PSEUDO BL-ALGEBRAS AND DRℓ-MONOIDS

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Abstract. It is shown that pseudo BL-algebras are categorically equivalent to certain bounded DRℓ-monoids. Using this result, we obtain some properties of pseudo BL-algebras, in particular, we can characterize congruence kernels by means of normal filters. Further, we deal with representable pseudo BL-algebras and, in conclusion, we prove that they form a variety.

Keywords: pseudo BL-algebra, DRℓ-monoid, filter, polar, representable pseudo BL-algebra

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1. Connections between pseudo BL-algebras and DRℓ-monoids

Recently, pseudo BL-algebras were introduced by A. Di Nola, G. Georgescu and A. Iorgulescu in [3] as a noncommutative extension of Hájek’s BL-algebras (see [6]).

An algebra \( \mathfrak{A} = (A, \vee, \wedge, \odot, \rightarrow, \sim, 0, 1) \) of type \( \langle 2, 2, 2, 2, 0, 0 \rangle \) is called a pseudo BL-algebra iff \( (A, \vee, \wedge, 0, 1) \) is a bounded lattice, \( (A, \odot, 1) \) is a monoid and the following conditions are satisfied for all \( x, y, z \in A \):

1. \( x \odot y \leq z \) iff \( x \leq y \Rightarrow z \) iff \( y \leq x \Rightarrow z \),
2. \( x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightarrow y) \),
3. \( (x \rightarrow y) \vee (y \rightarrow x) = (x \sim y) \vee (y \sim x) = 0 \).

By [3, Corollary 3.29], pseudo BL-algebras satisfying the identity

\[ (x \sim 0) \rightarrow 0 = (x \rightarrow 0) \sim 0 = x \]

are the duals of pseudo MV-algebras.

In the same way, (noncommutative) DRℓ-monoids extend Swamy’s DRℓ-semigroups which were introduced in [12] as a common generalization of abelian ℓ-groups and Brouwerian algebras.
An algebra $\mathfrak{A} = (A, +, 0, \vee, \wedge, \to, \leftarrow)$ of type $\langle 2, 0, 2, 2, 2 \rangle$ is a \textit{dually residuated lattice ordered monoid}, or simply a $\text{DRL}$-monoid, iff

1. $(A, +, 0, \vee, \wedge)$ is an $\ell$-monoid, that is, $(A, +, 0)$ is a monoid, $(A, \vee, \wedge)$ is a lattice and, for any $x, y, s, t \in A$, the following distributive laws are satisfied:

   \begin{align*}
   s + (x \vee y) + t &= (s + x + t) \vee (s + y + t), \\
   s + (x \wedge y) + t &= (s + x + t) \wedge (s + y + t);
   \end{align*}

2. for any $x, y \in A$, $x \to y$ is the least $s \in A$ such that $s + y \geq x$, and $x \leftarrow y$ is the least $t \in A$ such that $y + t \geq x$;

3. $\mathfrak{A}$ fulfills the identities

   \begin{align*}
   ((x \to y) \vee 0) + y &\leq x \vee y, \quad y + ((x \leftarrow y) \vee 0) \leq x \vee y, \\
   x \to x &\geq 0, \quad x \leftarrow x \geq 0.
   \end{align*}

Note that the inequalities $x \to x \geq 0$ and $x \leftarrow x \geq 0$ can be omitted, and the condition (2) is equivalent to the system of identities (see [10])

\begin{align*}
(x \to y) + y \geq x, \quad y + (x \leftarrow y) \geq x, \\
x \to y \leq (x \vee z) \to y, \quad x \leftarrow y \leq (x \vee z) \leftarrow y, \\
(x + y) \to y \leq x, \quad (y + x) \leftarrow y \leq x.
\end{align*}

In [11], mutual relationships between $\text{BL}$-algebras and bounded representable commutative $\text{DRL}$-monoids are described.

