Abstract. Let $G$ be a graph. A vertex subversion strategy of $G$, say $S$, is a set of vertices in $G$ whose closed neighborhood is removed from $G$. The survival-subgraph is denoted by $G/S$. The Neighbor-Integrity of $G$, $NI(G)$, is defined to be $NI(G) = \min_{S \subseteq V(G)} \{ |S| + c(G/S) \}$, where $S$ is any vertex subversion strategy of $G$, and $c(G/S)$ is the maximum order of the components of $G/S$. In this paper we give some results connecting the neighbor-integrity and binary graph operations.

Keywords: vulnerability, integrity, neighbor-integrity

MSC 2000: 05C40, 05C85

1. Introduction

If we think of a graph as a model of a network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. In order to measure the vulnerability we have some parameters like connectivity, toughness, binding number, integrity and tenacity [4], [5], [6], [10]. But these parameters do not consider the effect which removal of a vertex has on the neighbors of that vertex. In a Spy Network, vertices correspond to stations or operatives, and edges represent lines of communication. If a station or an operative is captured, the adjacent stations will be betrayed and are therefore useless in the whole network. Therefore, instead of removing only vertices from a graph, we remove vertices and all of their adjacent vertices. The concept of Neighbor-Integrity was introduced as a measure of graph vulnerability in this sense by Margaret B. Cozzens and Shu-Shih Y. Wu [5].

Let $G$ be a simple graph without loops and multiple edges and let $u$ be any vertex in $G$. The set $N(u) = \{ v \in V(G); v \neq u, v$ and $u$ are adjacent $\}$ is the open neighborhood of $u$, and $N[u] = \{ u \} \cup N(u)$ denotes the closed neighborhood of $u$. A
A vertex \( u \) in \( G \) is said to be suverted if the closed neighborhood of \( u, N[u] \), is removed from \( G \). A set of vertices \( S = \{u_1, u_2, \ldots, u_m\} \) is called a vertex suversion strategy of \( G \) if each of the vertices in \( S \) has been suverted from \( G \). If \( S \) has been suverted from the graph \( G \), then the survival subgraph is disconnected, a clique, or the empty graph (see [5]). The survival subgraph is denoted by \( G/S \). The Neighbor-Integrity of a graph \( G \) is defined to be

\[
\text{NI}(G) = \min_{S \subseteq V(G)} \{|S| + c(G/S)\},
\]

where \( S \) is any vertex suversion strategy of \( G \), and \( c(G/S) \) is the maximum order of the components of \( G/S \) [5].

Cozzens and Wu [5], [7], [8], [9] obtained several results on the neighbor-integrity. In Section 2 the known results on the neighbor-integrity are given. In Section 3 we give the neighbor-integrity of graphs obtained by binary graph operations.

2. Basic results

In this section we will review some of the known results.

**Theorem 2.1** [5], [8]. The neighbor-integrity of
(a) the complete graph \( K_n \) is 1.
(b) the path \( P_n \) is

\[
\text{NI}(P_n) = \begin{cases} 
2\sqrt{n+3} - 4, & \text{if } n \geq 2; \\
1, & \text{if } n = 1.
\end{cases}
\]

If \( S \) achieves the neighbor-integrity of the graph \( P_n \), then \(|S| = \lceil \sqrt{n+3} \rceil - 1 \).
(c) the cycle \( C_n \) is

\[
\text{NI}(C_n) = \begin{cases} 
2\sqrt{n} - 3, & \text{if } n \geq 5; \\
2, & \text{if } n = 4; \\
1, & \text{if } n = 3.
\end{cases}
\]

If \( S \) achieves the neighbor-integrity of the graph \( C_n \), then \(|S| = \lceil \sqrt{n} \rceil - 1 \).

