OPERATORS ON GMV-ALGEBRAS

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Abstract. Closure GMV-algebras are introduced as a commutative generalization of closure MV-algebras, which were studied as a natural generalization of topological Boolean algebras.

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1. Introduction

It is well known that Boolean algebras are algebraic counterparts of the classical propositional two-valued logic similarly as MV-algebras (see [1], [2]) are for Lukasiewicz infinite valued logic. Every MV-algebra contains a Boolean algebra, which is formed by the set of its idempotent elements. The same property is possessed also by GMV-algebras, the non-commutative generalization of MV-algebras (see [5] or [9]).

In the paper [11], closure MV-algebras are introduced and studied as a natural generalization of topological Boolean algebras (see [12]). The additive closure operator is here introduced as a natural generalization of the topological closure operator on topological Boolean algebras. The aim of this paper is to generalize the results of [11] to the case of GMV-algebras.

The paper is divided into Introduction and three main sections. In Section 2, the closure GMV-algebras are introduced and the relation between additive closure operators and multiplicative interior operators on GMV-algebras is described. In the case of closure MV-algebras there is a one-to-one correspondence between additive closure operators and multiplicative interior operators. In the paper, it is shown that this correspondence exists also for closure GMV-algebras, but the relation is there a little bit different.
In Section 3 one works with idempotent elements of a closure GMV-algebra, for example, it is shown that every idempotent element of a closure GMV-algebra induces a new closure GMV-algebra, similarly as is the case for closure MV-algebras.

Finally, in the last section GMV-algebras are factorized via their normal ideals and the connections between congruences and normal c-ideals of closure GMV-algebras are described with help of DRL-monoids, which are studied in [6] or in [13].

2. Closure GMV-algebras

**Definition 1.** An algebra $\mathcal{A} = (A, \oplus, \neg, \sim, 0, 1)$ of signature $\langle 2, 1, 1, 0, 0 \rangle$ is called a GMV-algebra, iff the following conditions are satisfied for each $x, y, z \in A$:

1. **(GMV1)** $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
2. **(GMV2)** $x \oplus 0 = 0 = 0 \oplus x$,
3. **(GMV3)** $x \oplus 1 = 1 = 1 \oplus x$,
4. **(GMV4)** $\neg 0 = 0 = 0 \neg$,
5. **(GMV5)** $\neg (x \oplus y) = \neg (x \oplus y)$,
6. **(GMV6)** $y \oplus (x \sim y) = (\neg y \oplus x) \oplus y = x \oplus (y \oplus \sim x) = (\neg x \oplus y) \oplus x$,
7. **(GMV7)** $y \oplus (x \sim y) = (\neg y \oplus x) \oplus y$,
8. **(GMV8)** $\neg \neg x = x$,

where $x \sim y := \neg (x \oplus y)$.

**Remark 1.** We can define the relation of the partial order $\leq$ on every GMV-algebra $\mathcal{A}$. We put $x \leq y \iff \neg x \oplus y = 1 \quad \forall x, y \in A$.

Then $(A, \leq)$ is a distributive lattice, where each $x, y$ satisfy

- $x \lor y = y \oplus (x \sim y) = (\neg y \oplus x) \oplus y$,
- $x \land y = y \oplus (x \sim y) = (\neg y \oplus x) \oplus y$.

**Definition 2.** An algebraic structure $G = (G, +, \lor, \land)$ of signature $\langle 2, 0, 2, 2 \rangle$ is called an $l$-group iff

1. $(G, +, 0)$ is a group,
2. $(G, \lor, \land)$ is a lattice,
3. $x + (y \lor z) + w = (x + y + w) \lor (x + z + w) \quad \forall x, y, z, w \in G$,
   
   $x + (y \land z) + w = (x + y + w) \land (x + z + w) \quad \forall x, y, z, w \in G$.