**Theorem 1.1.** Let $\mathfrak{A} = (A, \vee, \wedge, \circ, \to, \leftarrow, 0, 1)$ be a pseudo $\text{BL}$-algebra. If we set

\begin{align*}
x + y &:= x \circ y, \quad x \vee_d y := x \wedge y, \quad x \wedge_d y := x \vee y, \\
x \to y &:= y \to x, \quad x \leftarrow y := y \leftarrow x, \quad 0_d := 1, \quad 1_d := 0
\end{align*}

for any $x, y \in A$, then $\mathfrak{A}_d = (A, +, 0_d, \vee_d, \wedge_d, \to_d, \leftarrow_d)$ is a bounded $\text{DRL}$-monoid with the greatest element $1_d$. In addition, this $\text{DRL}$-monoid satisfies the identities

\begin{align*}
(x \to y) \wedge_d (y \to x) &= 0_d, \\
(x \leftarrow y) \wedge_d (y \leftarrow x) &= 0_d.
\end{align*}

**Proof.** Since $(A, \circ, 1, \vee, \wedge)$ is an $\ell$-monoid, by [3, Propositions 3.3, 3.9], so is $(A, +, 0_d, \vee_d, \wedge_d)$. The rest follows directly by the definitions. Note that if a $\text{DRL}$-monoid $\mathfrak{A}_d$ contains the greatest element $1_d$ then $0_d$ is its least element, by [8, Theorem 1.2.3]. \hfill \Box

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In view of Theorem 1.1, it is easily seen that in the definition of a pseudo BL-algebra, the condition (1) can be equivalently replaced by the following identities:

\[(x \rightarrow y) \circ x \leq y, \quad x \circ (x \rightarrow y) \leq y,\]
\[x \rightarrow y \geq x \rightarrow (y \land z), \quad x \rightarrow y \geq x \rightarrow (y \land z),\]
\[y \rightarrow (x \circ y) \geq x, \quad y \rightarrow (y \circ x) \geq x.\]

Consequently, pseudo BL-algebras form a variety of algebras of type \((2, 2, 2, 2, 0, 0)\). This variety is arithmetical; in accordance with [8, Theorem 3.1.1], the Pixley term of the variety of pseudo BL-algebras can be taken as follows:

\[p(x, y, z) = ((x \rightarrow y) \rightarrow z) \land ((z \rightarrow y) \rightarrow x) \land (x \lor z).\]

**Theorem 1.2.** Let \(\mathfrak{A} = (A, +, 0, \land, \rightarrow, \leftarrow)\) be a \(DR\ell\)-monoid with the greatest element 1. For any \(x, y \in A\) set

\[x \circ y := x + y, \quad x \lor y := x \land y, \quad x \land_d y := x \lor y,\]
\[x \rightarrow y := y \rightarrow x, \quad x \leftarrow y := y \leftarrow x, \quad 0_d := 1, \quad 1_d := 0.\]

Then \(\mathfrak{A}_d = (A, \land_d, \lor_d, \ominus, \rightarrow, \leftarrow, 0_d, 1_d)\) is a pseudo BL-algebra if and only if \(\mathfrak{A}\) satisfies (\(*\)).

**Proof.** In any \(DR\ell\)-monoid we have

\[x \lor y = ((y \rightarrow x) \lor 0) + x = x + ((y \leftarrow x) \lor 0).\]

Since \(\mathfrak{A}\) is bounded, that is, \(0 \leq x \leq 1\) for any \(x \in A\), it follows that

\[x \land_d y = (x \rightarrow y) \circ x = x \circ (x \rightarrow y).\]

The rest is obvious. \(\square\)

Let \(\mathcal{PBL}\) be the category of pseudo BL-algebras, that is, the category whose objects are pseudo BL-algebras and morphisms are homomorphisms of pseudo BL-algebras. Let \(\mathcal{DRL}_{1(\ast)}\) be the category of bounded \(DR\ell\)-monoids satisfying (\(*\)). Its morphisms are homomorphisms of \(DR\ell\)-monoids which preserve also 1, thus in the sequel, bounded \(DR\ell\)-monoids are regarded as algebras \((A, +, 0, \lor, \land, \rightarrow, \leftarrow, 1)\) of type \((2, 0, 2, 2, 2, 0)\).

