THE CONVERSE PROBLEM FOR A GENERALIZED DHOMBRES
FUNCTIONAL EQUATION

L. Reich, Graz, J. Šmítal, M. Štefánková, Opava

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Abstract. We consider the functional equation $f(xf(x)) = \varphi(f(x))$ where $\varphi: J \to J$ is a given homeomorphism of an open interval $J \subset (0, \infty)$ and $f: (0, \infty) \to J$ is an unknown continuous function. A characterization of the class $S(J, \varphi)$ of continuous solutions $f$ is given in a series of papers by Kahlig and Smítal 1998–2002, and in a recent paper by Reich et al. 2004, in the case when $\varphi$ is increasing. In the present paper we solve the converse problem, for which continuous maps $f: (0, \infty) \to J$, where $J$ is an interval, there is an increasing homeomorphism $\varphi$ of $J$ such that $f \in S(J, \varphi)$. We also show why the similar problem for decreasing $\varphi$ is difficult.

Keywords: iterative functional equation, equation of invariant curves, general continuous solution, converse problem

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1. Introduction

If not specified, by function we always mean a continuous function. We consider the functional equation

$$f(xf(x)) = \varphi(f(x)), \quad x \in (0, \infty)$$

where $\varphi: J \to J$ is a given (surjective) homeomorphism of an interval $J \subset (0, \infty)$ onto itself, and $f: (0, \infty) \to J$ is an unknown function. Denote by $S(J, \varphi)$ the class of solutions $f$ of (1.1) with the range $J$.

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This equation is a special case of equations of invariant curves. A survey of general results can be found in [6] and [7]. Solutions of (1.1) with increasing \( \varphi \) have been studied, e.g., in [1]–[5], where another references can be found. While [4] contains characterization of the equations which have only monotone solutions, our last paper [5] contains a characterization of the class of continuous solutions of (1.1). We recall the main results.

**1.1 Theorem** (Cf. [3], [4].) Let \( R_f \) denote the range of \( f \). Assume \( \varphi \) is increasing, and \( f \) is a nonconstant solution of (1.1).

(i) If \( 1 \notin R_f \) then \( R_f = (p, q) \) is an open interval, \( \varphi \) has no fixed point in \((p, q)\), and the case \( p > 1 \) can be reduced to \( q < 1 \) by a suitable transformation. Moreover, if \( q = 1 \) then \( f \) is monotone.

(ii) If \( 1 \in R_f \) then \( f \) is monotone, 1 is a fixed point of \( \varphi \), and \( R_f = (p, q) \). \( R_f = (p, 1) \) or \( R_f = [1, q) \). Hence, in either of the last two cases, \( f \) must be constant on an interval \((0, a)\) or \([a, \infty)\). Moreover, the case \( R_f = [1, q) \) can be reduced to \( R_f = (p, 1) \).

Thus, in view of the previous theorem, the case \( 1 \in R_f = (p, q) \) splits into two separate cases \( R_f = (p, 1) \) and \( R_f = [1, q) \), and \( f \) splits into two solutions \( f_p = \min \{f, 1\} \) and \( f_q = \max \{f, 1\} \). Consequently, the class \( S(J, \varphi) \) of solutions of (1.1) with \( R_f = J \) an arbitrary interval is determined by the classes \( S(J, \varphi) \) with \( J, \varphi \) satisfying the conditions

\[
J = (p, q), \quad 0 \leq p < q \leq 1, \text{ and } \varphi(y) \neq y \text{ for } y \in J.
\]

If not specified we assume (1.2) throughout the remainder of the paper.

The main result concerning monotone solutions is the following one.

