Abstract. A 2-stratified graph $G$ is a graph whose vertex set has been partitioned into two subsets, called the strata or color classes of $G$. Two 2-stratified graphs $G$ and $H$ are isomorphic if there exists a color-preserving isomorphism $\varphi$ from $G$ to $H$. A 2-stratified graph $G$ is said to be homogeneously embedded in a 2-stratified graph $H$ if for every vertex $x$ of $G$ and every vertex $y$ of $H$, where $x$ and $y$ are colored the same, there exists an induced 2-stratified subgraph $H'$ of $H$ containing $y$ and a color-preserving isomorphism $\varphi$ from $G$ to $H'$ such that $\varphi(x) = y$. A 2-stratified graph $F$ of minimum order in which $G$ can be homogeneously embedded is called a frame of $G$ and the order of $F$ is called the framing number $fr(G)$ of $G$. It is shown that every 2-stratified graph can be homogeneously embedded in some 2-stratified graph. For a graph $G$, a 2-stratified graph $F$ of minimum order in which every 2-stratification of $G$ can be homogeneously embedded is called a fence of $G$ and the order of $F$ is called the fencing number $fe(G)$ of $G$. The fencing numbers of some well-known classes of graphs are determined. It is shown that if $G$ is a vertex-transitive graph of order $n$ that is not a complete graph then $fe(G) = 2n$.

Keywords: stratified graph, homogeneous embedding

MSC 2000: 05C10, 05C15

1. Introduction

A common problem in graph theory concerns embedding one graph in another subject to certain conditions. For example, in 1936 König [8] showed that for every graph $G$ with maximum degree $r$, there exists an $r$-regular graph containing $G$ as an induced subgraph. In 1963 Erdős and Kelly [7] determined for each graph $G$ and
each integer \( r \geq \Delta(G) \), the minimum order of an \( r \)-regular graph containing \( G \) as an induced subgraph.

In 1992 a more restrictive embedding problem was introduced in [1]. A graph \( G \) is said to be **homogeneously embedded** in a graph \( H \) if for each vertex \( x \) of \( G \) and each vertex \( y \) of \( H \), there exists an embedding of \( G \) in \( H \) as an induced subgraph with \( x \) at \( y \). Equivalently, a graph \( G \) is **homogeneously embedded** in a graph \( H \) if for each vertex \( x \) of \( G \) and each vertex \( y \) of \( H \) there exists an induced subgraph \( H' \) of \( H \) containing \( y \) and an isomorphism \( \varphi \) from \( G \) to \( H' \) such that \( \varphi(x) = y \). A graph \( F \) of minimum order in which \( G \) can be homogeneously embedded is called a **frame** of (or for) \( G \) and the order of \( F \) is called the **framing number** \( \text{fr}(G) \) of \( G \). In [1] it was shown that every graph contains a frame and therefore a framing number.

For example, \( \text{fr}(P_3) = 4 \) since \( P_3 \) can be homogeneously embedded in \( C_4 \) (but not in any graph of order less than 4). Figure 1 shows homogeneous embeddings of \( P_3 \) in \( C_4 \) for two non-similar vertices of \( P_3 \).

![Homogeneously embedding P3 in C4](image)

In 1995 the concept of stratified graphs was introduced, inspired by the observation that in VLSI design, computer chips are designed so that its nodes are divided into layers. A graph \( G \) whose vertex set has been partitioned is called a **stratified graph**. If \( V(G) \) is partitioned into \( k \) subsets, then \( G \) is a **\( k \)-stratified graph**. The \( k \) subsets are called the **strata** or **color classes** of \( G \). If \( k = 2 \), then we customarily color the vertices of one subset red and the vertices of the other subset blue. Two 2-stratified graphs \( G \) and \( H \) are **isomorphic** if there exists a color-preserving isomorphism \( \varphi \) from \( G \) to \( H \). In this case, we write \( G \cong H \).

In [4] it was shown that there is a connection among embeddings, stratified graphs, and the area of domination. A vertex \( v \) in a graph \( G \) **dominates** itself and all of its neighbors. A set \( S \) of vertices in a graph \( G \) is a **dominating set** of \( G \) if every vertex of \( G \) is dominated by some vertex in \( S \). The minimum cardinality of a dominating set in \( G \) is the **domination number** \( \gamma(G) \) of \( G \). Although \( \gamma(G) \) is the standard domination number of a graph \( G \), there are many other domination parameters in graph theory, whose definitions depend on how the term **domination** is being interpreted in each case. For example, a vertex \( v \) in a graph \( G \) **openly dominates** (or **totally dominates**) each of its neighbors, but a vertex does not openly dominate itself. A set \( S \) of vertices in a graph \( G \) is an **open dominating set** if every vertex of \( G \) is openly dominated
by some vertex of $S$. A graph $G$ contains an open dominating set if and only if $G$ contains no isolated vertices. The minimum cardinality of an open dominating set is the open domination number $\gamma_o(G)$ of $G$.

