p = r$.

**Subcase 2.2.** $x \in W$. We may assume that $x = w_1 \in W$ and let $S$ be a set of three vertices in $D_1$ containing $w_1$. Again, if $S \cap V \neq \emptyset$ and $S \cap U \neq \emptyset$, then $d(S) = r$. So we may assume that $S$ is one of the following sets $S_1 = \{w_1, w_i, v_j\}$, where $2 \leq i < j \leq p$, $S_2 = \{w_1, w_i, u_j\}$, where $2 \leq i \leq p$ and $1 \leq j \leq p-1$, $S_3 = \{w_1, w_i, v_j\}$, where $2 \leq i \leq p$ and $1 \leq j \leq p-1$, $S_4 = \{w_1, u_i, v_j\}$.
where $1 \leq i < j \leq p - 1$, and $S_5 = \{w_1, v_i, v_j\}$, where $1 \leq i < j \leq p - 1$. An argument similar to the one in Subcase 1.1 shows that $d(S) \leq r$ for all possible choices for $S$.

Therefore, $se_3(x) = r$ for all $x \in V(D_1)$ and so $D_1$ is a strongly 3-self-centered digraph with 3-strong radius $r$.

For $n \geq 1$, we define the strong digraph $D_{n+1}$ recursively from $D_n$ by adding the $2(p-1)$ new vertices $x_1$, $x_2$, ..., $x_{p-1}$ and $y_1$, $y_2$, ..., $y_{p-1}$ and the new arcs (1) $(x_i, x_{i+1})$, $(y_i, y_{i+1})$ for $1 \leq i \leq p - 2$ and (2) $(v, x_1)$, $(x_{p-1}, v)$, $(v, y_1)$, and $(y_{p-1}, v)$ for all $v \in V(D_n)$. The digraph $D_{n+1}$ is shown in Figure 12. We assume that $D_n$ is a strongly 3-self-centered oriented graph of 3-strong radius $r$ for some integer $n \geq 1$ and show that $D_{n+1}$ is also a strongly 3-self-centered oriented graph of 3-strong radius $r$.

Let $X = \{x_1, x_2, \ldots, x_{p-1}\}$ and $Y = \{y_1, y_2, \ldots, y_{p-1}\}$. For $v \in V(D_{n+1})$, let $S$ be a set of three vertices in $D_{n+1}$ containing $v$. If $v \in V(D_n)$ and $S = \{v, x_1, y_1\}$, then $se_3(v) = d(S) = r$. So we may assume that $v \in X \cup Y$, say $v = x_1$. Let $S = \{v, y_1, z\}$, where $z \in V(D_n)$. Then $d(S) = se_3(v) = r$. Therefore, $se_3(v) = r$ for all $v \in V(D_{n+1})$ and so $D_{n+1}$ is also a strongly 3-self-centered oriented graph of 3-strong radius $r$.

Case 2. $r$ is odd. Let $r = 2p + 1$, where $p \geq 3$. Let $D_1$ be the digraph obtained from the directed $(p + 1)$-cycle $C_{p+1}$: $w_1, w_2, w_3, w_4, w_1$ by adding the $p - 1$ new vertices $u_1, u_2, \ldots, u_{p-1}$ and the new arcs (1) $(u_i, u_{i+1})$ for $1 \leq i \leq p - 2$ and (2) $(v, u_1)$ and $(u_{p-1}, v)$ for all $v \in V(C_{p+1})$. The digraph $D_1$ is shown in Figure 13 for $r = 7$.

Figure 12. The digraph $D_{n+1}$ in Case 1

Figure 13. The digraph $D_1$ in Case 2 for $r = 7$
For $n \geq 1$, we define $D_{n+1}$ recursively from $D_n$ by adding the $p - 1$ new vertices $x_1, x_2, \ldots, x_{p-1}$ and the new arcs (1) $(x_i, x_{i+1})$, for $1 \leq i \leq p - 2$ and (2) $(v, x_1)$ and $(x_{p-1}, v)$ for all $v \in V(D_n)$. The digraph $D_{n+1}$ is shown in Figure 14.

An argument similar to the one used in Case 1 shows that each strong digraph $D_n$ is a strongly 3-self-centered oriented graph of strong radius $r$ for all $n \geq 1$. □

Acknowledgements. The author is grateful to Professor Gary Chartrand for suggesting the concept of strong Steiner distance and kindly providing useful information on this topic.

References


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