BANACH-VALUED HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS ARE MCSHANE INTEGRABLE ON A PORTION

TUO-YEONG LEE, Singapore

(Received December 16, 2004)

Abstract. It is shown that a Banach-valued Henstock-Kurzweil integrable function on an $m$-dimensional compact interval is McShane integrable on a portion of the interval. As a consequence, there exist a non-Perron integrable function $f: [0,1]^2 \to \mathbb{R}$ and a continuous function $F: [0,1]^2 \to \mathbb{R}$ such that

$$
\left(\mathcal{P}\right) \int_0^x \left(\mathcal{P}\right) \int_0^y f(u,v) \, dv \, du = \left(\mathcal{P}\right) \int_0^y \left(\mathcal{P}\right) \int_0^x f(u,v) \, du \, dv = F(x,y)
$$

for all $(x,y) \in [0,1]^2$.

Keywords: Henstock-Kurzweil integral, McShane integral

MSC 2000: 28B05, 26A39

1. Introduction

It is well known that if $f$ is Denjoy-Perron integrable on an interval $[a,b] \subset \mathbb{R}$, then $f$ must be Lebesgue integrable on a portion of $[a,b]$. K. Karták [6] asked whether an analogous result holds for the multiple Perron integral. In a fairly recent paper [1] Buczolich gave an affirmative answer to this problem using the Henstock-Kurzweil integral. Nevertheless, his proof depends on the measurability of the integrand. Since a Banach-valued Henstock-Kurzweil integrable function need not be strongly measurable, see for instance [4, p. 567], it is natural to ask whether Buczolich’s result holds for Banach-valued Henstock-Kurzweil integrable functions. In this paper we give an affirmative answer to this problem. As an application, we answer another question of K. Karták [6, Problem 9.3] concerning the Perron integral; namely, there exist a non-Perron integrable function $f: [0,1]^2 \to \mathbb{R}$ and a continuous function
\[ F: [0,1]^2 \rightarrow \mathbb{R} \text{ such that} \]
\[
(P) \int_0^y \left\{ (P) \int_0^x f(u,v) \, du \right\} \, dv = (P) \int_0^y \left\{ (P) \int_0^x f(u,v) \, du \right\} \, dv = F(x,y)
\]
for all \((x,y) \in [0,1]^2\).

2. Preliminaries

Unless stated otherwise, the following conventions and notation will be used. The set of all real numbers is denoted by \(\mathbb{R}\), and the ambient space of this paper is \(\mathbb{R}^m\), where \(m\) is a fixed positive integer. The norm in \(\mathbb{R}^m\) is the maximum norm \(\| \cdot \|\), where \(\| (x_1, x_2, \ldots, x_m) \| = \max_{i=1}^m |x_i|\). For \(x \in \mathbb{R}^m\) and \(r > 0\), set \(B(x,r) := \{ y \in \mathbb{R}^m : \| y - x \| < r \}\). Let \(E := \prod [a_i, b_i]\) be a fixed non-degenerate interval in \(\mathbb{R}^m\). Let \(X\) be a Banach space equipped with a norm \(\| \cdot \|\). A function is always \(X\)-valued.

When no confusion is possible, we do not distinguish between a function defined on a set \(Z\) and its restriction to a set \(W \subseteq Z\).

An interval in \(\mathbb{R}^m\) is the cartesian product of \(m\) non-degenerate compact intervals in \(\mathbb{R}\). \(I\) denotes the family of all non-degenerate subintervals of \(E\). For each \(I \in I\), \(|I|\) denotes the volume of \(I\).

A partition \(P\) is a collection \(\{(I_i, \xi_i)\}_{i=1}^p\), where \(I_1, I_2, \ldots, I_p\) are non-overlapping non-degenerate subintervals of \(E\). Given \(Z \subseteq E\), a positive function \(\delta\) on \(Z\) is called a gauge on \(Z\). We say that a partition \(\{(I_i, \xi_i)\}_{i=1}^p\) is

(i) a partition in \(Z\) if \(\bigcup_{i=1}^p I_i \subseteq Z\),

(ii) a partition of \(Z\) if \(\bigcup_{i=1}^p I_i = Z\),

(iii) anchored in \(Z\) if \(\{\xi_1, \xi_2, \ldots, \xi_p\} \subseteq Z\),

(iv) \(\delta\)-fine if \(I_i \subseteq B(\xi_i, \delta(\xi_i))\) for each \(i = 1, 2, \ldots, p\),

(v) Perron if \(\xi_i \in I_i\) for each \(i = 1, 2, \ldots, p\),

(vi) McShane if \(\xi_i\) need not belong to \(I_i\) for all \(i = 1, 2, \ldots, p\).

According to Cousin’s Lemma [8, Lemma 6.2.6], for any given gauge \(\delta\) on \(E\), \(\delta\)-fine Perron partitions of \(E\) exist. Hence the following definition is meaningful.

