A NEW FORM OF FUZZY $\alpha$-COMPACTNESS

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Abstract. A new form of $\alpha$-compactness is introduced in $L$-topological spaces by $\alpha$-open $L$-sets and their inequality where $L$ is a complete de Morgan algebra. It doesn’t rely on the structure of the basis lattice $L$. It can also be characterized by means of $\alpha$-closed $L$-sets and their inequality. When $L$ is a completely distributive de Morgan algebra, its many characterizations are presented and the relations between it and the other types of compactness are discussed. Countable $\alpha$-compactness and the $\alpha$-Lindelöf property are also researched.

Keywords: L-topology, compactness, $\alpha$-compactness, countable $\alpha$-compactness, $\alpha$-Lindelöf property, $\alpha$-irresolute map, $\alpha$-continuous map

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1. Introduction

The notion of $\alpha$-open sets was introduced in [13]. The concept of $\alpha$-compactness for topological spaces was discussed in [12], and it was generalized to $[0, 1]$-topological spaces by Thakur, Saraf and Jabalpur [18]. The definition in [18] is based on Chang’s compactness which is not a good extension of ordinary compactness.

In [1], Aygün presented a new form of $\alpha$-compactness which is based on Kudri’s compactness [7] which is equivalent to strong compactness in [9], [19].

The concepts of SR-compactness and near SR-compactness were introduced by S. G. Li, S. Z. Bai and N. Li in terms of strongly semiopen $L$-sets [4], [8]. In fact, a strongly semiopen $L$-set is exactly an $\alpha$-open set in the sense of [14]. Thus both SR-compactness and near SR-compactness are extensions of $\alpha$-compactness. Moreover, the notion of SR-compactness was based on N-compactness and the notion of near

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SR-compactness was based on strong fuzzy compactness. This implies that near SR-compactness is equivalent to $\alpha$-compactness in [1] when the basis lattice $L$ is a complete distributive de Morgan algebra.

In [15], [16], a new definition of fuzzy compactness was presented in $L$-topological spaces by means of open $L$-sets and their inequality where $L$ is a complete de Morgan algebra. This new definition doesn’t depend on the structure of $L$. When $L$ is completely distributive, it is equivalent to the notion of fuzzy compactness in [9], [10], [19].

In this paper, following the lines of [15], [16], we will introduce a new form of $\alpha$-compactness in $L$-topological spaces by means of $\alpha$-open $L$-sets and their inequality. This new form of $\alpha$-compactness has many characterizations if $L$ is completely distributive. Moreover, we will compare our $\alpha$-compactness with other types of $\alpha$-compactness.

2. Preliminaries

Throughout this paper $(L, \vee, \wedge, \cdot)$ is a complete de Morgan algebra, $X$ is a non-empty set. $L^X$ is the set of all $L$-fuzzy sets (or $L$-sets for short) on $X$. The smallest element and the largest element in $L^X$ are denoted by $\chi_0$ and $\chi_X$. We often don’t distinguish a crisp subset $A$ of $X$ and its character function $\chi_A$.

An element $a$ in $L$ is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. An element $a$ in $L$ is called co-prime if $a'$ is prime [6]. The set of non-unit prime elements in $L$ is denoted by $P(L)$. The set of non-zero co-prime elements in $L$ is denoted by $M(L)$.

The binary relation $<$ in $L$ is defined as follows: for $a, b \in L$, $a < b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [5]. In a completely distributive de Morgan algebra $L$, each element $b$ is a sup of $\{a \in L; a < b\}$. A set $\{a \in L; a < b\}$ is called the greatest minimal family of $b$ in the sense of [9], [19], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L; a' < b\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$ we use the following notations from [17].

$$A_{[a]} = \{x \in X; A(x) \geq a\}, \quad A^{(a)} = \{x \in X; A(x) \not\geq a\},$$

$$A_{(a)} = \{x \in X; a \in \beta(A(x))\}.$$

An $L$-topological space (or $L$-space for short) is a pair $(X, \mathcal{F})$, where $\mathcal{F}$ is a subfamily of $L^X$ which contains $\chi_0$, $\chi_X$ and is closed for any suprema and finite infima. $\mathcal{F}$ is called an $L$-topology on $X$. Members of $\mathcal{F}$ are called open $L$-sets and their complements are called closed $L$-sets.
Definition 2.1 ([9], [19]). An L-space \((X, \mathcal{T})\) is called weakly induced if \(\forall a \in L,\forall A \in \mathcal{T}\), it follows that \(A(a) \in [\mathcal{T}]\), where \([\mathcal{T}]\) denotes the topology formed by all crisp sets in \(\mathcal{T}\).