**Theorem 2.2** [5]. (a) The size of a maximum matching in \( G \) is an upper bound for \( \text{NI}(G) \).
(b) The independence number of \( G \) is an upper bound for \( \text{NI}(G) \).
(c) \( \text{NI}(G) = 1 \) if and only if \( G \) contains a spanning subgraph that is a star or \( G \) is a set of isolated vertices.
Theorem 2.3 [5]. Let $C_n^k$ be the $k$-th power of a cycle, where $n \geq 3$ and $1 \leq k \leq \lfloor n/2 \rfloor$. Then
\[
\text{NI}(C_n^k) = \begin{cases} 
[2\sqrt{n} - (2k + 1)], & \text{if } 1 \leq k < \frac{\sqrt{n} - 1}{2} \\
\lfloor n/(2k + 1) \rfloor, & \text{otherwise}.
\end{cases}
\]

Theorem 2.4 [11].
(a) For any graph $G$, $\text{NI}(G \times P_n) \leq n \text{NI}(G)$.
(b) For any graphs $G$ and $H$, $\text{NI}(G \times H) \geq \max\{\text{NI}(G), \text{NI}(H)\}$.

3. Graph operations and neighbor-integrity

In this section we consider the binary graph operations. These operations are join, composition, product and corona of two graphs. The graphs $G_1$ and $G_2$ have disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively.

(a) Join

Definition 3.1. The union $G = G_1 \cup G_2$ of graphs $G_1$ and $G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The join $G = G_1 + G_2$ of graphs $G_1$ and $G_2$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$.

Definition 3.2. A subset $S$ of $V(G)$ such that every edge of $G$ has at least one end in $S$ is called a covering of $G$. The number of vertices in a minimum covering of $G$ is the covering number of $G$ and is denoted by $\beta(G)$.

Theorem 3.1. Let $G_1$ and $G_2$ be two graphs. Then
\[
\text{NI}(G_1 + G_2) = \begin{cases} 
1, & \text{if } \beta(G_1) = 1 \text{ or } \beta(G_2) = 1; \\
2, & \text{otherwise}.
\end{cases}
\]

Proof. This proof is also valid up to symmetry for $G_2$. If $\beta(G_1) = 1$, then we can find a vertex $u$ such that $u \in V(G_1)$ and $N[u] = V(G_1 + G_2)$. Hence $(G_1 + G_2)/\{u\}$ is empty and $\text{c}(G_1 + G_2)/\{u\}) = 0$. Therefore NI$(G_1 + G_2) = 1$, if $\beta(G_1) = 1$. On the other hand, it is always true that NI$(G_1 + G_2) \leq 2$ and it cannot be 1 if $\beta(G_1) \geq 2$.

This completes the proof.

(b) Composition

Definition 3.3. The composition $G_1[G_2]$ of two graphs $G_1$ and $G_2$ has its vertex set $V(G_1) \times V(G_2)$, with $(u_1, u_2)$ adjacent to $(v_1, v_2)$ if either $u_1$ is adjacent to $v_1$ in $G_1$ or $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ in $G_2$. 247
Definition 3.4. A vertex dominating set for a graph $G$ is a set $S$ of vertices such that every vertex of $G$ belongs to $S$ or is adjacent to a vertex of $S$. The minimum cardinality of a vertex dominating set in a graph $G$ is called the vertex dominating number of $G$ and is denoted by $\gamma(G)$.

Theorem 3.2. Let $H$ be a graph of order $n \geq 2$ and let $G$ be a graph. Then $\text{NI}(G[H]) = \min\{\text{NI}(G) + c(G/S)(n - 2), \gamma(G) + \min_{v \in V(H)}\{c(H/v)\}, \gamma(G)(1 + \min_{v \in V(H)}\{\text{NI}(H/v)\})\}$ where $S \subseteq V(G)$ and $|S| + c(G/S) = \text{NI}(G)$.