A red-blue coloring of a graph $G$ is an assignment of the colors red and blue to the vertices of $G$, one color to each vertex. If there is at least one red vertex and at least one blue vertex, then a 2-stratified graph results. Let $F$ be a 2-stratified graph, where some blue vertex $v$ of $F$ has been designated as the root. An $F$-coloring of a graph $G$ is a red-blue coloring of $G$ such that every blue vertex $v$ of $G$ belongs to a copy of $F$ rooted at $v$. The $F$-domination number $\gamma_F(G)$ of $G$ is the minimum number of red vertices in an $F$-coloring of $G$. For the 2-stratified rooted graphs $F_0$, $F_1$, and $F_2$ shown in Figure 2, it was shown in [4] that for every graph $G$ of order at least 3 containing no isolated vertices,

$$\gamma_{F_0}(G) = \gamma_{F_1}(G) = \gamma(G) \quad \text{and} \quad \gamma_{F_2}(G) = \gamma_o(G).$$

Other domination parameters can be expressed as $\gamma_F(G)$ for some 2-stratified rooted graph $F$. Furthermore, for every 2-stratified graph $F$, there is a domination theory corresponding to $F$.

![Figure 2. Three 2-stratified rooted graphs](image)

This suggests the idea of homogeneously embedding one 2-stratified graph in another. A 2-stratified graph $G$ is said to be homogeneously embedded in a 2-stratified graph $H$ if for every vertex $x$ of $G$ and every vertex $y$ of $H$, where $x$ and $y$ are colored the same, there exists an induced 2-stratified subgraph $H'$ of $H$ containing $y$ and a color-preserving isomorphism $\varphi$ from $G$ to $H'$ such that $\varphi(x) = y$. A 2-stratified graph $F$ of minimum order in which $G$ can be homogeneously embedded is called a frame of (or for) $G$ and the order of $F$ is called the framing number $\text{fr}(G)$ of $G$.

2. Frames

First we show that every 2-stratified graph has a frame and therefore a framing number.
**Theorem 1.** Every 2-stratified graph can be homogeneously embedded in some 2-stratified graph.

**Proof.** Let $G$ be a 2-stratified graph of order $n$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $v_1, v_2, \ldots, v_r$ are red and $v_{r+1}, v_{r+2}, \ldots, v_{r+b}$ are blue, where $r + b = n$. We may assume that $r \geq b$. We construct a 2-stratified graph $H$ in which $G$ can be homogeneously embedded. We begin with $2r - 1$ copies $G_1, G_2, \ldots, G_{2r-1}$ of $G$ with $V(G_j) = \{v_{1,j}, v_{2,j}, \ldots, v_{n,j}\}$ for $1 \leq j \leq 2r - 1$, as shown below, where $v_{i,j}$ (1 ≤ $i$ ≤ $n$) denotes the vertex $v_i$ of $G$ in the graph $G_j$.

![Figure 3. The 2r - 1 copies of G](image)

The vertex set of $H$ is $\bigcup_{j=1}^{2r-1} V(G_j)$ and every edge in $G_j$ (1 ≤ $j$ ≤ 2r - 1) is an edge of $H$. Additional edges are added to complete the construction of $H$. For each vertex $v_{i,j}$, where 1 ≤ $i$ ≤ $n$ and 1 ≤ $j$ ≤ 2r - 1, the vertex $v_{i,j}$ is joined to vertices of $H$ not in $G_j$ as follows:

1. First, suppose that $v_{i,j}$ is a red vertex, that is, 1 ≤ $i$ ≤ $r$. For each integer $k$ with 1 ≤ $k$ < $i$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k}$ in $G_{j+k}$. For each integer $k$ with $i$ < $k$ ≤ $r$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k-1}$ in $G_{j+k-1}$. (The subscripts $j+k$ and $j+k-1$ are expressed modulo 2r.)