Definition 2.2 ([9], [19]). For a topological space \((X, \tau)\), let \(\omega_L(\tau)\) denote the family of all lower semi-continuous maps from \((X, \tau)\) to \(L\), i.e., \(\omega_L(\tau) = \{A \in L^X; A(a) \in \tau, a \in L\}\). Then \(\omega_L(\tau)\) is an \(L\)-topology on \(X\); in this case, \((X, \omega_L(\tau))\) is called topologically generated by \((X, \tau)\). A topologically generated \(L\)-space is also called an induced \(L\)-space.

It is obvious that \((X, \omega_L(\tau))\) is weakly induced.

For a subfamily \(\Phi \subseteq L^X, 2^{|\Phi|}\) denotes the set of all finite subfamilies of \(\Phi\) and \(2^{|\Phi|}\) denotes the set of all countable subfamilies of \(\Phi\).

Definition 2.3 ([15], [16]). Let \((X, \mathcal{T})\) be an \(L\)-space. \(G \in L^X\) is called (countably) compact if for every (countable) family \(\mathcal{U} \subseteq \mathcal{T}\), it follows that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{y \in 2^{|\mathcal{U}|}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in y} A(x) \right).
\]

Definition 2.4 ([16]). Let \((X, \mathcal{T})\) be an \(L\)-space. \(G \in L^X\) is said to have the Lindelöf property if for every family \(\mathcal{U} \subseteq \mathcal{T}\), it follows that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{y \in 2^{|\mathcal{U}|}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right).
\]

Lemma 2.5 ([16]). Let \(L\) be a complete Heyting algebra, let \(f: X \to Y\) be a map and \(f^L_+: L^X \to L^Y\) the extension of \(f\). Then for any family \(\mathcal{P} \subseteq L^Y\), we have

\[
\bigvee_{y \in Y} \left( f^L_+(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f^L_-(B)(x) \right),
\]

where \(f^L_+: L^X \to L^Y\) and \(f^L_-: L^Y \to L^X\) are defined as follows:

\[
f^L_-(G)(y) = \bigvee_{x \in f^{-1}(y)} G(x), \quad f^L_-(B) = B \circ f.
\]

The notion of an \(\alpha\)-open set was introduced by Njástad in [13] and generalized to \([0,1]\)-topological spaces by Shahana in [14]. Analogously we can generalize it to \(L\)-fuzzy setting as follows:

Definition 2.6 ([14]). An \(L\)-set \(G\) in an \(L\)-space \((X, \mathcal{T})\) is called \(\alpha\)-open if \(G \leq \text{int} (\text{cl} (\text{int} (G)))\). \(G\) is called \(\alpha\)-closed if \(G'\) is \(\alpha\)-open.

17
Definition 2.7 ([18]). Let \((X, \mathcal{T}_1)\) and \((Y, \mathcal{T}_2)\) be two \(L\)-spaces. A map \(f: (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)\) is called \(\alpha\)-continuous if \(f^{-1}_L(G)\) is \(\alpha\)-open in \((X, \mathcal{T}_1)\) for every open \(L\)-set \(G\) in \((Y, \mathcal{T}_2)\).

It can be seen that an \(\alpha\)-continuous map was also said to be strongly semi-continuous in [14].

Definition 2.8 ([18]). Let \((X, \mathcal{T}_1)\) and \((Y, \mathcal{T}_2)\) be two \(L\)-spaces. A map \(f: (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)\) is called \(\alpha\)-irresolute if \(f^{-1}_L(G)\) is \(\alpha\)-open in \((X, \mathcal{T}_1)\) for every \(\alpha\)-open \(L\)-set \(G\) in \((Y, \mathcal{T}_2)\).