Proof. Let $X$ be a subset of $V(G[H])$ such that $|X| + c(G[H]/X) = \text{NI}(G[H])$. The graph $G[H]$ contains $n$ copies of $G$ and let $S$ be a set of removed vertices from any copy of $G$. Then we have two cases:

Case 1: If $S$ is not a dominating set, then $X$ must contain the vertices of every copy of $G$ in $G[H]$. Hence $|X| = |S| < \gamma(G)$ and $c(G[H]/X) = c(G/S)(n - 1)$. When $S$ realizes the neighbor-integrity of $G$, we have

$$
(1) \quad \text{NI}(G[H]) = \min_{S \subseteq V(G)}\{|S| + c(G/S)(n - 1)\} = \text{NI}(G) + c(G/S)(n - 2).
$$

Case 2: If $S$ is a dominating set, then $|X| \geq \gamma(G)$.

$$
(2) \quad \text{If } |X| = \gamma(G), \text{ then } c(G[H]/X) = \min_{v \in V(H)}\{c(H/v)\}.
$$

$$
(3) \quad \text{If } |X| > \gamma(G), \text{ then } c(G[H]/X) = \min_{v \in V(H)}\{\text{NI}(H/v)\}\gamma(G).
$$

The theorem follows from (1), (2) and (3).

Corollary 3.1. (a) $\text{NI}(P_m[P_n]) = \min\{\lceil 2\sqrt{m + 3} \rceil(n - 1) + \lceil \sqrt{m + 3} \rceil(2 - n) - 3n + 2, \lceil \frac{m}{2} \rceil + \lceil \frac{m - 2}{2} \rceil, \lceil \frac{m}{2} \rceil(2\sqrt{m} - 3)\}$,

(b) $\text{NI}(P_m[C_n]) = \min\{\lceil 2\sqrt{m + 3} \rceil(n - 1) + \lceil \sqrt{m + 3} \rceil(2 - n) - 3n + 2, \lceil \frac{m}{2} \rceil + n - 3, \lceil \frac{m}{2} \rceil(2\sqrt{m} - 3)\}$,

(c) $\text{NI}(C_m[P_n]) = \min\{\lceil 2\sqrt{m - 3} \rceil(n - 1) + \lceil \sqrt{m} \rceil(2 - n), \lceil \frac{m}{2} \rceil + \lceil \frac{m - 2}{2} \rceil, \lceil \frac{m}{2} \rceil(2\sqrt{m} - 3)\}$,

(d) $\text{NI}(C_m[C_n]) = \min\{\lceil 2\sqrt{m - 3} \rceil(n - 1) + \lceil \sqrt{m} \rceil(2 - n), \lceil \frac{m}{2} \rceil + n - 3, \lceil \frac{m}{2} \rceil(2\sqrt{m} - 3)\}$.

Proof. (a) follows from Theorem 3.2 and 2.1(b). The proof of the other parts is similar.
Definition 3.5. The (Cartesian) product $G_1 \times G_2$ of graphs $G_1$ and $G_2$ also has $V(G_1) \times V(G_2)$ as its vertex set, but here $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if either $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ or $u_2 = v_2$ and $u_1$ is adjacent to $v_1$.

In the next theorem we give $NI(K_2 \times P_n)$ for $5 \leq n < 34$ and a lower bound for $NI(K_2 \times P_n)$ where $n \geq 34$. In Theorem 3.4, we compute the exact result of $NI(K_2 \times P_n)$ for $n \geq 34$.

Theorem 3.3. Let $n$ be a positive integer. If $5 \leq n < 34$, then $NI(K_2 \times P_n) = \lceil n/3 \rceil + 1$. Moreover, if $n \geq 34$, then $NI(K_2 \times P_n) \geq \lceil 2\sqrt{2n+4} \rceil - 5$.

Proof. Let $S \subseteq V(K_2 \times P_n)$ and let $b$ be the maximum order of the components of $(K_2 \times P_n)/S$. Then we have two cases, depending on $b$:

Case 1: Let $b = 1$. In order to obtain the components of order 1, we have to remove $\lceil n/3 \rceil$ vertices. Hence $NI(K_2 \times P_n) = \lceil n/3 \rceil + 1$.

Case 2: Let $b \geq 2$. If we remove $|S| = a$ vertices, then the number of components is at most $a + 1$. So

$$c((K_2 \times P_n)/S) \geq \frac{2n - 4a}{a + 1},$$

and

$$NI(K_2 \times P_n) \geq \min_a \left\{ a + \frac{2n - 4a}{a + 1} \right\}.$$ 

The function $f(a) = a + \frac{2n - 4a}{a + 1}$ assumes its minimum value at $a = -1 + \sqrt{2n+4}$ and $f(-1 + \sqrt{2n+4}) = 2\sqrt{2n+4} - 5$. Since the neighbor-integrity is integer valued, we round this result up to get a lower bound and so $NI(K_2 \times P_n) \geq \lceil 2\sqrt{2n+4} \rceil - 5$ if $b \geq 2$.