2. Next, suppose that $v_{i,j}$ is a blue vertex, that is, $r + 1$ ≤ $i$ ≤ $n$. For each integer $k$ with $r + 1$ ≤ $k$ < $i$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k-r}$ in $G_{j+k-r}$. For each integer $k$ with $i$ < $k$ ≤ $n$, the vertex $v_{i,j}$ is joined to the neighbors of $v_{k,j+k-r-1}$ in $G_{j+k-r-1}$. (Again, the subscripts $j+k-r$ and $j+k-r-1$ are expressed modulo 2r - 1.)

We now show that $G$ can be homogeneously embedded in $H$. It suffices to show that for each vertex $v_k$ of $G$, where 1 ≤ $k$ ≤ $n$, and each vertex $y$ of $H$ such that $v_k$ and $y$ are colored the same, the graph $G$ can be embedded as an induced subgraph of $H$ with $v_k$ at $y$. We may assume that $y = v_{i,j}$, where 1 ≤ $i$ ≤ $n$ and 1 ≤ $j$ ≤ 2r - 1.
Thus, if $1 \leq i \leq r$, define
\[
U = \begin{cases} 
V(G_{j+k}) \cup \{v_{i,j}\} - \{v_{k,j+k}\} & \text{if } 1 \leq k < i \\
V(G_{j}) & \text{if } i = k \\
V(G_{j+k-1}) \cup \{v_{i,j}\} - \{v_{k,j+k-1}\} & \text{if } i < k \leq r;
\end{cases}
\]

while if $r + 1 \leq i \leq n$, define
\[
U = \begin{cases} 
V(G_{j+k-r}) \cup \{v_{i,j}\} - \{v_{k,j+k-r}\} & \text{if } r + 1 \leq k < i \\
V(G_{j}) & \text{if } i = k \\
V(G_{j+k-r-1}) \cup \{v_{i,j}\} - \{v_{k,j+k-r-1}\} & \text{if } i < k \leq n.
\end{cases}
\]

In each case, $(U)_H \equiv G$, as desired.

Figure 4 illustrates the construction of the 2-stratified graph $H$ described in Theorem 2.1 for a given graph $G$. Since $G$ has two red vertices and two blue vertices, the 2-stratified graph $H$ is constructed from three copies $G_1, G_2, G_3$ of $G$.

![Figure 4](image)

Figure 4. Constructing a 2-stratified graph $H$ in which $G$ can be homogeneously embedded

The construction of the 2-stratified graph $H$ in Theorem 2.1 gives the following upper bound for $\text{fr}(G)$ in terms of the number of red vertices and the number of blue vertices in a 2-stratified graph $G$. 

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Corollary 2.2. Let $G$ be a 2-stratified graph with $r$ red vertices and $b$ blue vertex. Then

$$\text{fr}(G) \leq \max\{2r - 1, 2b - 1\}|V(G)|.$$ 

The upper bound in Corollary 2.2 can be improved. In order to show this, we need some additional definitions. Let $G$ be a 2-stratified graph with coloring $c$. Two vertices $u$ and $v$ with $c(u) = c(v)$ in $G$ are similar if there exists a color-preserving automorphism $\varphi$ of $G$ such that $\varphi(u) = v$. A 2-stratified graph $G$ is color vertex-transitive if every two vertices of $G$ having the same color are similar. Similarity is an equivalence relation on the vertex set of $G$ and the resulting equivalence classes are referred to as the orbits of $G$. Clearly, every orbit contains vertices of a single color. Suppose that $G$ is 2-stratified graph with $k_r$ red orbits and $k_b$ blue orbits, where say $k_r \geq k_b$. By an argument similar to the one described in Theorem 2.1, we can construct a 2-stratified graph $H$ from the $2k_r - 1$ copies $G$ in which $G$ can be homogeneously embedded. Therefore, we have the following.

Corollary 2.3. Let $G$ be a 2-stratified graph with $k_r$ red orbits and $k_b$ blue orbits. Then

$$\text{fr}(G) \leq \max\{2k_r - 1, 2k_b - 1\}|V(G)|.$$ 

Corollary 2.4. If $G$ is a graph with two orbits and $G'$ is the 2-stratification of $G$ in which the vertices of one orbit are colored red and the vertices of the other orbit are colored blue, then $G'$ is a frame of itself.

By Theorem 2.1, for every 2-stratified graph $G$, there exists a 2-stratified graph in which $G$ can be homogeneously embedded. In fact, more can be said.

Corollary 2.5. For every 2-stratified graph $G$, there exists a positive integer $N$ such that for every integer $n \geq N$, there exists a 2-stratified graph $H$ of order $n$ in which $G$ can be homogeneously embedded, while for each positive integer $n < N$, no such graph $H$ of order $n$ exists.