**Definition 2.1.** A function \(f: E \rightarrow X\) is said to be Henstock-Kurzweil integrable (McShane integrable, respectively) on \(E\) if there exists \(A \in X\) with the following property: given \(\varepsilon > 0\) there exists a gauge \(\delta\) on \(E\) such that

\[
\left\| \sum_{i=1}^p f(\xi_i)[I_i] - A \right\| < \varepsilon
\]
for each $\delta$-fine Perron partition ($\delta$-fine McShane partition, respectively) $\{(I_i, \xi_i)\}_{i=1}^p$ of $E$. We write $A$ as $(\text{HK}) \int_E f$, $((\text{M}) \int_E f$, respectively).

It is well known that if $f$ is Henstock-Kurzweil integrable on $E$, then $f$ is Henstock-Kurzweil integrable on each subinterval $J$ of $E$. Moreover, the interval function $J \mapsto (\text{HK}) \int_J f$ is additive on $I$. This interval function is known as the indefinite Henstock-Kurzweil integral, or in short the indefinite HK-integral, of $f$.

**Theorem 2.2** (Saks-Henstock Lemma). Let $f : E \to X$ be Henstock-Kurzweil integrable on $E$ and let $F$ be the indefinite HK-integral of $f$. Then given $\varepsilon > 0$ there exists a gauge $\delta$ on $E$ such that

$$\left\| \sum_{(I,x) \in P} \{ f(x)|I| - F(I) \} \right\| < \varepsilon$$

for each $\delta$-fine Perron partition $P$ in $E$.

3. **Banach-valued Henstock-Kurzweil integrable functions are McShane integrable on a portion**

**Theorem 3.1.** Let $f : E \to X$ be Henstock-Kurzweil integrable on $E$ and let $F$ denote the indefinite Henstock-Kurzweil integral of $f$. Then the following conditions are equivalent:

(i) $f$ is McShane integrable on $E$;

(ii) $\sup \| \sum_{i=1}^q F(J_i) \|$ is finite, where the supremum is taken over all finite partitions $\{J_1, \ldots, J_q\}$ of pairwise non-overlapping subintervals of $E$.

**Proof.** Since $E$ is compact, the implication (i) $\implies$ (ii) follows from [11, Lemma 28].

(ii) $\implies$ (i). Assume (ii). If $x \in X^*$, then $x(f)$ is Henstock-Kurzweil integrable on $E$ and the indefinite Henstock-Kurzweil integral of $x(f)$ is of bounded variation on $E$. The rest of the proof is similar to that of the implication (iii) $\implies$ (i) of [2, Corollary 9]. The proof is complete.

In view of [3, Proposition 2B], the next theorem is a mild improvement of [2, Theorem 8].
Theorem 3.2. Let \( f: E \rightarrow X \) be Henstock-Kurzweil integrable on \( E \) and let \( F \) denote the indefinite Henstock-Kurzweil integral of \( f \). Then the following conditions are equivalent:

(i) \( f \) is McShane integrable on \( E \);
(ii) \( F \) is absolutely continuous on \( \mathcal{I} \), that is, given any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that the inequality \( \left\| \sum_{i=1}^{p} F(I_i) \right\| < \varepsilon \) holds whenever \( \{I_1, \ldots, I_p\} \) is a finite collection of pairwise non-overlapping subintervals of \( E \) with \( \sum_{i=1}^{p} |I_i| < \delta \).

Proof. (i) \( \Rightarrow \) (ii). This follows from [11, Lemma 28].
(ii) \( \Rightarrow \) (i). Since \( E \) is compact, this follows from Theorem 3.1.

It is well known that the real-valued McShane integral is equivalent to the Lebesgue integral. For a proof of this result, see, for example, [10]. Hence the following theorem is a generalization of [1, Theorem]. Recall that a portion of \( E \) is a set of the form \( E \setminus I \), where \( I \) is an open interval in \( \mathbb{R}^m \).

Theorem 3.3. If \( f: E \rightarrow X \) is Henstock-Kurzweil integrable on \( E \), then \( f \) is McShane integrable on a portion of \( E \).

Proof. Since \( f \) is assumed to be Henstock-Kurzweil integrable on \( E \), the Saks-Henstock Lemma (Theorem 2.2) holds. Therefore there exists a gauge \( \delta \) on \( E \) such that

\[
\left\| \sum_{(I,x) \in P} \{f(x)|I| - F(I)\} \right\| < 1
\]

for each \( \delta \)-fine Perron partition \( P \) in \( E \). For each \( n \in \mathbb{N} \), we set

\[
X_n = \left\{ x \in E : \|f(x)\| < n \text{ and } \delta(x) > \frac{1}{n} \right\}.
\]

Clearly \( \bigcup_{n \in \mathbb{N}} X_n = E \) and hence by Baire’s Category Theorem [5, Theorem 5.2] there exists \( N \in \mathbb{N} \) such that \( X_N \) is dense on some \( J \) belonging to \( \mathcal{I} \). Without loss of generality we may assume that \( \text{diam}(J) < 1/N \), where \( \text{diam}(J) \) denotes the diameter of \( J \).