**1.2 Theorem** (Cf. [4].) If \( q < 1 \) then any continuous solution of (1.1) is monotone if and only if

\[
\varphi(y) < y \text{ in } J \quad \text{and} \quad \prod_{k=0}^{\infty} \frac{\varphi^k(u)}{\varphi^k(v)} = \infty, \quad \text{for any } u > v \text{ in } J,
\]

or

\[
\varphi(y) > y \text{ in } J \quad \text{and} \quad \prod_{k=1}^{\infty} \frac{\varphi^{-k}(u)}{\varphi^{-k}(v)} = \infty, \quad \text{for any } u > v \text{ in } J.
\]

**1.3 Remark.** Assume (1.2). Then neither (1.3) nor (1.4) can be satisfied if \( p > 0 \), cf. [4]. Thus, by the above theorems, non-monotone solutions of (1.1) do exist if and only if one of the following three conditions is satisfied: (i) \( 0 < p < q < 1 \); (ii) \( 0 = p < q < 1 \), \( \varphi(y) < y \) in \( J \), and (1.3) is not true; (iii) \( 0 = p < q < 1 \), \( \varphi(y) > y \) in \( J \), and (1.4) is not true.
Recall that by a piecewise monotone function defined on an interval we always mean a function with finite number of monotone pieces.

**1.4 Theorem** (Cf. [5].) Assume \( q < 1 \). Then for any solution \( f \) of (1.1) there is a sequence \( \{ f_n \}_{n=1}^{\infty} \) of solutions converging uniformly on every compact set to \( f \), and such that any \( f_n \) is piecewise monotone on every compact interval.

**1.5 Theorem** (Cf. [5].) Assume (1.1) has a solution which is not monotone. Then there is a solution \( f \) of (1.1) and a compact interval \( I \subset (0, \infty) \) such that \( f \) is monotone on no subinterval of \( I \).

**1.6 Remark.** It is easy to see that the space \( S(J, \varphi) \) is closed with respect to the almost uniform convergence, i.e., convergence which is uniform on any compact subset of \((0, \infty)\). Consequently, by Theorem 1.5, the space \( S(J, \varphi) \) is the almost uniform closure of the set of solutions piecewise monotone on every compact interval. Theorems 1.4 and 1.5 imply that, for \( q < 1 \), this is a non-trivial statement.

The next section contains solution of the converse problem. Theorem 2.7 characterizes monotone functions which are solutions of (1.1) for a suitable \( \varphi \), while Theorem 2.8 gives a characterization of the continuous solutions. Theorem 2.9 then shows that a typical nondecreasing function is a solution of (1.1) while a typical nonincreasing function and a typical continuous function fail to be solutions. By “typical” function we mean a function from a residual subset of the space of functions under consideration. Finally, in Section 3 we show that, for decreasing \( \varphi \), the class \( S(J, \varphi) \) can be empty even in the case \( J = (0,1) \); note that for increasing \( \varphi \) there are always nonconstant monotone solutions [3].

**2. The converse problem for increasing \( \varphi \)**

Throughout this section we assume (1.2). For simplicity we shall say that a homeomorphism \( \psi \) of \( J \) is regular if it is increasing and has no fixed points in \( J \). For a function \( f : (0, \infty) \rightarrow J \), we let \( \tau_f \), or simply \( \tau \), denote the function given by \( \tau_f(x) = xf(x) \), for \( x > 0 \).

The next two results follow immediately from (1.1). Recall that \( S(J, \varphi) \) is the set of continuous solutions \( f \) of (1.1) with \( R_f = J \). Obviously, for distinct homeomorphisms \( \varphi \) and \( \psi \) of \( J \), \( S(J, \varphi) \cap S(J, \psi) = \emptyset \).

**2.1 Lemma.** Let \( f \in S(J, \varphi) \), and let \( \varphi \) be regular. Then \( \tau_f \) is increasing.

**Proof.** Assume that \( \tau_f(x_1) < \tau_f(x_2) \) and \( \tau_f(x_3) < \tau_f(x_2) \), for some \( x_1 < x_2 < x_3 \). Since \( \tau_f \) is continuous, there are \( u \) in \( (x_1, x_2) \) and \( v \) in \( (x_2, x_3) \) such that \( \alpha := \tau_f(u) = \tau_f(v) \). Then \( f(u) > f(v) \) and consequently, \( f(\alpha) = f(uf(u)) = \varphi(f(u)) > \)
\( \varphi(f(v)) = f(vf(v)) = f(\alpha) \) which is impossible. Similarly if \( \tau_f(x_1), \tau_f(x_3) < \tau_f(x_2) \). Thus, \( \tau_f \) is strictly monotone. To finish the argument assume that \( \tau_f \) is decreasing. Then, for \( u < v, \ u/v > f(v)/f(u) \), and since \( f < 1 \), letting \( u \to \infty \) we obtain \( 0 > f(v) > 0 \).