Proof. Suppose that $\text{fr}(G) = N$. Then there exists a 2-stratified graph $F$ of order $N$ in which $G$ can be homogeneously embedded. Let $v$ be a red vertex of $F$. Define $F_1$ be the 2-stratified graph of order $N + 1$ by adding a new red vertex $v_1$ to $F$ and joining $v_1$ to the neighbors of $v$. Then $v$ and $v_1$ are color-similar vertices and $G$ can be homogeneously embedded in $F_1$. Proceeding inductively, we see that for each integer $n \geq N$, there is 2-stratified graph $H$ of order $n$ in which $G$ can be homogeneously embedded. On the other hand, by the definition of $\text{fr}(G)$, there exists no 2-stratified graph $H$ of order $n < N$ in which $G$ can be homogeneously embedded. 

\[\square\]
Using the construction devised by König to produce a regular graph containing a given graph as an induced subgraph, we are able to show the following.

**Theorem 2.6.** Every 2-stratified graph can be homogeneously embedded in some 2-stratified regular graph.

**Proof.** Let $G$ be a 2-stratified graph. We show that $G$ can be homogeneously embedded in a 2-stratified regular graph $R$. By Theorem 2.1, the graph $G$ can be homogeneously embedded in some 2-stratified graph $H$. If $H$ is regular, then let $H = R$. Thus, we may assume that $H$ is not an regular graph. Suppose that $H$ is not regular.

Let $H$ be a 2-stratified graph. We show that $G$ can be homogeneously embedded in some 2-stratified regular graph $R$. By Theorem 2.1, the graph $G$ can be homogeneously embedded in some 2-stratified graph $H$. If $H$ is regular, then let $H = R$. Thus, we may assume that $H$ is not a regular graph. Suppose that $H$ is not regular.

Let $V(H) = \{v_1, v_2, \ldots, v_n\}$. Let $H'$ be another copy of $H$ with $V(H') = \{v'_1, v'_2, \ldots, v'_n\}$, where each vertex $v'_i$ in $H'$ corresponds to $v_i$ in $H$ for $1 \leq i \leq n$. Construct the graph $H_1$ from $H$ and $H'$ by adding the edges $v_iv'_i$ for all vertices $v_i$ ($1 \leq i \leq n$) such that $\deg v_i < \Delta(H)$. Then $H$ is an induced subgraph of $H_1$ and $\delta(H_1) = \delta(H) + 1$. If $H_1$ is regular, then we let $R = H_1$. If not, then we continue this procedure until we obtain a regular graph $H_k$, where $k = \Delta(H) - \delta(H)$. It is routine to verify that $G$ can be homogeneously embedded in $H_k$. \[\square\]

We now determine frames and the framing numbers of the 2-stratifications of some familiar graphs, beginning with a simple example.

**Proposition 2.7.** Every 2-stratification $G$ of a complete graph $K_n$ is its own frame and so $\text{fr}(G) = n$.

We now turn to complete bipartite graphs.

**Proposition 2.8.** Let $G$ be a 2-stratification of $K_{s+t}$ with partite sets $V_1$ and $V_2$, where $|V_1| = s$ and $|V_2| = t$. For $i = 1, 2$, let $r_i$ be the number of red vertices in $V_i$ and let $b_i$ the number of blue vertices in $V_i$ and let

\[
r = \max\{r_1, r_2\} \quad \text{and} \quad b = \max\{b_1, b_2\}.
\]

Then $\text{fr}(G) = s + t$ if the vertices of each set $V_i$, $i = 1, 2$, are colored the same and $\text{fr}(G) = 2(r + b)$ otherwise.

**Proof.** If the vertices of $V_1$ are colored the same and the vertices of $V_2$ are colored the same, then $G$ is the frame of itself by Corollary 2.4 and so $\text{fr}(G) = s + t$. Thus, we may assume that there are vertices in either $V_1$ or $V_2$ that are colored differently. Furthermore, we may assume, without loss of generality, that either $V_1$ or $V_2$ has all its vertices colored the same and this color is red.