An element $u \in G$ ($u > 0$) is said to be a strong unit of an $l$-group $G$ iff

$$(\forall a \in G)(\exists n \in \mathbb{N})(a \leq nu),$$

where $nu \overset{\text{def}}{=} u + u + \ldots + u$. 338
If an \( l \)-group \( G \) contains a strong unit \( u \), then we call it a \textit{unital} \( l \)-group and write \((G, u)\).

Let \( \leq \) be the lattice order on \((G, \lor, \land)\). Then for the \( l \)-group \( G \) we can use notation \( G = (G, +, 0, \leq) \), which is equivalent to the former notation.

Remark 2.

a) Let \((G, +, 0, \leq)\) be an \( l \)-group and let \( u \) be a strong unit of \( G \). If we put \( x \oplus y := (x + y) \land u \), \( \sim x := u - x \), \( \sim x := -x + u \), then \( \Gamma(G, u) = ([0, u], \oplus, \sim, \sim, 0, u) \) is a \( GMV \)-algebra.

b) On the other hand, A. Dvurečenskij has shown that for each \( GMV \)-algebra \( \mathcal{A} \) there exists a unital \( l \)-group \((G, u)\) such that \( \mathcal{A} \equiv \Gamma(G, u) \) — see [4].

We can now define the additive closure and the multiplicative interior operator in the same way as for the \( MV \)-algebras. From [12], Theorem 5 and Theorem 6, we know that additive closure operators on an \( MV \)-algebra \( \mathcal{A} \) generalize topological closure operators on the Boolean algebra \( B(\mathcal{A}) \) of its idempotent elements.

Definition 3.

a) Let \( \mathcal{A} = (A, \oplus, \neg, \sim, 0, 1) \) be a \( GMV \)-algebra and \( \text{Cl}: A \to A \) a mapping. Then 
\( \text{Cl} \) is called an \textit{additive closure operator} on \( \mathcal{A} \) iff for each \( a, b \in A \)
1. \( \text{Cl}(a \oplus b) = \text{Cl}(a) \oplus \text{Cl}(b) \);
2. \( a \leq \text{Cl}(a) \);
3. \( \text{Cl}(\text{Cl}(a)) = \text{Cl}(a) \);
4. \( \text{Cl}(0) = 0 \).

b) If \( \text{Cl} \) is an additive closure operator on \( \mathcal{A} \) then \( \mathcal{A} = (A, \oplus, \neg, \sim, 0, 1, \text{Cl}) \) is called a \textit{closure \( GMV \)-algebra} and \( \text{Cl}(a) \) is called the \textit{closure} of an element \( a \in A \). An element \( a \) is said to be \textit{closed} iff \( \text{Cl}(a) = a \).

Definition 4.

a) Let \( \mathcal{A} = (A, \ominus, \neg, \sim, 0, 1) \) be a \( GMV \)-algebra and \( \text{Int}: A \to A \) a mapping. Then \( \text{Int} \) is called a \textit{multiplicative interior operator} on \( \mathcal{A} \) if and only if for each \( a, b \in A \)
1. \( \text{Int}(a \circ b) = \text{Int}(a) \circ \text{Int}(b) \);
2. \( \text{Int}(a) \leq a \);
3. \( \text{Int}(\text{Int}(a)) = \text{Int}(a) \);
4. \( \text{Int}(1) = 1 \).

b) If \( \text{Int} \) is a multiplicative interior operator on \( \mathcal{A} \), then an algebra \( \mathcal{A} = (A, \ominus, \neg, \sim, 0, 1, \text{Int}) \) is called an \textit{interior \( GMV \)-algebra} and \( \text{Int}(a) \) is called the \textit{interior} of an element \( a \in A \). An element \( a \) is said to be \textit{open} iff \( \text{Int}(a) = a \).
Lemma 1. Let \( \mathcal{A} = (A, \oplus, \neg, 0, 1, \text{Cl}) \) be a closure GMV-algebra. We put
a) \( \text{Int}^\sim(a) = \neg\text{Cl}(\neg a) \),
b) \( \text{Int}^\sim(a) = \sim\text{Cl}(\neg a) \)
for each \( a \in A \). Then these two operators are multiplicative interior operators on \( \mathcal{A} \)
and for each \( a, b \in A \) we have
a) \( \text{Cl}(a) = \sim\text{Int}^\sim(\neg a) \),
b) \( \text{Cl}(a) = \neg\text{Int}^\sim(\neg a) \).