Consequently, we have

$$NI(K_2 \times P_n) \geq \min \{ \left\lceil n/3 \right\rceil + 1, \left\lceil 2\sqrt{2n+4} \right\rceil - 5 \} \text{ for every } b.$$ 

One can easily show that

$$\left\lceil n/3 \right\rceil + 1 \leq \left\lceil 2\sqrt{2n+4} \right\rceil - 5 \text{ for } 5 \leq n < 34.$$ 

Therefore

$$NI(K_2 \times P_n) = \left\lceil n/3 \right\rceil + 1 \text{ for } 5 \leq n < 34$$

and

$$NI(K_2 \times P_n) \geq \left\lceil 2\sqrt{2n+4} \right\rceil - 5 \text{ for } n \geq 34.$$ 

Hence the proof is completed. \qed

Before we prove Theorem 3.4, we need the following lemma.
Lemma 3.1. Let \( a = -1 + \lfloor \sqrt{2n + 4} \rfloor, b = -4 + \lfloor \sqrt{2n + 4} \rfloor \) and \( n \geq 0 \). The following inequalities hold.
(a) \( 2n - ab - 4a + 1 \leq 3b + 7 \),
(b) \( 2n - ab - 3a - 1 \leq 3b + 6 \).

Proof. (a) We shall show that \( 2n + 6 \leq \lfloor \sqrt{2n + 4} \rfloor (2 + \lfloor \sqrt{2n + 4} \rfloor) \). For every \( n \geq 0 \),
\[
2n + 6 \leq \sqrt{2n + 4} (1 + \sqrt{2n + 4}) \leq \sqrt{2n + 4} (2 + \lfloor \sqrt{2n + 4} \rfloor) \\
\leq \lfloor \sqrt{2n + 4} \rfloor (2 + \lfloor \sqrt{2n + 4} \rfloor).
\]
The proof of part (b) can be reduced to a sequence of inequalities similar to those in (a). \( \square \)

Theorem 3.4. Let \( a = -1 + \lfloor \sqrt{2n + 4} \rfloor, b = -4 + \lfloor \sqrt{2n + 4} \rfloor \) and \( n \geq 34 \). Then
\[
\text{NI}(K_2 \times P_n) = \begin{cases} 
  a + b, & \text{if } n \leq \frac{1}{2}(ab + 4a + b - 2); \\
  a + b + 1, & \text{if } n > \frac{1}{2}(ab + 4a + b - 2) \text{ and } n \leq \frac{1}{2}(ab + 4a + 2b + 2); \\
  a + b + 2, & \text{otherwise.}
\end{cases}
\]

Proof. Let \( S \subseteq V(K_2 \times P_n) \) and let \( b \) be the maximum order of the components of \( (K_2 \times P_n)/S \). If we remove \(|S| = a \) vertices from any copy of \( P_n \), then we have \( a + 1 \) components for \( n \geq 34 \). So we consider two cases, depending on \( b \):

Case 1: Let \( b \) be an even number. If we remove \(|S| = a \) vertices in such a way that the first component has \( b - 1 \) vertices and each of the \( a - 1 \) components have \( b \) vertices from any copy of \( P_n \), then we have \( a + 1 \) components as shown in Figure 1.

Figure 1 shows a specific situation and we can select the vertices \( x_i \) from different copies of \( P_n \). Notice that our aim is to investigate whether some vertices should be deleted or not from the last component.

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In this case, the last component has $2n - ab - 4a + 1$ vertices and $2n - ab - 4a + 1$ must be an odd number. By Lemma 3.1(a), we know that $2n - ab - 4a + 1 \leq 3b + 7$. That is, we must remove at most two vertices from the last component. Hence we have the following three possibilities for the last component:

(a) The last component has at most $b - 1$ vertices,
(b) The last component has at least $b$ and at most $2b + 3$ vertices,
(c) The last component has at least $2b + 4$ and at most $3b + 7$ vertices.