**Theorem 1.3.** The categories \(\mathcal{PBL}\) and \(\mathcal{DRL}_{1(\ast)}\) are equivalent.

**Proof.** Theorems 1.1 and 1.2 enable us to define a functor \(\mathcal{F}: \mathcal{PBL} \rightarrow \mathcal{DRL}_{1(\ast)}\) as follows: (i) \(\mathcal{F}(\mathfrak{A}) = \mathfrak{A}_d\) for any pseudo BL-algebra \(\mathfrak{A}\), and (ii) \(\mathcal{F}(h) = h\) for any pseudo BL-homomorphism \(h\). It is easy to see that \(\mathcal{F}\) is really a categorical equivalence. \(\square\)
2. Filters

According to [3], a subset \( F \) of a pseudo BL-algebra \( A \) with the following properties is said to be a filter of \( A \):

\begin{itemize}
  \item[(F1)] \( 1 \in F \);
  \item[(F2)] \( \forall x, y \in F; x \wedge y \in F \);
  \item[(F3)] \( \forall x \in F \forall y \in A; x \leq y \implies y \in F \).
\end{itemize}

For any subset \( M \subseteq A \), the intersection of all filters containing \( M \) is called a filter generated by \( M \) and denoted by \([M]\). It is clear that

\[ [M] = \{ x \in A; x \geq a_1 \ldots \wedge a_n \text{ for some } a_1, \ldots, a_n \in M \text{ and } n \geq 1 \}, \]

and if we write briefly \([a]\) for \([\{a\}]\) then

\[ [a] = \{ x \in A; x \geq a^n \text{ for some } n \geq 1 \}. \]

In Section 1, we have already proved that \( \text{DR}_t \)-monoids include the duals of pseudo BL-algebras. It is obvious that \( F \subseteq A \) is a filter of a pseudo BL-algebra \( \mathfrak{A} \) iff it is an ideal of the induced bounded \( \text{DR}_t \)-monoid \( \mathfrak{A}_d \), that is,

\begin{itemize}
  \item[(I1)] \( 0_d \in F \);
  \item[(I2)] \( \forall x, y \in F; x + y \in F \);
  \item[(I3)] \( \forall x \in F \forall y \in A; x \geq_d y \implies y \in F \).
\end{itemize}

Ideals of noncommutative \( \text{DR}_t \)-monoids were studied in [9]. Considering the above facts, we immediately obtain the following results.

**Proposition 2.1.** The set of all filters of any pseudo BL-algebra \( \mathfrak{A} \), ordered by set inclusion, is an algebraic Brouwerian lattice. For any filters \( F, G \) of \( \mathfrak{A} \), the relative pseudocomplement of \( F \) with respect to \( G \) is given by

\[ F \ast G = \{ a \in A; a \lor x \in G \text{ for all } x \in F \}. \]

Let \( \mathfrak{A} \) be a pseudo BL-algebra and \( X \subseteq A \). The set

\[ X^\perp = \{ a \in A; a \lor x = 1 \text{ for any } x \in X \} \]

is called the polar of \( X \). For any \( x \in A \) we write \( x^\perp \) instead of \( \{x\}^\perp \).

A subset \( X \) of \( A \) is a polar in \( \mathfrak{A} \) iff \( X = Y^\perp \) for some \( Y \subseteq A \).

**Proposition 2.2** [3, Propositions 4.38, 4.39]. For all subsets \( X, Y \) of a pseudo BL-algebra \( \mathfrak{A} \), (i) \( X^\perp \) is a filter of \( \mathfrak{A} \), (ii) \( X \subseteq X^{\perp \perp} \), (iii) \( X \subseteq Y \) implies \( Y^\perp \subseteq X^\perp \), (iv) \( X^\perp = X^{\perp \perp \perp} \).
Proposition 2.3. For any subset $X$ of a pseudo BL-algebra $\mathfrak{A}$, $X$ is a polar in $\mathfrak{A}$ iff $X = X^\perp$. 