Proof. We restrict ourselves to the case a), since b) can be proved analogously.
1. \( \text{Int}^\sim(a \odot b) = \neg\text{Cl}(\neg a \odot \neg b) = \neg\text{Cl}(\neg a \odot \neg b) = \neg\text{Cl}(\neg a \odot \text{Cl}(\neg b)) = \neg\text{Cl}(\neg a \odot \text{Cl}(\neg b)); \)
2. \( \text{Int}^\sim(a) = \neg\text{Cl}(\neg a) \leq \sim\neg a = a; \)
3. \( \text{Int}^\sim(\text{Int}^\sim(a)) = \neg\text{Cl}(\neg \text{Cl}(\neg a)) = \neg\text{Cl}(\text{Cl}(\neg a)) = \neg\text{Cl}(\neg a) = \text{Int}^\sim(a); \)
4. \( \text{Int}^\sim(1) = \neg\text{Cl}(1) = \neg\text{Cl}(0) = \neg 0 = 1. \)

The next lemma shows that the operator \( \text{Cl} \) from Definition 3 and the operators \( \text{Int}^\sim, \text{Int}^\sim \) from Lemma 1 are all isotone.

Lemma 2. If \( a \leq b \) for any \( a, b \in A \), then \( \text{Cl}(a) \leq \text{Cl}(b) \) and \( \text{Int}^\sim(a) \leq \text{Int}^\sim(b) \),
as well as \( \text{Int}^\sim(a) \leq \text{Int}^\sim(b) \).

Proof. Let \( a \leq b \). Then \( \text{Cl}(b) = \text{Cl}(a \lor b) = \text{Cl}(a \oplus (b \odot \neg a)) \). Therefore
\( \text{Cl}(b) = \text{Cl}(a) \oplus \text{Cl}(b \odot \neg a) \geq \text{Cl}(a) \lor \text{Cl}(b \odot \neg a) \), and so \( \text{Cl}(a) \leq \text{Cl}(b) \).

Similarly from \( a \leq b \) we have \( \text{Int}^\sim(a) = \text{Int}^\sim(a \land b) = \text{Int}^\sim(b \odot (a \oplus \neg b)) = \text{Int}^\sim(b) \odot \text{Int}^\sim(a \oplus \neg b) \leq \text{Int}^\sim(b) \land \text{Int}^\sim(a \oplus \neg b) \), hence \( \text{Int}^\sim(a) \leq \text{Int}^\sim(b) \) and analogously for \( \text{Int}^\sim \).

In the case of closure MV-algebras, here we were able to construct from one closure operator just one interior operator by the rule \( \text{Int}(x) = \neg\text{Cl}(\neg x) \) and then get back to the original one. Now, let us try to describe the situation for closure GMV-algebras.

Remark 3. Let us consider a closure GMV-algebra \( \mathcal{A} \) and a mapping \( f : A \to A \). We can define two new operators \( \Phi^\sim(f) \) and \( \Phi^\sim(f) \) on \( A \) by the rules \( \Phi^\sim(f)(a) = \neg f(\neg a) \) and \( \Phi^\sim(f)(a) = \neg f(\neg a) \). Then we clearly have that \( \Phi^\sim \circ \Phi^\sim = \text{id} = \Phi^\sim \circ \Phi^\sim \) and if we take an additive closure operator \( \text{Cl} \) on \( \mathcal{A} \) instead of the arbitrary mapping \( f \) on \( \mathcal{A} \), then (by Lemma 1) we see that there exists a one-to-one correspondence between the additive closure operators and the multiplicative interior operators on the closure GMV-algebras. Compared to closure MV-algebras, the relation is here a little bit different as we are going to show.
Let us denote for each even non-negative integer $i$ and for an operator $\text{Cl}_0$