Let $F$ be a frame of $G$. Since $G$ can be homogeneously embedded in $F$, every red vertex of $F$ is (1) adjacent to at least $r$ red vertices in $F$ and not adjacent to at
least \( r - 1 \) red vertices in \( F \) and (2) adjacent to at least \( b \) blue vertices in \( F \) and not adjacent to at least \( b \) blue vertices in \( F \). Hence \( F \) contains at least \( 2r \) red vertices and at least \( 2b \) blue vertices and so \( \text{fr}(G) \geq 2(r + b) \). On the other hand, let \( F' \) be the 2-stratification of the complete bipartite graph \( K_{r+b,r+b} \) in which each partite set of \( F' \) contains \( r \) red vertices and \( b \) blue vertices. Since \( G \) can be homogeneously embedded in \( F' \), it follows that \( \text{fr}(G) \leq 2(r + b) \). Therefore, \( \text{fr}(G) = 2(r + b) \).

This gives us the framing numbers of all stars.

**Corollary 2.9.** For each integer \( n \geq 2 \), the framing number of a 2-stratification of \( K_{1,n-1} \) is either \( n \) or \( 2(n-1) \).

We now determine frames and the framing numbers of all connected 2-stratified graphs of order 4 or less. Since every connected graph of order 3 or less is either complete or a star, we know the framing numbers of the 2-stratifications of all such graphs. The following result will be useful in determining the framing numbers of 2-stratifications of connected graphs of order 4.

**Theorem 2.10.** If \( F \) is a frame of a stratified graph \( G \), then \( \bar{F} \) is a frame of \( \bar{G} \).

**Proof.** Suppose that the order of \( F \) is \( n \). Thus for every vertex \( x \) of \( G \) and every vertex \( y \) of \( F \), where \( x \) and \( y \) are colored the same, there exists an induced stratified subgraph \( H \) of \( F \) containing \( y \) and a color-preserving isomorphism \( \varphi \) from \( G \) to \( H \) such that \( \varphi(x) = y \). Therefore, there exists a set \( U \subseteq V(F) \) for which \( H = \langle U \rangle_F \). Then \( U \subseteq V(F) \) and \( \langle U \rangle_F = \bar{H} \). Thus for each vertex \( x \) of \( \bar{G} \) and each vertex \( y \) of \( \bar{F} \), \( \bar{H} \) is an induced stratified subgraph of \( \bar{F} \) containing \( y \) and \( \varphi \) is a color-preserving isomorphism from \( \bar{G} \) to \( \bar{H} \) such that \( \bar{\varphi}(x) = y \). Therefore, \( \bar{G} \) can be homogeneously embedded in \( \bar{F} \), implying that \( \text{fr}(\bar{G}) \leq \text{fr}(\bar{F}) \). Then we have \( \text{fr}(G) = \text{fr}(\bar{G}) \leq \text{fr}(\bar{F}) \).

Therefore, \( \text{fr}(G) = \text{fr}(\bar{G}) = n \). Since the order of \( \bar{F} \) is \( n = \text{fr}(\bar{G}) \), it follows that \( \bar{F} \) is a frame of \( \bar{G} \).

First, we consider the paths \( P_4 \) of order 4.

**Proposition 2.11.** If \( G \) is a 2-stratification of \( P_4 \), then \( \text{fr}(G) = 4 \) or \( \text{fr}(G) = 6 \).

**Proof.** The graph \( P_4 \) is self-complementary and has the five 2-stratifications (up to color interchange) shown in Figure 5. Observe that \( G_3 \cong \bar{G}_2 \) and \( G_5 \cong \bar{G}_4 \). By Corollary 2.4, the 2-stratification \( G_1 \) is a frame of itself and so \( \text{fr}(G_1) = 4 \). Moreover, by Theorem 2.10, \( \text{fr}(G_3) = \text{fr}(G_2) \) and \( \text{fr}(G_5) = \text{fr}(G_4) \). Thus, it remains to consider \( \text{fr}(G_2) \) and \( \text{fr}(G_4) \). Let \( H \) be a frame of \( G_2 \). Then every red vertex of \( H \) is adjacent to two independent blue vertices and is not adjacent to a blue vertex. This implies that \( H \) contains at least three blue vertices. Similarly, \( H \) contains at least three red
vertices. Therefore, the order of $H$ is at least 6. Since $G_2$ can be homogeneously embedded in the 2-stratified graph $H_2$ of order 6, it follows that $H_2$ is a frame of $G_2$ and $\text{fr}(G_2) = 6$. By Theorem 2.10, $\overline{H}_2$ is a frame of $G_3$ and $\text{fr}(G_3) = 6$.