The next lemma follows directly from (1.1).

2.2 Lemma. Let \( f \) be a continuous increasing map \((0, \infty), \) with \( R_f = J, \) and let \( \varphi(y) := f(yf^{-1}(y)) \), for \( y \in J \). Then \( \varphi \) is a regular homeomorphism of \( J \) such that \( f \in S(J, \varphi) \).

2.3 Lemma. Let \( f \) be a decreasing continuous function on \((0, \infty), \) with \( R_f = J \). Then there is a regular homeomorphism \( \varphi \) of \( J \) such that \( f \in S(J, \varphi) \) if and only if \( \tau_f \) is strictly increasing.

Proof. One implication follows since if \( \tau \) is strictly increasing then, similarly as in Lemma 2.2 it suffices to take \( \varphi(y) := f(yf^{-1}(y)) \), for \( y \in J \). The other implication follows by Lemma 2.1.

2.4 Definition. Let \( L \) be a family of level sets of a function \( f: (0, \infty) \to J \). Thus, \( L \) consists of sets \( f^{-1}(\{y\}) \), for \( y \) in \( A \subset J \). Then \( L \) is said to be \( \tau_f \)-consistent if it has a decomposition \( L = \bigcup_{t \in T} L_t \) into \( \tau_f \)-orbits \( L_t = \{\tau^0_f(f^{-1}(\{y\}))\} \times_{y \in J}, \) for any \( t \in T \).

2.5 Lemma. Let \( f \) be a continuous function on \((0, \infty), \) with \( R_f = J \). If \( f \in S(J, \varphi) \) for a regular homeomorphism \( \varphi \) of \( J \) then the system \( L \) of level sets of \( f \) is \( \tau_f \)-consistent.

Proof. Assume that \( f \in S(J, \varphi) \). Since, by Lemma 2.1, \( \tau \) is increasing, \( f \) is constant on \( \tau(K) \), for any \( K \in L \). Since \( \tau \) is invertible, (1.1) implies \( f(z) = \varphi^{-1}(f(\tau^{-1}(\{z\}))) \) and consequently, \( f \) is constant on the \( \tau \)-preimage of any level set \( K \). Hence, \( L \) is \( \tau \)-consistent.

2.6 Lemma. Let \( f \) be a continuous function on \((0, \infty) \) with \( R_f = J \), and such that

\[
\lim_{x \to 0} f(x) \in \{p, q\}.
\]

Assume that \( \tau \) is strictly increasing, and that the system \( L \) of the level sets of \( f \) is \( \tau \)-consistent. Then there is a regular homeomorphism \( \varphi \) of \( J \) such that \( f \in S(J, \varphi) \).

Proof. Put \( B = \min\{f^{-1}(\{y\}); y \in J\} \). Since \( f|B \) is a bijection \( B \to J \), it has the inverse \( g: J \to B \), not necessarily continuous. Let

\[
\varphi(y) = f(yg(y)), \text{ for } y \in J.
\]
For any $y \in J$ let $x = g(y)$. Then (2.2) can be rewritten to

\[(2.3) \quad \varphi(f(x)) = f(\tau(x)), \quad \text{for } x \in B.\]

For an $x \in B$, let $K(x) \in \mathcal{L}$ be the level set containing $x$. Since $\mathcal{L}$ is $\tau$-consistent and $\tau$ is continuous and increasing, (2.3) implies

\[(2.4) \quad f(\tau(K(x))) = \varphi(f(K(x))), \quad \text{for any } x \in B.\]

Since any $K \in \mathcal{L}$ is of the form $K = K(x)$, for some $x \in B$, (2.3) and (2.4) imply that (1.1) is satisfied for any $x > 0$.