According to these possibilities, the neighbor-integrity of $K_2 \times P_n$ is equal to

$$
\begin{cases}
  a + b, & \text{if } n \leq \frac{1}{2}(ab + 4a + b - 2); \\
  a + b + 1, & \text{if } n > \frac{1}{2}(ab + 4a + b - 2) \text{ and } n \leq \frac{1}{2}(ab + 4a + 2b + 2); \\
  a + b + 2, & \text{if } n > \frac{1}{2}(ab + 4a + 2b + 2) \text{ and } n \leq \frac{1}{2}(ab + 4a + 3b + 6).
\end{cases}
$$

(4)

Case 2: Let $b$ be an odd number. If we remove $|S| = a$ vertices in such a way that the first component has $b$ vertices and each of the $a - 1$ components have $b - 1$ vertices from any copy of $P_n$, then we have $a + 1$ components as shown in Figure 2. Let $\{x_1, x_2, \ldots, x_a\}$ be a set of removed vertices from any copy of $P_n$. Notice that Figure 2 shows a specific situation and we can select the vertices $x_i$ from different copies of $P_n$.

![Figure 2](image)

Our aim is to investigate whether some vertices should be deleted or not from the last component.

In this case, the last component has $2n - ab - 3a - 1$ vertices. By Lemma 3.1(b), we know that $2n - ab - 3a - 1 \leq 3b + 6$. That is, we must remove at most two vertices from the last component. Hence we have three possibilities for the last component and so the neighbor-integrity of $K_2 \times P_n$ is equal to

$$
\begin{cases}
  a + b, & \text{if } n \leq \frac{1}{2}(ab + 3a + b + 1); \\
  a + b + 1, & \text{if } n > \frac{1}{2}(ab + 3a + b + 1) \text{ and } n \leq \frac{1}{2}(ab + 3a + 2b + 4); \\
  a + b + 2, & \text{if } n > \frac{1}{2}(ab + 3a + 2b + 4) \text{ and } n \leq \frac{1}{2}(ab + 3a + 3b + 7).
\end{cases}
$$

(5)
By elementary arithmetical operations it follows from (4) and (5) that we have

$$
\text{NI}(K_2 \times P_n) = \begin{cases} 
  a + b, & \text{if } n \leq \frac{1}{2}(ab + 4a + b - 2); \\
  a + b + 1, & \text{if } n > \frac{1}{2}(ab + 4a + b - 2) \text{ and } n \leq \frac{1}{2}(ab + 4a + 2b + 2); \\
  a + b + 2, & \text{otherwise}.
\end{cases}
$$

\[ \blacksquare \]

**Theorem 3.5.** Let \( n \) be a positive integer. If \( 5 \leq n < 39 \), then

$$
\text{NI}(K_2 \times C_n) = \begin{cases} 
  (n/3) + 2, & \text{if } n = 3k \text{ and } k \text{ is odd}; \\
  [n/3] + 1, & \text{otherwise}.
\end{cases}
$$

Moreover, if \( n \geq 39 \), then \( \text{NI}(K_2 \times C_n) \geq \lfloor 2\sqrt{2n} \rfloor - 4 \).

**Proof.** Let \( S \subseteq V(K_2 \times C_n) \) and let \( b \) be the maximum order of the components of \((K_2 \times C_n)/S\). Then we have two cases, depending on \( b \):

**Case 1:** Let \( b = 1 \). If \( n = 3k \) and \( k \) is odd, then we have to remove \((n/3) + 1\) vertices and so \( \text{NI}(K_2 \times C_n) = (n/3) + 2 \). Otherwise we have to remove \([n/3]\) vertices and so \( \text{NI}(K_2 \times C_n) = [n/3] + 1 \).