Proof. Let $X = Y^\perp$; then $X^\perp = Y^\perp \cap Y^\perp = Y^\perp = X$. □ 

By Proposition 2.1, the pseudocomplement of a filter $F$ is 

$$F^* = \{a \in A; \quad a \lor x = 1 \text{ for any } x \in F\}.$$ 

Moreover, it is clear that $F^\perp = F^*$ whenever $F$ is a filter, and conversely, any polar is the pseudocomplement of some filter; in fact, $X = (X^\perp)^*$. Thus the polars in any pseudo BL-algebra are precisely the pseudocomplements in the lattice of its filters. Therefore, by the Glivenko-Frink Theorem, we directly obtain 

Theorem 2.4. The set of all polars in any pseudo BL-algebra, ordered by set inclusion, is a complete Boolean algebra. 

A filter $F$ of a pseudo BL-algebra $\mathfrak{A}$ is said to be normal iff it satisfies the following condition for each $x, y \in A$: 

$$x \rightarrow y \in F \iff x \leftarrow y \in F.$$ 

Proposition 2.5. For any filter $F$, the following conditions are equivalent: 

(i) $F$ is normal; 

(ii) $x \circ F = F \circ x$ for each $x \in A$. 

Proposition 2.6. If $F$ and $G$ are normal filters of $\mathfrak{A}$ then 

$$F \lor G = \{x \in A; \quad x \supseteq a \circ b \text{ for some } a \in F, b \in G\}.$$ 

In addition, $F \lor G$ is a normal filter. Consequently, normal filters of any pseudo BL-algebra form a complete sublattice of the lattice of all its filters. 

Theorem 2.7. In any pseudo BL-algebra, there is a one-to-one correspondence between the normal filters and the congruence relations. In fact, $F$ corresponds to $\Theta(F)$ defined by 

$$\langle x, y \rangle \in \Theta(F) = \Theta_1(F) \iff (x \rightarrow y) \land (y \rightarrow x) \in F,$$

or equivalently, 

$$\langle x, y \rangle \in \Theta(F) = \Theta_2(F) \iff (x \leftarrow y) \land (y \leftarrow x) \in F.$$ 

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As proved in [3], and in general for noncommutative $DR$-monoids in [9], if $F$ is not a normal filter then the binary relations defined in the previous theorem, $\Theta_1(F)$ and $\Theta_2(F)$, are two distinct congruence relations on the distributive lattice $\mathcal{L}(\mathfrak{A}) = (A, \lor, \land, 0, 1)$. In the quotient lattices $\mathcal{L}(\mathfrak{A})/\Theta_1(F)$ and $\mathcal{L}(\mathfrak{A})/\Theta_2(F)$ we have

\[(2.1) \quad [x]_{\Theta_1(F)} \leq [y]_{\Theta_1(F)} \iff x \sim y \in F\]

and

\[(2.2) \quad [x]_{\Theta_2(F)} \leq [y]_{\Theta_2(F)} \iff x \sim y \in F,\]

respectively.

Let $\mathfrak{A}$ be a pseudo $BL$-algebra. A filter $F$ of $\mathfrak{A}$ is said to be prime if it is a finitely meet-irreducible element in the lattice of filters of $\mathfrak{A}$.

By [3, Theorem 4.28], for any filter $F$ of a pseudo $BL$-algebra $\mathfrak{A}$ and for each ideal $I$ of the lattice $\mathcal{L}(\mathfrak{A})$, if $F \cap I = \emptyset$ then there exists a prime filter $P$ of $\mathfrak{A}$ with $F \subseteq P$ and $P \cap I = \emptyset$. Consequently, every proper filter is the intersection of all prime filters including it. In particular, the intersection of all prime filters is equal to $\{1\}$.