Since $\mathcal{L}$ is $\tau$-consistent we have $\tau(B) = B$ whence, by (2.3), $R_\varphi = J$. Hence to show that $\varphi$ is a homeomorphism it suffices to show that it is increasing. For, $f|B$ being strictly monotone on $B$, it is increasing if the limit in (2.1) equals $p$, and it is decreasing if the limit equals $q$. Since $\tau$ is increasing, (2.3) implies that $\varphi$ in either case is increasing. The regularity of $\varphi$ (or rather, the fact that $\varphi$ has no fixed point) now follows by Theorem 1.1 since $f \in \mathcal{S}(J, \varphi)$. □

2.7 Theorem. Let $f$ be a continuous monotone function on $(0, \infty)$, with $R_f = J$. Then $f \in \mathcal{S}(J, \varphi)$, for a regular homeomorphism $\varphi$ of $J$ if and only if one of the following conditions is satisfied:

(i) $f$ is increasing;

(ii) $f$ is decreasing and $\tau$ is increasing;

(iii) $f$ is nondecreasing and the system of maximal intervals of constancy of $f$ is $\tau$-consistent;

(iv) $f$ is nonincreasing, $\tau$ is increasing, and the system of intervals of constancy of $f$ is $\tau$-consistent.

Proof. The first two conditions are given in Lemmas 2.2 and 2.3. Condition (iii) follows by Lemmas 2.5 and 2.6 since, for a monotone function, the level sets are $\tau$-consistent if and only if the maximal intervals of constancy are consistent. Similarly, (iv) follows by Lemmas 2.1, 2.5 and 2.6. □

The condition (iv) of Theorem 2.7 can be easily modified to an arbitrary continuous function $f$ satisfying the necessary condition that $\lim_{x \to 0} f(x)$ and $\lim_{x \to \infty} f(x)$ are distinct points in $\{p, q\}$.

2.8 Theorem. Let $f$ be a continuous function on $(0, \infty)$, with $R_f = J$. Then $f \in \mathcal{S}(J, \varphi)$, for a regular homeomorphism $\varphi$ of $J$ if and only if $\lim_{x \to \infty} f(x) \in \{p, q\}$, $\tau$ is increasing, and the system of level sets of $f$ is $\tau$-consistent.

Proof. Condition $\lim_{x \to \infty} f(x) \in \{p, q\}$ is necessary whenever $\varphi$ is a regular homeomorphism; this can be easily verified (see also [3]). The necessity of the other
two conditions follows by Lemmas 2.1 and 2.5. Finally, Lemma 2.6 gives the sufficient condition.

In the sequel let, for an open interval $J$, $\mathcal{M}_+(J)$ and $\mathcal{M}_-(J)$ denote the class of continuous nondecreasing, resp. nonincreasing functions from $(0, \infty)$ onto the closure of $J$. Let $\mathcal{G}(J)$ be the class of continuous functions $f$ from $(0, \infty)$ onto the closure of $J$ such that $\lim_{x \to \infty} f(x) \in [p, q]$. Let $S(J) = \bigcup_{\varphi} S(J, \varphi)$, where the union is taken over all regular homeomorphisms $\varphi$ of $J$. Finally, let $S_+(J) = S(J) \cap \mathcal{M}_+(J)$, and similarly define $S_-(J)$. Obviously, $\mathcal{M}_+(J)$ and $\mathcal{M}_-(J)$ are complete metric spaces with respect to the uniform metric. The following result states that, roughly speaking, a typical nondecreasing function is a solution of (1.1) for some $\varphi$, but a typical nonincreasing function as well as a typical continuous “globally” monotone function is not a solution.

2.9 Theorem.
(i) $S_+(J)$ is residual in $\mathcal{M}_+(J)$;
(ii) $S_-(J)$ is nowhere dense in $\mathcal{M}_-(J)$;
(iii) $S(J)$ is nowhere dense in $\mathcal{G}(J)$.

