Abstract. Recently, Rim and Teply [8], using the notion of $\tau$-exact modules, found a necessary condition for the existence of $\tau$-torsionfree covers with respect to a given hereditary torsion theory $\tau$ for the category $R$-mod of all unitary left $R$-modules over an associative ring $R$ with identity. Some relations between $\tau$-torsionfree and $\tau$-exact covers have been investigated in [5]. The purpose of this note is to show that if $\sigma = (T, F)$ is Goldie’s torsion theory and $\mathcal{F}_\sigma$ is a precover class, then $\mathcal{F}_\sigma$ is a precover class whenever $\tau \supseteq \sigma$. Further, it is shown that $\mathcal{F}_\sigma$ is a cover class if and only if $\sigma$ is of finite type and, in the case of non-singular rings, this is equivalent to the fact that $\mathcal{F}_\tau$ is a cover class for all hereditary torsion theories $\tau \supseteq \sigma$.

Keywords: hereditary torsion theory, Goldie’s torsion theory, non-singular ring, precover class, cover class

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In what follows, $R$ stands for an associative ring with identity and $R$-mod denotes the category of all unitary left $R$-modules. The basic properties of rings and modules can be found in [1]. A class $\mathcal{G}$ of modules is called abstract, if it is closed under isomorphic copies, co-abstract, if its members are pairwise non-isomorphic and complete with respect to a given property, if every module with this property is isomorphic to a member of the class $\mathcal{G}$.

Recall that a hereditary torsion theory $\tau = (T, F)$ for the category $R$-mod consists of two abstract classes $\mathcal{T}_\tau$ and $\mathcal{F}_\tau$, the $\tau$-torsion class and the $\tau$-torsionfree class, respectively, such that $\text{Hom}(T, F) = 0$ whenever $T \in \mathcal{T}_\tau$ and $F \in \mathcal{F}_\tau$, the class $\mathcal{F}_\tau$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathcal{T}_\tau$ is closed under submodules, extensions and arbitrary direct products.

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and for each module $M$ there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$. For two hereditary torsion theories $\tau$ and $\tau'$ the symbol $\tau \leq \tau'$ means that $\mathcal{T}_\tau \subseteq \mathcal{T}_{\tau'}$ and consequently $\mathcal{F}_{\tau'} \subseteq \mathcal{F}_\tau$. Associated with each hereditary torsion theory $\tau$ is the Gabriel filter $\mathcal{L}_\tau$ of left ideals of $R$ consisting of all left ideals $I \subseteq R$ with $R/I \in \mathcal{T}_\tau$. Recall that $\tau$ is said to be of finite type, if $\mathcal{L}_\tau$ contains a cofinal subset $\mathcal{L}'_\tau$ of finitely generated left ideals. A submodule $N$ of the module $M$ is called $\tau$-closed (or $\tau$-pure), if $N$ belongs to $\mathcal{F}_\tau$. Associated with each hereditary torsion theory is the Gabriel filter $\mathcal{L}_\sigma$ of all left ideals $I$ of $R$ with $R=I \leq \mathcal{T}_\sigma$. Recall that $I$ is said to be of finite type, if $\mathcal{L}_\sigma$ contains a cofinal subset $\mathcal{L}'_\sigma$ of finitely generated left ideals. A submodule $N$ of the module $M$ is called $\sigma$-closed (or $\sigma$-pure), if $N$ belongs to $\mathcal{F}_\sigma$. Associated with each hereditary torsion theory is the Gabriel filter $\mathcal{L}_\sigma$ of all left ideals $I$ of $R$ with $R=I \leq \mathcal{T}_\sigma$. Recall that $I$ is said to be of finite type, if $\mathcal{L}_\sigma$ contains a cofinal subset $\mathcal{L}'_\sigma$ of finitely generated left ideals. A submodule $N$ of the module $M$ is called $\sigma$-closed (or $\sigma$-pure), if $N$ belongs to $\mathcal{F}_\sigma$. Associated with each hereditary torsion theory is the Gabriel filter $\mathcal{L}_\sigma$ of all left ideals $I$ of $R$ with $R=I \leq \mathcal{T}_\sigma$. Recall that $I$ is said to be of finite type, if $\mathcal{L}_\sigma$ contains a cofinal subset $\mathcal{L}'_\sigma$ of finitely generated left ideals. A submodule $N$ of the module $M$ is called $\sigma$-closed (or $\sigma$-pure), if $N$ belongs to $\mathcal{F}_\sigma$.
Let $\tau \geq \sigma$ and the same holds for cover classes provided that the ring $R$ is non-singular. More precisely, we are going to prove the following two theorems.