\[
\text{Cl}_i^- = \Phi^- \circ \ldots \circ \Phi^-(\text{Cl}_0),
\]

\[
\text{Cl}_i^+ = \Phi^+ \circ \ldots \circ \Phi^+(\text{Cl}_0)
\]

and for each odd non-negative integer $i$

\[
\text{Int}_i^+ = \Phi^+ \circ \ldots \circ \Phi^+(\text{Cl}_0),
\]

\[
\text{Int}_i^- = \Phi^- \circ \ldots \circ \Phi^-(\text{Cl}_0).
\]

The following theorem is an easy consequence of the preceding Remark 3 and of Lemma 1.

**Theorem 3.** Let $\text{Cl}_0$ be an additive closure operator on a GMV-algebra $\mathcal{A}$. Then we have for each $k \in \mathbb{N} \cup \{0\}$

a) $\text{Cl}_{2k}$ and $\text{Cl}_{2k}^+$ are additive closure operators on $\mathcal{A}$;

b) $\text{Int}_{2k+1}^-$ and $\text{Int}_{2k+1}^+$ are multiplicative interior operators on $\mathcal{A}$.

### 3. Idempotent elements of closure GMV-algebras

Now, we can consider the set $B(\mathcal{A}) = \{a \in A; \ a \oplus a = a\}$ of additively idempotent elements of a GMV-algebra $\mathcal{A}$. One can show that $B(\mathcal{A})$ is just the set of multiplicatively idempotent elements in $\mathcal{A}$. $B(\mathcal{A})$ is a sublattice of the lattice $(A, \vee, \wedge)$, contains 0 a 1 and is also a Boolean algebra. Analogously as for $MV$-algebras one can show that the operations $\oplus$, $\odot$ coincide on $B(\mathcal{A})$ with the lattice operations $\vee$, $\wedge$—see [10].

**Lemma 4.** Let $\mathcal{A}$ be a GMV-algebra and let $a$ be an idempotent element in $\mathcal{A}$. Then

a) $y \odot a = a \odot y = a \wedge y$,

b) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$,

c) $(x \oplus y) \odot a = (x \odot a) \oplus (y \odot a)$

for each $x, y \in A$.

**Proof.**

a) Let $y \leq a$. Then $a \leq y \oplus a \leq a \odot a = a$, thus $y \oplus a = a$ and hence,

by [9], Theorem 7, $y \odot a = y = y \wedge a$. 

341
Let now $y \in A$ be arbitrary. Clearly $y \circ a \leq y, a$. Let $z \in A, z \leq y, a$. Then also $z = z \circ a \leq y \circ a$, and consequently $y \circ a = y \land a$. Similarly $a \circ y = a \land y$.

b) Let $a \in B(\mathcal{A})$. Using distributivity of \( \oplus \) over \( \land \), we obtain
\[
(a \land x) \oplus (a \land y) = (a \oplus a) \land (x \oplus a) \land (a \oplus y) \land (x \oplus y),
\]
hence by a), \( a \circ (x \oplus y) = (a \circ x) \oplus (a \circ y) \).

c) Analogously to the case b).

Similarly as for closure $MV$-algebras, we can show that every idempotent element $a$ in a closure $GMV$-algebra $\mathcal{A}$ determines a new closure $GMV$-algebra on the interval $[0, a]$.