3. Definition and characterizations of \(\alpha\)-compactness

Definition 3.1. Let \((X, \mathcal{T})\) be an \(L\)-space. \(G \in L^X\) is called (countably) \(\alpha\)-compact if for every (countable) family \(\mathcal{U}\) of \(\alpha\)-open \(L\)-sets, it follows that

\[
\bigcap_{x \in X} \left( G'(x) \cup \bigcup_{A \in \mathcal{U}} A(x) \right) \subseteq \bigcup_{\mathcal{V} \in 2^{\mathcal{U}}} \left( \bigcap_{x \in X} \left( G'(x) \cup \bigcup_{A \in \mathcal{V}} A(x) \right) \right).
\]

Definition 3.2. Let \((X, \mathcal{T})\) be an \(L\)-space. \(G \in L^X\) is said to have the \(\alpha\)-Lindelöf property (or be an \(\alpha\)-Lindelöf \(L\)-set) if for every family \(\mathcal{U}\) of \(\alpha\)-open \(L\)-sets, it follows that

\[
\bigcap_{x \in X} \left( G'(x) \cup \bigcup_{A \in \mathcal{U}} A(x) \right) \subseteq \bigcup_{\mathcal{V} \in 2^{\mathcal{U}}} \left( \bigcap_{x \in X} \left( G'(x) \cup \bigcup_{A \in \mathcal{V}} A(x) \right) \right).
\]

Obviously we have the following theorem.

Theorem 3.3. \(\alpha\)-compactness implies countable \(\alpha\)-compactness and the \(\alpha\)-Lindelöf property. Moreover, an \(L\)-set having the \(\alpha\)-Lindelöf property is \(\alpha\)-compact if and only if it is countably \(\alpha\)-compact.

Since an open \(L\)-set is \(\alpha\)-open, we have the following theorem.

Theorem 3.4. \(\alpha\)-compactness implies compactness, countable \(\alpha\)-compactness implies countable compactness, and the \(\alpha\)-Lindelöf property implies the Lindelöf property.

From Definition 3.1 and Definition 3.2 we can obtain the following two theorems by simply using complements.
Theorem 3.5. Let \((X, \mathcal{T})\) be an \(L\)-space. \(G \in L^X\) is (countably) \(\alpha\)-compact if and only if for every (countable) family \(\mathcal{B}\) of \(\alpha\)-closed \(L\)-sets, it follows that

\[
\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right) \supseteq \bigwedge_{\mathcal{F} \in 2^{\mathcal{B}}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right).
\]

Theorem 3.6. Let \((X, \mathcal{T})\) be an \(L\)-space. \(G \in L^X\) has the \(\alpha\)-Lindelöf property if and only if for every family \(\mathcal{B}\) of \(\alpha\)-closed \(L\)-sets, it follows that

\[
\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right) \supseteq \bigwedge_{\mathcal{F} \in 2^{\mathcal{B}}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right).
\]

In order to present characterizations of \(\alpha\)-compactness, countable \(\alpha\)-compactness and the \(\alpha\)-Lindelöf property, we generalize the notions of an \(a\)-shading and an \(a\)-\(R\)-neighborhood family in [15], [16] as follows:

Definition 3.7. Let \((X, \mathcal{T})\) be an \(L\)-space, \(a \in L \setminus \{1\}\) and \(G \in L^X\). A family \(\mathcal{A} \subseteq L^X\) is said to be

1. an \(a\)-shading of \(G\) if for any \(x \in X\), \(\left( G'(x) \lor \bigvee_{A \in \mathcal{A}} A(x) \right) \not\ll a\).

2. a strong \(a\)-shading of \(G\) if \(\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{A}} A(x) \right) \not\ll a\).

3. an \(a\)-remote family of \(G\) if for any \(x \in X\), \(\left( G(x) \land \bigwedge_{B \in \mathcal{A}} B(x) \right) \not\ll a\).

4. a strong \(a\)-remote family of \(G\) if \(\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{A}} B(x) \right) \not\ll a\).