**Theorem 1.** Let $\sigma = (\mathcal{F}_\sigma, \mathcal{F}_\sigma)$ be Goldie's torsion theory for the category $R$-mod. If $\mathcal{F}_\tau$ is a precover class, then $\mathcal{F}_\tau$ is a precover class for any hereditary torsion theory $\tau \geq \sigma$.

**Theorem 2.** Let $\sigma = (\mathcal{F}_\sigma, \mathcal{F}_\sigma)$ be Goldie's torsion theory for the category $R$-mod. The following conditions are equivalent:

(i) $\mathcal{F}_\sigma$ is a cover class;

(ii) $\sigma$ is of finite type;

(iii) $\sigma$ is perfect.

If, moreover, the ring $R$ is non-singular ($Z(R) = 0$), then these conditions are equivalent to the following three conditions:

(iv) every non-zero left ideal of $R$ contains a finitely generated essential left ideal;

(v) $rR$ is $\sigma$-noetherian;

(vi) for every hereditary torsion theory $\tau \geq \sigma$ the class $\mathcal{F}_\tau$ is a cover class.

We start with some preliminary lemmas, the symbol $\sigma$ will always denote Goldie’s torsion theory.

**Lemma 1.** Let $\tau \geq \sigma$ be a hereditary torsion theory for the category $R$-mod. Then

(i) a module $Q \in \mathcal{F}_\tau$ is $\tau$-injective if and only if it is injective;

(ii) a submodule $K \subseteq Q$ with $Q \in \mathcal{F}_\tau$ injective is $\tau$-closed if and only if it is injective. In this case the factor-module $Q/K$ is also injective.

**Proof.** (i) If $Q \in \mathcal{F}_\tau$ is $\tau$-injective and $E(Q)$ is the injective hull of $Q$, then $E(Q)/Q \in \mathcal{F}_\tau \subseteq \mathcal{F}_\sigma$ by [7; Corollary 44.3]. In view of the obvious fact $E(Q)/Q \in \mathcal{F}_\sigma$ we have $Q = E(Q)$. The converse is obvious.

(ii) If $K$ is $\tau$-closed in $Q$, then $Q/K \in \mathcal{F}_\tau \subseteq \mathcal{F}_\sigma$. Hence $K$ has no proper essential extension in $Q$ and consequently it is injective. The rest is clear.

**Lemma 2.** Let $\tau \geq \sigma$ be a hereditary torsion theory for the category $R$-mod. If every module has an $\mathcal{F}_\tau$-cover, then every directed union of $\tau$-torsionfree injective modules is $\tau$-torsionfree injective.

**Proof.** Let $K = \bigcup_{\alpha \in \Lambda} K_\alpha$ be a directed union of $\tau$-torsionfree injective modules, let $M = E(K)$ be the injective hull of $K$ and let $\varphi: G \to M/K$ be an $\mathcal{F}_\tau$-cover of the module $M/K$. Denoting by $\pi_\alpha: M/K_\alpha \to M/K$ the corresponding natural projections, there are homomorphisms $f_\alpha: M/K_\alpha \to G$ such that $\varphi f_\alpha = \pi_\alpha$ for
every $\alpha \in \Lambda$. Obviously, $\text{Ker } f_\alpha \subseteq K/K_\alpha$ and we are going to show that the equality holds for each $\alpha \in \Lambda$. If not, then $K_\beta/K_\alpha \not\subseteq \text{Ker } f_\alpha$ for some $\alpha, \beta \in \Lambda$ and so $0 \neq f_\alpha(K_\beta/K_\alpha) \cong K_\beta/L_\beta \in \mathcal{F}_r \subseteq \mathcal{F}_\sigma$ yields according to Lemma 1 that $0 \neq f_\alpha(K_\beta/K_\alpha) \subseteq \text{Ker } \varphi$ is injective. This contradicts the fact that $\varphi$ is an $\mathcal{F}_r$-cover of the module $M/K$ and consequently $\text{Im } f_\alpha \cong M/K \in \mathcal{F}_r$ for each $\alpha \in \Lambda$. Thus $M/K \in \mathcal{F}_\sigma \cap \mathcal{F}_r = 0$, $M = K$ and we are through. \hfill $\square$