**Theorem 5.** Let $\mathcal{A} = (A, \oplus, \neg, \sim, 0, 1, \text{Cl})$ be a closure $GMV$-algebra and let $a$ be an idempotent element in $\mathcal{A}$. We put

- $x \oplus_a y = x \oplus y,$
- $\neg_a x = \neg (x \oplus \neg a),$ 
- $\sim_a x = \sim (\neg a \oplus x),$ 
- $0_a = 0,$ 
- $1_a = a,$ 
- $\text{Cl}_a(x) = a \circ \text{Cl}(x)$

for each $x, y \in A$. Then $\mathcal{A}_a = ([0, a], \oplus_a, \neg_a, \sim_a, 0_a, 1_a, \text{Cl}_a)$ is a closure $GMV$-algebra and we have

- $x \oplus_a y = x \oplus y,$
- $\text{Int}_a^\neg (x) = a \circ \text{Int}^\neg (\neg a \oplus x),$
- $\text{Int}_a^\sim (x) = a \circ \text{Int}^\sim (x \oplus \neg a).$

**Proof.** Availability of axioms (GMV1)–(GMV8) from Definition 1 for the algebra $([0, a], \oplus_a, \neg_a, \sim_a, 0, a)$ are proved in [9], so $\mathcal{A}_a$ is a $GMV$-algebra. In the second part of the proof we need to show that $\text{Cl}_a$ is an additive closure operator on $\mathcal{A}_a$.

1. $\text{Cl}_a(x \oplus y) = a \circ \text{Cl}(x \oplus y) = a \circ (\text{Cl}(x) \oplus \text{Cl}(y)) = (a \circ \text{Cl}(x)) \oplus (a \circ \text{Cl}(y)) = \text{Cl}_a(x) \oplus \text{Cl}_a(y);$ 
2. $\text{Cl}_a(x) = a \circ \text{Cl}(x) \geq a \circ x = a \land x = x;$
3. $\text{Cl}_a(\text{Cl}_a(x)) = a \circ \text{Cl}(a \circ \text{Cl}(x)) \leq a \circ \text{Cl}(\text{Cl}(x)) = a \circ \text{Cl}(x) = \text{Cl}_a(x);$ on the other hand, according to 2 we get $\text{Cl}_a(x) = a \circ \text{Cl}(x) \leq \text{Cl}_a(a \circ \text{Cl}(x)) = \text{Cl}_a(\text{Cl}_a(x))$, so, together we have $\text{Cl}_a(\text{Cl}_a(x)) = \text{Cl}_a(x);$ 
4. $\text{Cl}_a(0) = a \circ \text{Cl}(0) = a \circ 0 = a \land 0 = 0.$

Further, $\text{Int}_a^\neg (x) = \neg_a \text{Cl}_a(\neg_a x) = \neg ((a \circ \text{Cl}(\neg (\neg a \oplus x)) \oplus \neg a) = (\neg a \oplus \neg \text{Cl}(\neg (\neg a \oplus x))) \circ a = (\neg a \oplus \text{Int}^\neg (\neg a \oplus x)) \circ a = \text{Int}^\neg (\neg a \oplus x) \land a = a \circ \text{Int}^\neg (\neg a \oplus x).$

Analogously for $\text{Int}_a^\sim$. \( \square \)
Corollary 6. Let $\mathcal{A}$ be a GMV-algebra and $a \in A$ an idempotent element. Then a mapping $h$ given by the formula $h(x) = a \circ x$ for each $x \in A$ is a homomorphism from $\mathcal{A}$ onto $\mathcal{A}_a$.