Proof. It is well-known that the class of strictly increasing functions from $(0, \infty)$ onto $J$ is residual in $\mathcal{M}_+(J)$. This follows by the fact that the set $\mathcal{M}_+(J)$ consisting of $f$ in $\mathcal{M}_+(J)$ which have no interval of constancy $K \subset (0, n]$ of length greater than $1/n$, is nowhere dense in $\mathcal{M}_+(J)$. This proves (i), by Lemma 2.2.

To prove (ii), let $G$ be an open (in the uniform topology) neighborhood of an $f \in \mathcal{M}_-(J)$. It is easy to see that there is a function $g \in \mathcal{M}_-(J) \cap G$ such that $ug(u) > vg(v)$, for some $u < v$. But then $\tau_g$ is not increasing and, by the continuity, the same is true for any $h$ belonging to an open neighborhood $H \subset G$ of $g$ in $\mathcal{M}_-(J)$. Consequently, by Lemma 2.1, $H \cap S(J) = \emptyset$.

Proof of (iii) is similar and we omit it.

3. The converse problem for decreasing $\varphi$

While the solutions of (1.1) in the regular case (i.e., with $\varphi$ an increasing homeomorphism) are completely characterized, it seems to be difficult to obtain similar results as, e.g., in Theorems 1.1, 1.2, 1.4 and 1.5 for $\varphi$ decreasing. On the other hand, we conjecture that the converse problem for $\varphi$ decreasing is solvable and characterization as in Theorems 2.7 and 2.8 would be possible.

The essential difference between this and the regular case is that for certain decreasing homeomorphisms $\varphi$ there are no nonconstant continuous solutions at all. This can be indicated by the following two examples. They exhibit decreasing homeomorphisms of an open interval $J$ such that any point is periodic with period 1 or 2.
In the first case there is a $\varphi$ and an uncountable nested family of compact subintervals of $J$, each being the range of a continuous solution of (1.1); note that by Theorem 1.1, this is impossible if $\varphi$ is increasing. In the second case, (1.1) has only a constant solution.

3.1 Example. Let $\alpha \in (0,1)$, and let $\varphi(y) = \alpha/y$, for $y \in J = (0, \infty)$. Thus, $\sqrt{\alpha}$ is a fixed point and any $y \neq \alpha$ in $J$ is a periodic point of $\varphi$ of period 2. However, for any $\beta$ such that $0 < \beta < \alpha/\beta < 1$ there is a continuous solution $f$ of (1.1) with $R_f = [\beta, \alpha/\beta]$.

Proof. It is easy to see that both mappings

$$\Phi(x, y) = \left(\frac{xy}{\alpha}, \frac{\alpha}{y}\right) \quad \text{and} \quad \Phi^{-1}(x, y) = \left(\frac{xy}{\alpha}, \frac{\alpha}{y}\right)$$

are continuous bijections of the strip $H_\beta = (0, \infty) \times [\beta, \alpha/\beta]$ onto itself. Thus, $\Phi$ is a homeomorphism of $H_\beta$. It is easy to see that a not necessarily surjective function $f$: $(0, \infty) \to J$ is a solution of (1.1) if $\Phi(f) \subset f$ (cf. also [3]); here we identify a function with its graph. Moreover, the second iterates of the functions (3.1) are given by

$$\Phi^2(x, y) = (x\alpha, y), \quad \text{and} \quad \Phi^{-2}(x, y) = (x/\alpha, y).$$