**Lemma 3.** Let $\tau = (\mathcal{F}_r, \mathcal{F}_\tau)$ be an arbitrary hereditary torsion theory for the category $R$-mod. The following conditions are equivalent:

(i) every module has a $\tau$-torsionfree precover;

(ii) every injective module has a $\tau$-torsionfree precover;

(iii) every injective module has an injective $\tau$-torsionfree precover.

**Proof.** For an arbitrary injective module $M$ we obviously have the commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi} & M \\
\downarrow{i} & & \downarrow{j} \\
E(G) & \xrightarrow{\psi} & M
\end{array}
$$

where $i$ is the inclusion map of $G$ into its injective hull $E(G)$ and $\varphi$ is an $\mathcal{F}_\tau$-precover of the module $M$. Then $\psi$ is obviously an $\mathcal{F}_\tau$-precover of $M$ and consequently (ii) implies (iii).

Assuming (iii) let us consider the pullback diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\varphi} & M \\
\downarrow{i} & & \downarrow{j} \\
G & \xrightarrow{\psi} & E(M)
\end{array}
$$

where $M \in R$-mod is arbitrary and $\psi$ is an $\mathcal{F}_\tau$-precover of $E(M)$ with $G$ injective. Clearly, $i$ is injective, hence $F \in \mathcal{F}_\tau$ and the pullback property yields that $\varphi$ is an $\mathcal{F}_\tau$-precover of the module $M$. The rest is clear. \hfill $\square$

**Lemma 4.** Let $\tau = (\mathcal{F}_r, \mathcal{F}_\tau)$ be a hereditary torsion theory for the category $R$-mod. A homomorphism $\varphi: G \rightarrow M$ with $G \in \mathcal{F}_r$ and $M$ injective is an $\mathcal{F}_\tau$-precover of the module $M$ if and only if to each homomorphism $f: Q \rightarrow M$ with $Q \in \mathcal{F}_r$ injective, there exists a homomorphism $g: Q \rightarrow G$ such that $\varphi g = f$. 398
\textbf{Proof.} Only the sufficiency requires verification. So, let us consider the commutative diagram

\[
\begin{array}{ccc}
E(F) & \xrightarrow{i} & F \\
\downarrow{g} & & \downarrow{f} \\
G & \xrightarrow{\varphi} & M
\end{array}
\]

with the given \( \varphi, M \) injective and \( f : F \to M, F \in \mathcal{F}_\tau \), arbitrary. Then there is \( h : E(F) \to M \) with \( hi = f, M \) being injective, and \( g : E(F) \to G \) with \( \varphi g = h \) by the definition of a precovers. Thus \( \varphi(gi) = hi = f \) and the proof is complete. \( \square \)

\textbf{Proof (of Theorem 1).} Let \( \lambda \) be an arbitrary infinite cardinal and let \( \mathcal{M}_\lambda \) be any complete co-abstract set of modules of cardinalities at most \( \lambda \). For any \( M \in \mathcal{M}_\lambda \) we fix an \( \mathcal{F}_\sigma \)-precover \( \varphi_M : G_M \to M \) and denote by \( \kappa \) the first cardinal with \( \kappa > |G_M| \) for each \( M \in \mathcal{M}_\lambda \).

Further, let \( Q \in \mathcal{F}_\tau \) be an arbitrary injective module with \( |Q| \geq \kappa \) and let \( K \leq Q \) be its submodule such that \( |Q/K| \leq \lambda \). Then, obviously, \( Q \in \mathcal{F}_\sigma \) and consequently, by the above part, the factor-module \( Q/K \) has an \( \mathcal{F}_\sigma \)-precover \( \varphi : G \to Q/K \) with \( |G| < \kappa \). Thus, there is a homomorphism \( f : Q \to G \) such that \( \varphi f = \pi, \pi \) being the canonical projection \( Q \to Q/K \). Now \( \text{Ker } f = L \) is contained in \( K \) and it is a direct summand of \( Q \) by Lemma 1 (ii) owing to the fact that \( Q/L \cong \text{Im } f \in \mathcal{F}_\sigma \). Moreover, \( |Q/L| = |\text{Im } f| \leq |G| < \kappa \).