Consider any finite collection \( \{J_1, \ldots, J_q\} \) of pairwise non-overlapping subintervals of \( J \). For each \( i \in \{1, \ldots, q\} \) we invoke the density of \( X_N \cap J \) in \( J \) to pick \( x_i \in X_N \cap J \). Since \( \text{diam}(J) < 1/N \), we see that \( \{(J_1, x_1), \ldots, (J_q, x_q)\} \) is a \( (1/N) \)-fine, and hence \( \delta \)-fine, Perron partition anchored in \( X_N \cap J \). Hence, by our choice of \( \delta \),

\[
\left\| \sum_{i=1}^{q} \{f(x_i)|J_i| - F(J_i)\} \right\| < 1
\]
and so
\[ \left\| \sum_{i=1}^{q} F(J_i) \right\| < 1 + \sum_{i=1}^{q} \|f(x_i)\| |J_i| < 1 + N|J|. \]

As \( \{J_1, \ldots, J_q\} \) is an arbitrary finite collection of pairwise non-overlapping subintervals of \( J \), an appeal to Theorem 3.1 completes the proof of the theorem.

In [7], Kurzweil and Jarník proved that if \( f \) is a real-valued Henstock-Kurzweil integrable function on \( E \), then there exists an increasing sequence \( \{X_n\}_{n=1}^{\infty} \) of closed sets whose union is \( E \), and for each \( n \in \mathbb{N} \), \( f \) is Lebesgue integrable on \( X_n \). Hence it is natural to pose the following problem.

Problem 3.4. Let \( f: E \rightarrow X \) be Henstock-Kurzweil integrable on \( E \). Can we find an increasing sequence \( \{X_n\}_{n=1}^{\infty} \) of closed sets whose union is \( E \), and for each \( n \in \mathbb{N} \), \( f \) is McShane integrable on \( X_n \)?

4. On a question of K. Karták concerning the Perron integral

K. Karták posed the following problem for the Perron integral:

Problem 4.1 [6, Problem 9.3]. Is there a function \( f: [0,1]^2 \rightarrow \mathbb{R} \) such that

\[ (\text{P}) \int_0^x \left\{ (\text{P}) \int_0^y f(u,v) \, dv \right\} \, du = (\text{P}) \int_0^y \left\{ (\text{P}) \int_0^x f(u,v) \, du \right\} \, dv = F(x,y) \]

for all \( (x,y) \in [0,1]^2 \) and that the function \( F \) is continuous on \([0,1]^2\) while \( f \) is not Perron integrable on \([0,1]^2\)?

Recall that the real-valued Henstock-Kurzweil integral is equivalent to the Perron integral. Hence we may use the Henstock-Kurzweil integral to answer the above question of K. Karták.

Theorem 4.2. There exist \( f: [0,1]^2 \rightarrow \mathbb{R} \) and a continuous function \( F: [0,1]^2 \rightarrow \mathbb{R} \) such that

\[ (\text{HK}) \int_0^x \left\{ (\text{HK}) \int_0^y f(u,v) \, dv \right\} \, du = (\text{HK}) \int_0^y \left\{ (\text{HK}) \int_0^x f(u,v) \, du \right\} \, dv = F(x,y) \]

for all \( (x,y) \in [0,1]^2 \) but \( f \) is not Henstock-Kurzweil integrable on \([0,1]^2\).

Proof. Let \( f \) be given as in [12, Chapter VI]. Then there exist a continuous function \( F: [0,1]^2 \rightarrow \mathbb{R} \) and \( f: [0,1]^2 \rightarrow \mathbb{R} \) such that

\[ \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 F(x,y)}{\partial y \partial x} = f(x,y) \]

353
for all \((x, y) \in (0,1)^2\). Moreover, \(f\) is not Lebesgue integrable, and hence not Mc-
Shane integrable, on any non-degenerate subinterval of \([0,1]^2\). It is clear that (1) holds for all \((x, y) \in [0,1]^2\). Using Theorem 3.3 with \(E = [0,1]^2\) and \(X = \mathbb{R}\), we conclude that \(f\) cannot be Henstock-Kurzweil integrable on \([0,1]^2\). The proof is complete.

In view of [9, Theorem 4.3] and [9, Theorem 4.1], we see that every real-valued
indefinite Henstock-Kurzweil integral generates a \(\sigma\)-finite Henstock variational mea-
Sure. Thus it is natural to pose the following problem.

**Problem 4.3.** Let \(F\) be given as in Theorem 4.2, and let \(\widetilde{F}\) be the additive
interval function induced by \(F\). Must the Henstock variational measure \(V_{HK}\widetilde{F}\) be
\(\sigma\)-finite on \([0,1]^2\)?

**References**


(1990), 557–567.


400–414. (In Czech.)

[7] J. Kurzweil, J. Jarník: Equi-integrability and controlled convergence of Perron-type in-

[8] Peng-Yee Lee, R. Výborný: The integral, An Easy Approach after Kurzweil and Hen-
stock. Australian Mathematical Society Lecture Series 14 (Cambridge University Press,
2000).


functions defined on \(R^m\). Illinois J. Math. 46 (2002), 1125–1144.

(1950), 102 pp. (In Russian.)

Author’s address: Tuo-Yeong Lee, Mathematics and Mathematics Education, National
Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore
637616, Republic of Singapore, e-mail: tylee@nie.edu.sg.