Proof. Let $x, y \in A$. Then

$$h(x \circ y) = a \circ (x \circ y) = a \circ a \circ (x \circ y) = a \circ (a \circ x) \circ y.$$  

By Lemma 4a) we have

$$a \circ (a \circ x) \circ y = a \circ (x \circ a) \circ y = (a \circ x) \circ (a \circ y) = h(x) \circ_a h(y).$$

Further,

- $h(\sim_a x) = a \circ \sim x = a \wedge \sim x = \sim x \wedge a = a \circ (\sim x \oplus \sim a) = a \circ \sim (x \circ a) = a \circ \sim (a \circ x) = a \circ \sim h(x) = \sim_a h(x)$,
- $h(\sim a x) = a \circ \sim x = a \wedge \sim x = \sim x \wedge a = (\sim a \oplus \sim x) \circ a = \sim (a \circ x) \circ a = \sim h(x) \circ a = \sim h(x) \circ \sim a = \sim_a h(x)$,
- $h(0) = 0 = 0_a$

and finally

$$h(x \oplus y) = h(\sim (\sim x \oplus \sim y)) = \sim a h(\sim x \circ \sim y) = \sim a (h(\sim x) \circ_a h(\sim y)) = \sim a (\sim_a h(x) \circ_a \sim_a h(y)) = h(x) \oplus_a h(y).$$

So $h$ is a homomorphism from the GMV-algebra $\mathcal{A}$ into the GMV-algebra $\mathcal{A}_a$ and since $x = a \circ x = h(x)$ for each $x \in [0, a]$, $h$ is surjective.

Definition 5. Let $\mathcal{A}_1 = (A_1, \oplus_1, \sim_1, \circ_1, 0_1, 1_1, \text{Cl}_1)$ and $\mathcal{A}_2 = (A_2, \oplus_2, \sim_2, \circ_2, 0_2, 1_2, \text{Cl}_2)$ be closure GMV-algebras and let $h: A_1 \rightarrow A_2$ be a homomorphism from $\mathcal{A}_1$ into $\mathcal{A}_2$. Then $h$ is said to be a $c$-homomorphism from $\mathcal{A}_1$ into $\mathcal{A}_2$ iff

(C1) $h(\text{Cl}_1(x)) = \text{Cl}_2(h(x))$

for each $x \in A_1$.

Lemma 7. Let us consider closure GMV-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. A homomorphism $h$ from the GMV-algebra $\mathcal{A}_1$ into the GMV-algebra $\mathcal{A}_2$ is a $c$-homomorphism from $\mathcal{A}_1$ into $\mathcal{A}_2$ if and only if one of the following two equivalent conditions is satisfied:

(C2) $h(\text{Int}_1(x)) = \text{Int}_2(h(x))$,
(C3) $h(\text{Int}_1^*(x)) = \text{Int}_2^*(h(x))$

for each $x \in A_1$.

Proof. A homomorphism $h$ from $\mathcal{A}_1$ into $\mathcal{A}_2$ is a $c$-homomorphism iff

$$h(\text{Cl}_1(x)) = \text{Cl}_2(h(x))$$
for each \( x \in A_1 \), so for \( \neg_1 x \), too. From the last equation we get
\[
\sim_2 h(\text{Cl}_1(\neg_1 x)) = \sim_2 \text{Cl}_2(h(\neg_1 x)).
\]
Since \( h \) is a homomorphism from \( \mathcal{A}_1 \) into \( \mathcal{A}_2 \), we have got \( h(\neg_1 x) = \sim_2 h(x) \) and also \( h(\sim_1 x) = \sim_2 h(x) \) for each \( x \in A_1 \). Therefore we can write instead of the last equation
\[
h(\sim_1 \text{Cl}_1(\neg_1 x)) = \sim_2 \text{Cl}_2(\sim_2 h(x)),
\]
which is equivalent to the axiom (C3), thus
\[
h(\text{Int}_1^\sim(x)) = \text{Int}_2^\sim(h(x)).
\]
The equivalence of the conditions (C1), (C2) we can be proved analogously. \( \square \)

The following theorem refers to Theorem 5 and Corollary 6 and completes our description of the relation of closure GMV-algebras \( \mathcal{A} = (\mathcal{A}, \oplus, \neg, 0, 1, \text{Cl}) \) and \( \mathcal{A}_a = ([0, a], \oplus_a, \neg_a, \sim_a, 0_a, 1_a, \text{Cl}_a) \).