Next we consider $G_4$. Let $H$ be a frame of $G_4$. Then every red vertex of $H$ is adjacent to two independent red vertices and is not adjacent to a red vertex. This implies that $H$ contains at least four red vertices. Furthermore, every red vertex of $H$ is adjacent to a blue vertex and not adjacent to a blue vertex, implying that $H$ has at least two blue vertices. Hence the order of $H$ is at least 6. Since $G_4$ can be homogeneously embedded in the 2-stratified graph $H_4$, it follows that $H_4$ is a frame of $G_4$ and $\text{fr}(G_4) = 6$. By Theorem 2.10, $\overline{H}_4$ is a frame of $G_5$ and $\text{fr}(G_5) = 6$.

For the graphs $K_4 - e$ and $K_1 + (K_2 \cup K_1)$ of order 4, we only state the framing numbers and give a frame in Figures 6 and 7. For these next two results, $H_i$ is a frame of $G_i$ in each case.

**Proposition 2.12.** If $G$ is a 2-stratification of $K_4 - e$, then $\text{fr}(G) \in \{4, 5, 6\}$.

For the graphs $K_4 - e$ and $K_1 + (K_2 \cup K_1)$ of order 4, we only state the framing numbers and give a frame in Figures 6 and 7. For these next two results, $H_i$ is a frame of $G_i$ in each case.

**Proposition 2.12.** If $G$ is a 2-stratification of $K_4 - e$, then $\text{fr}(G) \in \{4, 5, 6\}$.
Proposition 2.13. If $G$ is a 2-stratification of $K_1 + (K_2 \cup K_1)$, then $\text{fr}(G) = 5$ or $\text{fr}(G) = 6$.

\[ G_1 \quad G_2 \quad G_3 \quad G_4 \quad G_5 \]

\[ H_1 \quad H_2 \quad H_3 \quad H_4 \quad H_5 \]

Figure 7. 2-stratifications of $K_1 + (K_2 \cup K_1)$ and their frames

Since we now know the framing number of every 2-stratification of every connected graph of order 4 or less and since the complement of every disconnected graph is connected, it follows by Theorem 2.10 that we know the framing number of every 2-stratification of every graph of order 4 or less.

3. Fences

For a graph $G$, a 2-stratified graph $F$ of minimum order in which every 2-stratification of $G$ can be homogeneously embedded is called a fence of $G$ and the order of $F$ is called the fencing number $\text{fe}(G)$ of $G$. The following observation is useful.

Observation 3.1. Let $G_1$ and $G_2$ be two 2-stratified connected graphs. If the disconnected graph $G_1 \cup G_2$ can be homogeneously embedded in a 2-stratified graph $H$, so can $G_1$ and $G_2$ individually. More generally, if a 2-stratified graph $G$ can be homogeneously embedded in a 2-stratified graph $H$, then every induced subgraph of $G$ can be homogeneously embedded in $H$.

It is a consequence of Theorem 2.1 and Observation 3.1 that every graph has a fence and therefore a fencing number. For example, every 2-stratification of $P_3$ can be homogeneously embedded in the 2-stratification of $Q_3$ shown in Figure 8. Thus, $\text{fe}(P_3) \leq 8$.

To show that $\text{fe}(P_3) \geq 8$, let $F$ be a fence of $P_3$. We show that $F$ contains at least 4 blue vertices. Since $G_3$ and $G_4$ are homogeneously embedded in $F$, it follows that every blue vertex in $F$ must be adjacent to a blue vertex and not adjacent to a blue vertex. Let $u$ be a blue vertex of $F$. Suppose that $u$ is adjacent to the blue vertex
v and is not adjacent to the blue vertex w. If v and w are adjacent, then there is a blue vertex x that is not adjacent to v; while if v and w are not adjacent, then there exists a blue vertex x that is adjacent to w. In each case, x is distinct from u, v, and w. Therefore, F contains at least four blue vertices. Similarly, F contains at least four red vertices. Therefore, \( f_e(P_3) > 8 \) and so \( f_e(P_3) = 8 \). Hence the 2-stratification of \( Q_3 \) in Figure 8 is a fence of \( P_3 \).

![Figure 8. The four 2-stratifications of \( P_3 \)](image)

First, we determine the fencing numbers of all complete graphs and complete bipartite graphs.

**Proposition 3.2.** For each integer \( n \geq 2 \), the fencing number of \( K_n \) is \( 2n - 2 \).

**Proof.** First, we show that \( f_e(K_n) \leq 2n - 2 \). Let \( G_0 \) be the 2-stratification of \( K_{2n-2} \) that contains \( n - 1 \) red vertices and \( n - 1 \) blue vertices. Since every 2-stratification of \( K_n \) can be homogeneously embedded in \( G_0 \), it follows that \( f_e(K_n) \leq 2n - 2 \).