**Theorem 2.8.** For any filter $F$ of a pseudo $BL$-algebra $\mathfrak{A}$, the following conditions are equivalent:

(i) $F$ is prime;

(ii) for all filters $G, H$ of $\mathfrak{A}$, $G \cap H \subseteq F$ implies $G \subseteq F$ or $H \subseteq F$;

(iii) for any $x, y \in A$, $x \lor y \in F$ implies $x \in F$ or $y \in F$;

(iv) for any $x, y \in A$, $x \lor y = 1$ implies $x \in F$ or $y \in F$;

(v) for any $x, y \in A$, $x \rightarrow y \in F$ or $y \rightarrow x \in F$;

(vi) for any $x, y \in A$, $x \sim y \in F$ or $y \sim x \in F$;

(vii) $\mathcal{L}(\mathfrak{A})/\Theta_1(F)$ is totally ordered;

(viii) $\mathcal{L}(\mathfrak{A})/\Theta_2(F)$ is totally ordered;

(ix) the set of all filters including $F$ is totally ordered under set inclusion.

**Remark.** The equivalence of (iii), (v), (vi), (vii) and (viii) is due to [3, Proposition 4.25].

**Proof.** (i) $\Rightarrow$ (ii): Using the distributivity of the lattice of filters, $G \cap H \subseteq F$ implies $F = F \lor (G \cap H) = (F \lor G) \cap (F \lor H)$, whence $F = F \lor G$ or $F = F \lor H$, that is, $F \supseteq G$ or $F \supseteq H$.

(ii) $\Rightarrow$ (iii): Obviously, $x \lor y \in F$ yields $[x] \cap [y] = [x \lor y] \subseteq F$. Hence, by (ii), $[x] \subseteq F$ or $[y] \subseteq F$ and thus $x \in F$ or $y \in F$.

(iii) $\Rightarrow$ (iv): This is evident since $1 \in F$. 

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(iv) \implies (v) and (iv) \implies (vi): By the definition of a pseudo BL-algebra,

\[(x \rightarrow y) \lor (y \rightarrow z) = (x \leftarrow y) \lor (y \leftarrow z) = 1,
\]

which implies the assertion by (iv).

(v) \implies (vii) and (vi) \implies (viii): This is obvious from (2.1) and (2.2), respectively.

(vii) \implies (ix): If \( F \subseteq G, H \) and neither \( G \subseteq H \) nor \( H \subseteq G \) then there exist \( a, b \in A \) with \( a \in G \setminus H \) and \( b \in H \setminus G \). For instance, let \( a \rightarrow b \in F \). Then \( b \geq a \land b = (a \rightarrow b) \circ a \in G \), whence \( b \in G \); a contradiction. Similarly (viii) \implies (ix).

(ix) \implies (i): \( F = G \cap H \) entails \( F = G \) or \( F = H \), because either \( G \subseteq H \) or \( H \subseteq G \).

\[\Box\]

3. Representable pseudo BL-algebras

**Proposition 3.1.** If \( P \) is a minimal prime filter of a pseudo BL-algebra \( \mathfrak{A} \) then \( A \setminus P \) is a maximal ideal of the lattice \( \mathfrak{L}(\mathfrak{A}) \).

**Proof.** By Zorn’s Lemma, there is a maximal ideal \( I \) of \( \mathfrak{L}(\mathfrak{A}) \) with \( A \setminus P \subseteq I \).

Since \( P \) is also a prime filter of \( \mathfrak{L}(\mathfrak{A}) \), it follows that \( A \setminus P \) is a prime ideal of \( \mathfrak{L}(\mathfrak{A}) \) which is included in some maximal (prime) ideal. We will show that \( I = A \setminus P \).

Denote \( Q = \bigcup \{a^\perp; a \in I\} \). We claim that \( P = Q \).

If \( x \in a^\perp \) for some \( a \in I \), then \( x \lor a = 1 \) and \( x \notin I \). Indeed, if \( x \in I \) then \( x \lor a \neq 1 \) since \( x \lor a = 1 \) would mean \( I = A \). Thus \( x \in A \setminus I \subseteq A \setminus (A \setminus P) = P \), whence \( a^\perp \subseteq A \setminus I \subseteq P \) and consequently, \( Q \subseteq A \setminus I \subseteq P \).