To prove the theorem it suffices to find a continuous map $f$ with range $[\beta, \alpha/\beta]$ such that $\Phi(f) = f$. Fix an $x_0 > 0$, put $y_0 = \alpha/\beta$, and let $\{(x_n, y_n)\}_{n=-\infty}^{\infty}$ be the full orbit of $(x_0, y_0)$, i.e., the sequence such that $\Phi(x_n, y_n) = (x_{n+1}, y_{n+1})$, for any integer $n$. Then $\{y_n\}_{n=-\infty}^{\infty}$ is an alternating sequence with terms $\beta$ and $\alpha/\beta$, and $\{x_n\}_{n=0}^{\infty}$ is a decreasing sequence tending to 0. Let $f_0$ be any increasing continuous function on $[x_1, x_0]$, with values $y_1 = f_0(x_1)$, and $y_0 = f_0(x_0)$ at the endpoints. For any integer $n$ define $f_n$ as $\Phi^n(f_0)$. Since $x \mapsto xy$ is increasing on $[x_1, x_0]$ and maps this interval onto $[x_2, x_1]$, $f_1: [x_2, x_1] \to [\beta, \alpha/\beta]$ is continuous and decreasing. Similarly, $f_{-1}: [x_0, x_{-1}] \to [\beta, \alpha/\beta]$ is continuous and decreasing. By (3.2), $f_2$ and $f_{-2}$ are increasing, etc. An induction argument yields that for any integer $n$, $f_n = \Phi^n(f_0)$ is a continuous strictly monotone map $[x_{n+1}, x_n] \to [\beta, \alpha/\beta]$, attaining the values $\beta$, $\alpha/\beta$ only at the endpoints $x_{n+1}$ and $x_n$. To finish the construction put $f = \bigcup_{n=-\infty}^{\infty} f_n$.

It is easy to see that $\Phi(f) = f$. $\square$

3.2 Example. Let $\varphi(y) = 1 - y$ in $J = (0, 1)$. Then $\frac{1}{2}$ is a fixed point and any $y \neq \frac{1}{2}$ in $J$ is a periodic point of $\varphi$ of period 2, but (1.1) has no continuous solution different from $f \equiv \frac{1}{2}$.

Proof. Assume there is a nonconstant solution $f$. Then there is an $\alpha > \frac{1}{2}$ in $J$ such that $f(x_0) = \alpha$, for some $x_0$. Since the range $R_f$ of $f$ is an interval,
and $\varphi(R_f) \subset R_f$, we have $\frac{1}{2} \in R_f$ (note that $\alpha = \varphi^2(\alpha)$ is a periodic point of $\varphi$). Let $\Phi$ be a homeomorphism of the strip $H = (0, \infty) \times (0, 1)$ onto itself, given by $\Phi(x, y) = (xy, 1 - y)$. Then $\Phi^2(x, y) = (xy(1 - y), y)$, and since $f$ is a solution, $\Phi(f) \subset f$ whence, $\Phi^2(f) \subset f$. By induction,

$\Phi^{2k}(x, y) = (x[y(1 - y)]^k, y)$ and $\Phi^{2k}(f) \subset f$.

Let $\{x_n, y_n\}^\infty_{n=-\infty}$ be the $\Phi$-orbit of $(x_0, y_0)$, with $y_0 = \alpha$. Let $z_0 < x_0$ be the maximal point such that $f(z_0) = \frac{1}{2}$. By (3.3), $z_{2k} = z_04^{-k}$ and $x_{2k} = x_0[\alpha(1 - \alpha)]^k$, and since $\alpha(1 - \alpha) < \frac{1}{4}$, we have $\lim_{k \to \infty} z_{2k}/x_{2k} = +\infty$. Thus, there is an $n$ such that $x_{2n} < x_{2n-2} < z_{2n}$. Then the range of $\Phi^{2n} = f|_{(x_{2n}, z_{2n})}$ contains the fixed point $\frac{1}{2}$, while the range $(\frac{1}{4}, \alpha)$ of $\Phi^{2n}(f([z_0, x_0]))$ does not (note that $X = (\frac{1}{4}, \alpha)$). But $\Phi^{2n}(f([z_0, x_0]))$ is a (graph of) function $g_{2n} : (x_{2n}, z_{2n}) \to J$. This contradicts the fact that $\Phi^{2n}(f) \subset f$ since $f_{2n}$ and $g_{2n}$ are continuous.

References


Authors’ addresses: Ludwig Reich, Institut f. Mathematik, Karl-Franzens-Universität Graz, A-8010 Graz, Austria, e-mail: ludwig.reich@uni-graz.at; Jaroslav Smítal, Marta Štefánková, Mathematical Institute, Silesian University, CZ-746 01 Opava, Czech Republic, e-mail: jaroslav.smital@math.slu.cz, marta.stefankova@math.slu.cz.