It is obvious that a strong \(a\)-shading of \(G\) is an \(a\)-shading of \(G\), a strong \(a\)-remote family of \(G\) is an \(a\)-remote family of \(G\), and \(\mathcal{P}\) is a strong \(a\)-remote family of \(G\) if and only if \(\mathcal{P}'\) is a strong \(a'\)-shading of \(G\). Moreover, a closed \(a\)-remote family is exactly an \(a\)-\(R\)-neighborhood family and a closed strong \(a\)-remote family is exactly an \(a'\)-\(R\)-neighborhood family in the sense of [19].

Definition 3.8. Let \(a \in L \setminus \{0\}\) and \(G \in L^X\). A subfamily \(\mathcal{A}\) of \(L^X\) is said to have a weak \(a\)-nonempty intersection in \(G\) if \(\bigvee_{x \in X} \left( G(x) \land \bigwedge_{A \in \mathcal{A}} A(x) \right) \not\ll a\). \(\mathcal{A}\) is said to have the finite (countable) weak \(a\)-intersection property in \(G\) if every finite (countable) subfamily \(\mathcal{P}\) of \(\mathcal{A}\) has a weak \(a\)-nonempty intersection in \(G\).

Definition 3.9. Let \(a \in L \setminus \{0\}\) and \(G \in L^X\). A subfamily \(\mathcal{A}\) of \(L^X\) is said to be a weak \(a\)-filter relative to \(G\) if any finite intersection of members in \(\mathcal{A}\) is weak \(a\)-nonempty in \(G\). A subfamily \(\mathcal{B}\) of \(L^X\) is said to be a weak \(a\)-filterbase relative to \(G\) if

\[\{ A \in L^X ; \text{there exists } B \in \mathcal{B} \text{ such that } B \subseteq A \}\]

is a weak \(a\)-filter relative to \(G\).
From Definition 3.1, Definition 3.2, Theorem 3.5 and Theorem 3.6 we immediately obtain the following two results.

**Theorem 3.10.** Let \((X, \mathcal{T})\) be an \(L\)-space and \(G \in L^X\). Then the following conditions are equivalent:

(1) \(G\) is (countably) \(\alpha\)-compact.

(2) For any \(a \in L \setminus \{1\}\), each (countable) \(\alpha\)-open strong \(a\)-shading \(\mathcal{U}\) of \(G\) has a finite subfamily which is a strong \(a\)-shading of \(G\).

(3) For any \(a \in L \setminus \{0\}\), each (countable) \(\alpha\)-closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a finite subfamily which is a strong \(a\)-remote family of \(G\).

(4) For any \(a \in L \setminus \{0\}\), each (countable) family of \(\alpha\)-closed \(L\)-sets which has the finite weak \(a\)-intersection property in \(G\) has a weak \(a\)-nonempty intersection in \(G\).

(5) For each \(a \in L \setminus \{0\}\), every \(\alpha\)-closed (countable) weak \(a\)-filterbase relative to \(G\) has a weak \(a\)-nonempty intersection in \(G\).

**Theorem 3.11.** Let \((X, \mathcal{T})\) be an \(L\)-space and \(G \in L^X\). Then the following conditions are equivalent:

(1) \(G\) has the \(\alpha\)-Lindelöf property.

(2) For any \(a \in L \setminus \{1\}\), each \(\alpha\)-open strong \(a\)-shading \(\mathcal{U}\) of \(G\) has a countable subfamily which is a strong \(a\)-shading of \(G\).

(3) For any \(a \in L \setminus \{0\}\), each \(\alpha\)-closed strong \(a\)-remote family \(\mathcal{P}\) of \(G\) has a countable subfamily which is a strong \(a\)-remote family of \(G\).

(4) For any \(a \in L \setminus \{0\}\), each family of \(\alpha\)-closed \(L\)-sets which has the countable weak \(a\)-intersection property in \(G\) has a weak \(a\)-nonempty intersection in \(G\).

### 4. Properties of (countable) \(\alpha\)-compactness

**Theorem 4.1.** Let \(L\) be a complete Heyting algebra. If both \(G\) and \(H\) are (countably) \(\alpha\)-compact, then \(G \lor H\) is (countably) \(\alpha\)-compact.