We shall now prove that \( Q \) is a prime filter of \( \mathfrak{A} \). (F1): Since any principal polar \( a^\perp \) contains 1, so does \( Q \). (F2): If \( x, y \in Q \), that is, \( x \in a^\perp, y \in b^\perp \) for some \( a, b \in I \), then \( a \lor b \in I \) and

\[(x \land y) \lor a \lor b \geq (x \lor a \lor b) \circ (y \lor a \lor b) = 1 \circ 1 = 1.
\]

Therefore \( x \lor y \in (a \lor b)^\perp \subseteq Q \). (F3): It is obvious since \( a^\perp \) is a filter of \( \mathfrak{A} \) for each \( a \in I \).

To prove that \( Q \) is prime, suppose \( x \lor y = 1 \) and \( x \notin Q \), that is, \( x \lor a \neq 1 \) for all \( a \in I \). If \( x \notin I \) then the ideal in the lattice \( \mathfrak{L}(\mathfrak{A}) \) generated by \( I \cup \{x\} \), \( (I \cup \{x\}) \), is proper, i.e., \( A \setminus P \subseteq I \subseteq (I \cup \{x\}) \neq A \), since \( (I \cup \{x\}) = A \) would entail \( 1 \leq x \lor a \) for some \( a \in I \); a contradiction. Hence \( x \in I \) and thus \( y \in x^\perp \subseteq Q \), proving that \( Q \) is prime.

However, \( P \) is a minimal prime filter of \( \mathfrak{A} \); thus \( Q \subseteq A \setminus I \subseteq P \) yields \( Q = A \setminus I = P \) as claimed. Therefore \( I = A \setminus P \).

\[\Box\]
Corollary 3.2. If $P$ is a minimal prime filter then

$$P = \bigcup \{a^\perp; a \notin P\}.$$ 

Proof. By the proof of the previous proposition, $P = \bigcup \{a^\perp; a \in I\}$, where $I = A \setminus P$. 

A pseudo $BL$-algebra is said to be representable if it is a subdirect product of linearly ordered pseudo $BL$-algebras.

By Theorems 2.7 and 2.8, subdirect representations by totally ordered pseudo $BL$-algebras are associated with families of normal prime filters whose intersections are precisely $\{1\}$. Therefore it is obvious that every $BL$-algebra is representable (see also [11]). In contrast, for pseudo $BL$-algebras, this assertion fails.

The following results generalize the similar properties of pseudo $MV$-algebras, [4, Theorem 2.20], [1, Theorem 5.9], and [2, Theorem 6.11].

Theorem 3.3. For any pseudo $BL$-algebra $\mathfrak{A}$, the following statements are equivalent.

(i) $\mathfrak{A}$ is representable.

(ii) There exists a family $\{P_i\}_{i \in I}$ of normal prime filters of $\mathfrak{A}$ such that

$$\bigcap_{i \in I} P_i = \{1\}.$$ 

(iii) Any polar of $\mathfrak{A}$ is a normal filter of $\mathfrak{A}$.

(iv) Any principal polar is a normal filter.

(v) Any minimal prime filter is normal.

Proof. As argued above, the equivalence of (i) and (ii) is clear.

(i) $\Rightarrow$ (iii): Suppose that $\mathfrak{A}$ is a subdirect product of linearly ordered pseudo $BL$-algebras $\{\mathfrak{A}_i\}_{i \in I}$. Observe that

$$x \vee y = 1 \text{ iff } \{i \in I; x_i \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} = \emptyset$$

for all $x, y \in A$, since $\mathfrak{A}_i$ are totally ordered.