**Theorem 8.** Let \( \mathcal{A} \) be a closure GMV-algebra and let \( a \) be its idempotent element, which is open to at least one of multiplicative interior operators \( \text{Int}^\sim \) and \( \text{Int}^\sim \) on \( \mathcal{A} \). Finally, let \( h : A \to [0, a] \) be a mapping such that \( h(x) = a \circ x \) for each \( x \in A \). Then \( h \) is a surjective \( c \)-homomorphism \( \mathcal{A} \) onto \( \mathcal{A}_a \).

**Proof.** Let us consider a mapping \( h : A \to [0, a] \) such that \( h(x) = a \circ x \) for each \( x \in A \). We know from Lemma 6 that \( h \) is a surjective homomorphism of GMV-algebras \( \mathcal{A} \) and \( \mathcal{A}_a \).

We need to show now that \( h \) is a \( c \)-homomorphism. Let \( a \) be open for example with respect to \( \text{Int}^\sim \). Then it is enough to check availability of the condition (C3) from Lemma 7. For each \( x \in A \) we have
\[
h(\text{Int}^\sim(x)) = a \circ \text{Int}^\sim(x) = \text{Int}^\sim(a) \circ \text{Int}^\sim(x) = \text{Int}^\sim(a \circ x) = \text{Int}^\sim(h(x)).
\]
Let \( y \leq a \). Then
\[
\text{Int}^\sim(y) = \text{Int}^\sim(a \land y) = \text{Int}^\sim(a \circ (y \uplus \sim a)) = a \circ \text{Int}^\sim(y \uplus \sim a) = \text{Int}^\sim_a(y).
\]
Altogether we have
\[
h(\text{Int}^\sim(x)) = \text{Int}^\sim(h(x)) = \text{Int}^\sim_a(h(x))
\]
for each \( x \in A \). \( \square \)

**Note.** If \( a \) is open with respect to \( \text{Int}^\sim \), then we check availability of the condition (C2) from Lemma 7.

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344
4. Factorization on closure GMV-algebras

**Definition 6.** Let us consider a GMV-algebra $\mathcal{A}$. Then a set $I \subseteq A$, $\emptyset \neq I$ is called an *ideal* of the GMV-algebra $\mathcal{A}$ iff

(I1) $0 \in I$;

(I2) if $x, y \in I$, then $x \oplus y \in I$;

(I3) if $x \in I$, $y \in A$, $y \leq x$, then $y \in I$.

An ideal $I$ of a GMV-algebra $\mathcal{A}$ is called a normal ideal iff for each $x, y \in A$

(I4) $\neg x \odot y \in I \iff y \odot \neg x \in I$.

**Definition 7.** A normal ideal $I$ of a closure GMV-algebra $\mathcal{A}$ is called a normal c-ideal if $\text{Cl}(a) \in I$ for each $a \in I$.

**Remark 4.** Normal ideals of GMV-algebra $\mathcal{A}$ are in a one-to-one correspondence with congruences on $\mathcal{A}$.

a) If $\equiv$ is a congruence on $\mathcal{A}$, then $0/\equiv = \{x \in A; x \equiv 0\}$ is a normal ideal of $\mathcal{A}$.

b) Let $H$ be a normal ideal of $\mathcal{A}$. The relation $\equiv_H$, where

$$x \equiv_H y \iff (\neg y \odot x) \oplus (\neg x \odot y) \in H,$$

or equivalently

$$x \equiv_H y \iff (y \odot \neg x) \oplus (x \odot \neg y) \in H,$$

is a congruence on $\mathcal{A}$ and $H = \{x \in A; x \equiv_H 0\} = 0/\equiv_H$ holds.

More detail is found in [5].