**Proof.** For any (countable) family \(\mathcal{P}\) of \(\alpha\)-closed \(L\)-sets, we have by Theorem 3.5 that

\[
\bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) = \left\{ \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\} \lor \left\{ \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\}
\]
\[
\left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \lor \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\}
= \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right).
\]

This shows that \( G \lor H \) is (countably) \( \alpha \)-compact. \( \square \)

Analogously we have the following result.

**Theorem 4.2.** Let \( L \) be a complete Heyting algebra. If both \( G \) and \( H \) have the \( \alpha \)-Lindelöf property, then so does \( G \lor H \).

**Theorem 4.3.** If \( G \) is (countably) \( \alpha \)-compact and \( H \) is \( \alpha \)-closed, then \( G \land H \) is (countably) \( \alpha \)-compact.

**Proof.** For any (countable) family \( \mathcal{P} \) of \( \alpha \)-closed \( L \)-sets, we have by Theorem 3.5 that

\[
\bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right)
= \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P} \cup (H)} B(x) \right)
\supseteq \bigwedge_{\mathcal{F} \in 2(\mathcal{P} \cup (H))} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right)
= \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P} \cup (H))} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\}
\land \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left( G(x) \land H(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\}
= \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \left\{ \bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\}
= \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \left\{ \bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\}
\]

This shows that \( G \land H \) is (countably) \( \alpha \)-compact. \( \square \)

Analogously we have the following result.

**Theorem 4.4.** If \( G \) has the \( \alpha \)-Lindelöf property and \( H \) is \( \alpha \)-closed, then \( G \land H \) has the \( \alpha \)-Lindelöf property.
Theorem 4.5. Let $L$ be a complete Heyting algebra and let $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be an $\alpha$-irresolute map. If $G$ is an $\alpha$-compact (or a countably $\alpha$-compact, an $\alpha$-Lindelöf) $L$-set in $(X, \mathcal{T}_1)$, then so is $f^{-}\!_{L}(G)$ in $(Y, \mathcal{T}_2)$.

Proof. We only prove that the theorem is true for $\alpha$-compactness. Suppose that $\mathcal{P}$ is a family of $\alpha$-closed $L$-sets in $(Y, \mathcal{T}_2)$. Then by Lemma 2.5 and $\alpha$-compactness of $G$ we have that

$$\bigvee_{y \in Y} \left( f^{-}\!_{L}(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f^{-}\!_{L}(B)(x) \right)$$

$$\geq \bigwedge_{\mathcal{F} \in \mathcal{P}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} f^{-}\!_{L}(B)(x) \right) = \bigwedge_{\mathcal{F} \in \mathcal{P}} \bigvee_{y \in Y} \left( f^{-}\!_{L}(G)(y) \land \bigwedge_{B \in \mathcal{F}} B(y) \right).$$

Therefore $f^{-}\!_{L}(G)$ is $\alpha$-compact.

Analogously we have the following result.

Theorem 4.6. Let $L$ be a complete Heyting algebra and let $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be an $\alpha$-continuous map. If $G$ is an $\alpha$-compact (a countably $\alpha$-compact, an $\alpha$-Lindelöf) $L$-set in $(X, \mathcal{T}_1)$, then $f^{-}\!_{L}(G)$ is a compact (countably compact, Lindelöf) $L$-set in $(Y, \mathcal{T}_2)$.

Definition 4.7. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two $L$-spaces. A map $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called strongly $\alpha$-irresolute if $f^{-}\!_{L}(G)$ is open in $(X, \mathcal{T}_1)$ for every $\alpha$-open $L$-set $G$ in $(Y, \mathcal{T}_2)$.

It is obvious that a strongly $\alpha$-irresolute map is $\alpha$-irresolute and continuous.

Analogously we have the following result.

Theorem 4.8. Let $L$ be a complete Heyting algebra and let $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a strongly $\alpha$-irresolute map. If $G$ is a compact (countably compact, Lindelöf) $L$-set in $(X, \mathcal{T}_1)$, then $f^{-}\!_{L}(G)$ is an $\alpha$-compact (a countably $\alpha$-compact, an $\alpha$-Lindelöf) $L$-set in $(Y, \mathcal{T}_2)$.