Let now $P$ be a polar in $\mathfrak{A}$, i.e. $P = P^\perp \perp$. Let $x \in A, a \in P$ and $y \in P^\perp$. Then $x \circ a \leq x$ implies $x \circ a = (x \circ a) \land x = (x \rightarrow (x \circ a)) \circ x$. Further, $\{i \in I; x_i \rightarrow (x_i \circ a_i) \neq 1_i\} \subseteq \{i \in I; a_i \neq 1_i\}$. Indeed, if $a_i = 1_i$ then $x_i \rightarrow (x_i \circ a_i) = x_i \rightarrow (x_i \circ 1_i) = x_i \rightarrow x_i = 1_i$. Hence we obtain

$$\{i \in I; x_i \rightarrow (x_i \circ a_i) \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} \subseteq \{i \in I; a_i \neq 1_i\} \cap \{i \in I; y_i \neq 1_i\} = \emptyset$$

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by (3.1), since \( a \in P \) and \( y \in P^\perp \). Therefore \( (x \rightarrow (x \circ a)) \vee y = 1 \), and thus \( x \rightarrow (x \circ a) \in P^\perp = P \). Hence \( x \circ a = (x \rightarrow (x \circ a)) \circ x \in P \circ x \), proving \( x \circ P \subseteq P \circ x \).

(iii) \( \Rightarrow \) (iv): Obvious.

(iv) \( \Rightarrow \) (v): By Corollary 3.2, \( P = \bigcup \{ a^\perp; a \notin P \} \) for any minimal prime filter \( P \). If \( x \rightarrow y \in P \) then there is \( a \notin P \) with \( x \rightarrow y \in a^\perp \) which is a normal filter, and hence \( x \rightarrow y \in a^\perp \subseteq P \). Summarizing, \( x \rightarrow y \in P \) if \( x \rightarrow y \in P \).

(v) \( \Rightarrow \) (i): Since any prime filter contains a minimal prime filter and the intersection of all prime filters of \( \mathcal{A} \) is obviously \( \{1\} \), so does the intersection of minimal prime filters. Thus, by (ii), \( \mathcal{A} \) is representable.

\[ \textbf{Theorem 3.4.} \quad \text{A pseudo BL-algebra is representable if and only if it satisfies the identities} \]

\[
(y \rightarrow x) \lor (z \rightarrow ((x \rightarrow y) \circ z)) = 1, \tag{3.2}
\]

\[
(y \rightarrow x) \lor (z \rightarrow (z \circ (x \rightarrow y))) = 1. \tag{3.3}
\]

Consequently, the class of representable pseudo BL-algebras is a variety.

\[ \textbf{Proof.} \quad \text{In any linearly ordered pseudo BL-algebra} \ \mathcal{A}, \text{ either} \ y \rightarrow x = 1 \text{ or } x \rightarrow y = 1 \text{ (and also } y \rightarrow x = 1 \text{ or } x \rightarrow y = 1), \text{ and so it is easy to verify that} \ \mathcal{A} \text{ fulfils (3.2) and (3.3). Therefore the part “only if” is obvious.} \]

\[ \text{Conversely, suppose that (3.2) and (3.3) are satisfied by } \mathcal{A}. \text{ In view of Theorem 3.3 (iv), it suffices to prove that any principal polar } x^\perp \text{ is a normal filter of } \mathcal{A}. \]

\[ \text{Let } y \in x^\perp, \text{ that is, } y \lor x = 1. \text{ Observe that in this case} \]

\[ x = 1 \rightarrow x = (y \lor x) \rightarrow x = (y \rightarrow x) \land (x \rightarrow x) = (y \rightarrow x) \land 1 = y \rightarrow x \]

\[ \text{and similarly } y = x \rightarrow y. \text{ Hence, by (3.2),} \]

\[ x \lor (z \rightarrow (y \circ z)) = (y \rightarrow x) \lor (z \rightarrow ((x \rightarrow y) \circ z)) = 1, \]

\[ \text{thus } z \rightarrow (y \circ z) \in x^\perp. \text{ Further, } y \circ z \leq z \text{ implies } y \circ z = (y \circ z) \land z = z \circ (z \rightarrow (y \circ z)) \in z \circ x^\perp, \text{ which shows } x^\perp \circ z \subseteq z \circ x^\perp. \text{ The other inclusion follows similarly by (3.3).} \]

\[ \square \]
References


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