**Note.**

a) We denote by $\mathcal{A}/I = \mathcal{A}/\equiv_I$ the factor GMV-algebra of a GMV-algebra $\mathcal{A}$ according to a congruence $\equiv_I$ on $\mathcal{A}$ and by $\mathfrak{a}$ the class of $A/I$ which contains the element $x$.

b) Let $\mathcal{A}$ be a closure GMV-algebra and let $I$ be its normal c-ideal. Let us put $\text{Cl}_I(\mathfrak{a}) := \text{Cl}(x)$ for each $x \in A$. This definition of the operator $\text{Cl}_I$ is correct as we will show in the proof of Theorem 9.

**Remark 5.** A DRl-monoid is an algebraic structure $\mathcal{A} = (A, +, 0, \lor, \land, \rightarrow, \leftarrow)$ of signature $\langle 2, 0, 2, 2, 2, 2 \rangle$, where $(A, +, 0)$ is a monoid, $(A, \lor, \land)$ is a lattice, $(A, +, \lor, \land, 0)$ is a lattice ordered monoid and the operations $\rightarrow$ and $\leftarrow$ are left and right dual residuations—see e.g. [6].

There are mutual relations between GMV-algebras and DRl-monoids which are described in [9], Theorems 12, 13.

345
Theorem 9. Let $\mathcal{A}$ be a closure GMV-algebra and let $I$ be its normal c-ideal. Then the factor GMV-algebra $\mathcal{A}/I$ endowed with the operator $\text{Cl}_I$ from the preceding Note b) is a closure GMV-algebra.

Proof. Let us consider $x \equiv_I y$. Then $(\neg x \odot y) \oplus (\neg y \odot x) \in I$, therefore $\neg x \odot y, \neg y \odot x \in I$ and $\text{Cl}(\neg x \odot y), \text{Cl}(\neg y \odot x) \in I$. Further we have

$$\text{Cl}(\neg y \odot x) \oplus \text{Cl}(y) = \text{Cl}((\neg y \odot x) \oplus y) = \text{Cl}(x \lor y) \supseteq \text{Cl}(x).$$

Since $\mathcal{A}$ is actually a $DRl$-monoid, we get

$$\text{Cl}(\neg y \odot x) \supseteq \text{Cl}(x) \rightarrow \text{Cl}(y) = \neg \text{Cl}(y) \circ \text{Cl}(x).$$

So we have $\neg \text{Cl}(y) \circ \text{Cl}(x) \in I$, since $\text{Cl}(\neg y \odot x) \in I$. We can show analogously that $\neg \text{Cl}(x) \circ \text{Cl}(y) \in I$. Therefore we can see that $(\neg \text{Cl}(x) \circ \text{Cl}(y)) \oplus (\neg \text{Cl}(y) \circ \text{Cl}(x)) \in I$, so $\text{Cl}(x) \equiv I \text{Cl}(y)$, and the operation $\text{Cl}_I$ is therefore correctly defined on $A/I$.

Moreover, $\text{Cl}_I: A/I \rightarrow A/I$ satisfies axioms 1–4 from Definition 3, because

1. $\text{Cl}_I(\pi \oplus b) = \text{Cl}_I(a \oplus b) = \text{Cl}(a \oplus b) = \text{Cl}(a) \oplus \text{Cl}(b) = \text{Cl}(a \oplus b) = \text{Cl}_I(\pi) \oplus \text{Cl}_I(b)$,
2. $\text{Cl}_I(\pi) = \overline{\text{Cl}(\pi)} = \pi$,
3. $\text{Cl}_I(\text{Cl}_I(\pi)) = \text{Cl}_I(\text{Cl}(a)) = \text{Cl}(\text{Cl}(a)) = \text{Cl}(\text{Cl}(a)) = \text{Cl}(a) = \text{Cl}_I(\pi)$,
4. $\text{Cl}_I(0) = \overline{\text{Cl}(0)} = 0$.

Corollary 10. There is a one-to-one correspondence between the normal c-ideals and the congruences of the closure GMV-algebras.

References


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