**Case 2:** Let \( b \geq 2 \). If we remove \(|S| = a\) vertices, then the number of components is at most \( a \). So

$$
\text{NI}(K_2 \times C_n) \geq \min_a \left\{ a + \frac{2n - 4a}{a} \right\}
$$

and

$$
\text{NI}(K_2 \times P_n) \geq \lfloor 2\sqrt{2n + 4} \rfloor - 5 \text{ if } b \geq 2.
$$

The rest of the proof is very similar to that of Theorem 3.3. \[ \blacksquare \]

**Theorem 3.6.** Let \( a = \lfloor \sqrt{2n} \rfloor \), \( b = \lceil \sqrt{2n} \rceil - 4 \) and \( n \geq 39 \). Then

$$
\text{NI}(K_2 \times C_n) = \begin{cases} 
  a + b - 1, & \text{if } n \leq \frac{1}{2}(ab + 3a); \\
  a + b, & \text{if } n > \frac{1}{2}(ab + 3a) \text{ and } n \leq \frac{1}{2}(ab + 4a); \\
  a + b + 1, & \text{if } n > \frac{1}{2}(ab + 4a) \text{ and } n \leq \frac{1}{2}(ab + 4a + b + 4); \\
  a + b + 2, & \text{otherwise}.
\end{cases}
$$

**Proof.** Let \( S \subseteq V(K_2 \times C_n) \) and let \( b \) be the maximum order of the components of \((K_2 \times C_n)/S\). If we remove \(|S| = a\) vertices from any copy of \( C_n \), then the number of components is \( a \). Now remove \(|S| = a\) vertices in such a way that each of the \( a - 1 \) components will have \( b \) vertices.

Our aim is to investigate whether some vertices should be deleted or not from the last component. Then we have two cases, depending on \( b \):
Case 1: If $b$ is an even number, then the last component has $2n - ab - 4a + b$ vertices.

Case 2: If $b$ is an odd number, then the last component has $2n - ab - 3a + b - 1$ vertices.

The rest of the proof is very similar to that of Theorem 3.4.

(d) Corona

**Definition 3.6.** The *corona* of two graphs $G_1$ (on $n$ vertices) and $G_2$ is defined as the graph $G$ obtained by taking one copy of $G_1$ of order $n$ and $n$ copies of $G_2$, and then joining the $i$'th vertex of $G_1$ to every vertex in the $i$'th copy of $G_2$. The corona of two graphs $G_1$ and $G_2$ is denoted by $G_1 \circ G_2$.

**Theorem 3.7.** Let $G_1$ and $G_2$ be graphs with orders $m$ and $n$, respectively. Then

(a) If $m \leq n$, then $\text{NI}(G_1 \circ G_2) = m$.
(b) If $m > n$, then $\text{NI}(G_1 \circ G_2) \geq n + 1$.

**Proof.** Let $S \subseteq V(G_1 \circ G_2)$. If $m \leq n$, then $S = V(G_1)$ and $c((G_1 \circ G_2)/S) = 0$. So $\text{NI}(G_1 \circ G_2) = m$. Otherwise $S \subset V(G_1)$ and hence $c((G_1 \circ G_2)/S) \geq n$. Then $\text{NI}(G_1 \circ G_2) \geq n + 1$.

The proof is completed.

**Definition 3.7.** The *wheel* with $m$ spokes, $W_{1,m}$, is a graph that contains an $m$-cycle and one additional vertex that is adjacent to all vertices of the cycle.

**Theorem 3.8.** Let $G$ be a graph of order $n$ and $W_{1,m}$ a wheel graph. Then

$$\text{NI}(W_{1,m} \circ G) = \begin{cases} m + 1, & \text{if } m + 1 \leq n; \\ n + 1, & \text{otherwise}. \end{cases}$$

**Proof.** Let $S \subseteq V(W_{1,m})$ and $u$ be a vertex which is adjacent to all the vertices of the $m$-cycle. The first part of the proof follows from Theorem 3.7. Otherwise, $S = \{u\}$ and $\text{NI}(W_{1,m} \circ G) = n + 1$.

The proof is completed.
References


Author’s address: Alpay Kirlangic, Department of Mathematics, Science Faculty, Ege University-35100, Bornova-Izmir, Turkey, e-mail: kirlan@bornova.ege.edu.tr.