Next, we show that \( f_e(K_n) \geq 2n - 2 \). Let \( F \) be a fence of \( K_n \). We show that \( F \) contains at least \( n - 1 \) blue vertices. Let \( H \) be the 2-stratification of \( K_n \) with exactly one red vertex. Since every blue vertex of \( H \) is adjacent to \( n - 2 \) blue vertices in \( H \), it follows that \( F \) contains at least \( n - 1 \) blue vertices. Similarly, \( F \) contains at least \( n - 1 \) red vertices. Therefore, the order of \( F \) is at least \( 2n - 2 \) and so \( f_e(K_n) \geq 2n - 2 \). \( \square \)

**Proposition 3.3.** For each pair \( r, t \) of integers with \( 1 \leq s \leq t \), the fencing number of \( K_{s,t} \) is \( 4t \).

**Proof.** First, let \( G_0 \) be the 2-stratification of the complete bipartite graph \( K_{2t,2t} \) for which each partite set of \( G_0 \) has exactly \( t \) red vertices and \( t \) blue vertices. Since every 2-stratification of \( K_{s,t} \) can be homogeneously embedded in \( G_0 \), it follows that \( f_e(K_{s,t}) \leq 4t \).

Next, we show that \( f_e(K_{s,t}) \geq 4t \). Let \( F \) be a fence of \( K_{s,t} \). We show that \( F \) contains at least \( 2t \) blue vertices. Suppose that \( U \) and \( V \) are the partite sets of \( K_{s,t} \) with \( |U| = s \) and \( |V| = t \). Let \( H_1 \) and \( H_2 \) be the 2-stratifications of \( K_{s,t} \) containing exactly one red vertex, where the red vertex of \( H_1 \) is in \( V \) and the red vertex of \( H_2 \) is in \( U \). In \( H_1 \), every blue vertex in \( U \) is adjacent to \( t - 1 \) blue vertices in \( V \); while
in $H_2$, every blue vertex in $V$ is not adjacent to $t - 1$ blue vertices in $V$. Since $H_1$ and $H_2$ can be homogeneously embedded in $F$, every blue vertex in $F$ is adjacent to at least $t - 1$ blue vertices and not adjacent to at least $t - 1$ blue vertices. Thus, $F$ contains at least $2t - 1$ blue vertices.

Suppose that $F$ contains exactly $2t - 1$ blue vertices. Since every blue vertex in $F$ is adjacent to at least $t - 1$ blue vertices, not adjacent to at least $t - 1$ blue vertices, and $F$ contains exactly $2t - 1$ blue vertices, every blue vertex of $F$ is adjacent to exactly $t - 1$ blue vertices. Let $B$ be the set of blue vertices of $F$ and let $\langle B \rangle$ be the subgraph of $F$ induced by $B$. Then $\langle B \rangle$ is $(t - 1)$-regular. Let $u$ be the red vertex in $H_2$. Then $u \in U$ and $u$ is adjacent to the $t$ blue vertices in the independent set $V$. Since $H_2$ can be homogeneously embedded in $F$, every red vertex in $F$ is adjacent to at least $t$ independent blue vertices. This implies that $B$ contains an independent subset $B'$ with $|B'| = t$. Since (1) $\langle B \rangle$ is $(t - 1)$-regular, (2) $B'$ is independent, and (3) $B - B'$ contains exactly $t - 1$ vertices, each blue vertex in $B'$ must be adjacent to every vertex in $B - B'$. However then, each vertex in $B - B'$ has degree $t$, contradicting the fact that $\langle B \rangle$ is $(t - 1)$-regular. Therefore, as claimed, $F$ contains at least $2t$ blue vertices. Similarly, $F$ contains at least $2t$ red vertices. Therefore, $\text{fe}(K_{s,t}) \geq 4t$.

In the case when $s = t$, then the fencing number of the regular graph $K_{s,t} = K_{s,s}$ is exactly twice of the order of $K_{s,t}$. We now show that the fencing number of every regular graph $G$ that is not complete is at least twice of the order of $G$.