Now let \( M \in R\text{-mod} \) be an arbitrary injective module, \( \lambda = \max(|M|, \aleph_\alpha) \), and let \( \kappa \) be the cardinal corresponding to \( \lambda \) by the beginning of this proof. Further, let \( \mathcal{N}_\kappa \) be any complete co-abstract set of \( \tau \)-torsionfree injective modules of cardinalities less than \( \kappa \). We put \( G = \bigoplus_{Q \in \mathcal{N}_\kappa} Q^{(\text{Hom}(Q,M))} \) and \( \varphi : G \to M \) will denote the corresponding natural evaluation map. To verify that \( \varphi \) is a \( \tau \)-torsionfree precover of the module \( M \) we shall use Lemma 4. So, let \( Q \in \mathcal{F}_\tau \) be an arbitrary injective module and let \( f : Q \to M \) be an arbitrary homomorphism. For \( |Q| < \kappa \) there exists an isomorphic copy of \( Q \) lying in \( \mathcal{N}_\kappa \) and the existence of the homomorphism \( g : Q \to G \) with \( \varphi g = f \) can be easily verified. In the opposite case, for \( |Q| \geq \kappa \), denoting \( K = \text{Ker } f \) we have \( |Q/K| = |\text{Im } f| \leq |M| \leq \lambda \). Thus, by the above part, there is a direct summand \( L \) of \( Q \) contained in \( K \) and such that \( |Q/L| < \kappa \). Moreover, \( f \) naturally induces the homomorphism \( \tilde{f} : Q/L \to M \) such that \( \tilde{f} \pi = f, \pi : Q \to Q/L \) being the canonical projection. Thus there is \( \mathcal{F}_\tau \)-injective \( Q/L \to G \) with \( \varphi \tilde{f} = \tilde{f} \) by the previous case, so \( \varphi(\tilde{f} \pi) = \tilde{f} \pi = f \) and to complete the proof it suffices now to apply Lemma 3. \( \square \)

\textbf{Proof (of Theorem 2).} (i) implies (ii). It suffices to use Lemma 2 and [7; Proposition 42.9].

(ii) implies (i). This has been proved in [9] in the case of a faithful torsion theory and in [2] in the general case.
(ii) is equivalent to (iii). This is obvious, $\sigma$ being exact by Lemma 1 (see also [7; Corollary 44.3]).

Assume now that the ring $R$ is non-singular.

(ii) implies (iv). Since $R$ is non-singular, the Gabriel filter $\mathcal{L}_\sigma$ consists of essential left ideals only, and consequently every essential left ideal contains an essential finitely generated left ideal by the hypothesis. So, let $0 \neq I \subseteq R$ be an arbitrary non-essential left ideal of $R$ and let $J \subseteq R$ be any left ideal maximal with respect to $I \cap J = 0$. Then $I \oplus J$ is essential in $R$ and consequently there is a finitely generated left ideal $K = \sum_{i=1}^{n} R a_i \subseteq I \oplus J$ essential in $R$. Now $a_i = b_i + c_i$, $b_i \in I$, $c_i \in J$, $i = 1, \ldots, n$, and it remains to verify that the left ideal $\sum_{i=1}^{n} R b_i$ is essential in $I$. However, for an arbitrary element $0 \neq u \in I$ we have $0 \neq ru = \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} r_i b_i + \sum_{i=1}^{n} r_i c_i$ for suitable elements $r, r_1, \ldots, r_n \in R$, and consequently, $0 \neq ru = \sum_{i=1}^{n} r_i b_i$, as we wished to show.

(iv) is equivalent to (v). See [7; Proposition 20.1].

(iv) implies (vi). Let $I \in \mathcal{L}_\tau$ be arbitrary and let $K \subseteq I$ be a finitely generated left ideal essential in $I$. Then $I/K \in \mathcal{F}_\tau \subseteq \mathcal{F}_\tau$, hence $K \in \mathcal{L}_\tau$ and the torsion theory $\tau$ is of finite type. Now it suffices to use [2].

(vi) implies (i). This is trivial.

References


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