**Proposition 3.4.** If $G$ is a regular graph of order $n$ that is not a complete graph, then

$$\text{fe}(G) \geq 2n.$$ 

**Proof.** Suppose that $F$ is a fence of an $r$-regular graph $G$ of order $n$ such that $G$ is not complete. We show that $F$ contains at least $n$ blue vertices. Let $v \in V(G)$. Let $H_1$ be the 2-stratification of $G$ in which every vertex in $N[v]$ is blue and the remaining $n - (r + 1) \geq 1$ vertices are red, and let $H_2$ be the 2-stratification of $G$ in which every vertex in $N(v)$ is red and the remaining vertices are blue. Thus, in $H_1$ the blue vertex $v$ is adjacent to $r$ blue vertices; while in $H_2$, the blue vertex $v$ is not adjacent to $n - (r + 1)$ blue vertices. Since $H_1$ and $H_2$ are homogeneously embedded in $F$, it follows that each blue vertex in $F$ is adjacent to at least $r$ blue vertices and not adjacent to at least $n - (r + 1)$ blue vertices. This implies that $F$ has at least $n$ blue vertices. Similarly, $F$ has at least $n$ red vertices. Therefore, the order of $F$ is at least $2n$.

For a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the reflection graph $\text{Ref}(G)$ of $G$ is constructed from $G$ by taking another copy $G'$ of $G$ with $V(G') = \{v'_1, v'_2, \ldots, v'_n\}$,
where \( v'_{i} \) corresponds to \( v_i \) for \( 1 \leq i \leq n \), and (1) joining each vertex \( v_i \) in \( G \) to the neighbors of \( v'_{i} \) in \( G' \) and (2) assigning the color red to every vertex in \( G \) and the color blue to every vertex in \( G' \).

**Theorem 3.5.** If \( G \) is a vertex-transitive graph of order \( n \) that is not a complete graph, then

\[
fe(G) = 2n.
\]

**Proof.** Since every vertex-transitive graph is regular, it follows by Proposition 3.4 that \( fe(G) \geq 2n \). Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \). We show that every 2-stratification of \( G \) can be homogeneously embedded in \( \text{Ref}(G) \), which has order \( 2n \). Let \( H \) be a 2-stratification of \( G \) and \( v \in V(H) \). Assume, without loss of generality, that \( v \) is blue. Let \( y \) be a blue vertex in \( \text{Ref}(G) \). Then \( v = v_i \) for some \( i \) (\( 1 \leq i \leq n \)) and \( y = v_j \) for some \( j \) (\( 1 \leq j \leq n \)). Since \( G \) is vertex-transitive, there exists an automorphism \( \alpha \) of \( G \) such that \( \alpha(v_i) = v_j \). Let \( F \) be the 2-stratified subgraph of \( \text{Ref}(G) \) with \( V(F) = \{v'_1, v'_2, \ldots, v'_n\} \), where

\[
v'_{i} = \begin{cases} 
\alpha(v_i) & \text{if } v_i \text{ is red} \\
\alpha(v_i)' & \text{if } v_i \text{ is blue} 
\end{cases}
\]

Then \( \langle V(F) \rangle_{\text{Ref}(G)} \cong G \). \( \square \)

**Corollary 3.6.** For each integer \( n \geq 4 \), the fencing number of \( C_n \) is \( 2n \).

The following observation is useful.

**Observation 3.7.** If \( H \) is an induced subgraph of a graph \( G \), then

\[
fe(H) \leq fe(G).
\]

**Proposition 3.8.** For each integer \( n \geq 4 \), the fencing number of \( P_n \) is \( 2(n + 1) \).

**Proof.** By Observation 3.7 and Corollary 3.6, \( fe(P_n) \leq fe(C_{n+1}) = 2(n + 1) \). To show that \( fe(P_n) \geq 2(n + 1) \), let \( F \) be a fence of \( P_n \). We show that \( F \) contains at least \( n + 1 \) blue vertices. Let \( P_n : v_1, v_2, \ldots, v_n \), let \( H_1 \) be the 2-stratification in which \( v_1 \) is the only red vertex, and let \( H_2 \) be the 2-stratification in which \( v_2 \) is the only red vertex. In \( H_1 \), the blue vertex \( v_3 \) is adjacent to blue vertices \( v_2 \) and \( v_4 \), while in \( H_2 \), the blue vertex \( v_1 \) is not adjacent to \( n - 2 \) blue vertices \( v_3, v_4, \ldots, v_n \). Since \( H_1 \) and \( H_2 \) are homogeneously embedded in \( F \), each blue vertex in \( F \) is adjacent to at least two blue vertices and not adjacent to at least \( n - 2 \) blue vertices. This implies that \( F \) has at least \( n + 1 \) blue vertices. Similarly, \( F \) has at least \( n + 1 \) red vertices. Therefore, the order of \( F \) is at least \( 2(n + 1) \). \( \square \)
References


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