5. Further characterizations of $\alpha$-compactness and goodness

In this section we assume that $L$ is a completely distributive de Morgan algebra.

Now we generalize the notions of a $\beta_\alpha$-open cover and a $Q_\alpha$-open cover [16] as follows:
**Definition 5.1.** Let $(X, \mathcal{F})$ be an $L$-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{W} \subseteq L^X$ is called a $\beta_a$-cover of $G$ if for any $x \in X$, it follows that $a \in \beta(G'(x) \cup \bigvee_{A \in \mathcal{W}} A(x))$. $\mathcal{W}$ is called a strong $\beta_a$-cover of $G$ if $a \in \beta\left(\bigcap_{x \in X} \left(G'(x) \cup \bigvee_{A \in \mathcal{W}} A(x)\right)\right)$.

**Definition 5.2.** Let $(X, \mathcal{F})$ be an $L$-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{W} \subseteq L^X$ is called a $\mathcal{Q}_a$-cover of $G$ if for any $x \in X$, it follows that $G'(x) \cup \bigvee_{A \in \mathcal{W}} A(x) \geq a$.

It is obvious that a strong $\beta_a$-cover of $G$ is a $\beta_a$-cover of $G$, and a $\beta_a$-cover of $G$ is a $\mathcal{Q}_a$-cover of $G$.

Analogously to the proof of Theorem 2.9 in [16] we can obtain the following theorem.

**Theorem 5.3.** Let $(X, \mathcal{F})$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent.

1. $G$ is $\alpha$-compact.
2. For any $a \in L \setminus \{0\}$ (or $a \in M(L)$), each $\alpha$-closed strong $a$-remote family of $G$ has a finite subfamily which is an $a$-remote (a strong $a$-remote) family of $G$.
3. For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $\alpha$-closed strong $a$-remote family $\mathcal{P}$ of $G$, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ (or $b \in \beta^*(a)$) such that $\mathcal{F}$ is a (strong) $b$-remote family of $G$.
4. For any $a \in L \setminus \{1\}$ (or $a \in P(L)$), each $\alpha$-open strong $a$-shading of $G$ has a finite subfamily which is an $a$-shading (a strong $a$-shading) of $G$.
5. For any $a \in L \setminus \{1\}$ (or $a \in P(L)$) and any $\alpha$-open strong $a$-shading $\mathcal{V}$ of $G$, there exist a finite subfamily $\mathcal{Y}$ of $\mathcal{V}$ and $b \in \alpha(a)$ (or $b \in \alpha^*(a)$) such that $\mathcal{Y}$ is a (strong) $b$-shading of $G$.
6. For any $a \in L \setminus \{0\}$ (or $a \in M(L)$), each $\alpha$-open strong $\beta_a$-cover of $G$ has a finite subfamily which is a (strong) $\beta_a$-cover of $G$.
7. For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $\alpha$-open strong $\beta_a$-cover $\mathcal{W}$ of $G$, there exist a finite subfamily $\mathcal{Y}$ of $\mathcal{W}$ and $b \in L$ (or $b \in M(L)$) with $a \in \beta(b)$ such that $\mathcal{Y}$ is a (strong) $\beta_b$-cover of $G$.
8. For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$, each $\alpha$-open $\mathcal{Q}_b$-cover of $G$ has a finite subfamily which is a $\mathcal{Q}_b$-cover of $G$.
9. For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$ (or $b \in \beta^*(a)$), each $\alpha$-open $\mathcal{Q}_b$-cover of $G$ has a finite subfamily which is a (strong) $\beta_b$-cover of $G$.

Analogously we also can present characterizations of countable $\alpha$-compactness and the $\alpha$-Lindelöf property.

Now we consider the goodness of $\alpha$-compactness.
For $a \in L$ and a crisp subset $D \subset X$, we define $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases}$$

$$(a \vee D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases}$$

**Theorem 5.4** ([17]). For an $L$-set $A \in L^X$, the following facts are true.

1. $A = \bigvee_{a \in L} (a \wedge A_{[a]}^-) = \bigvee_{a \in L} (a \wedge A_{[a]}^-)$.
2. $A = \bigwedge_{a \in L} (a \vee A_{[a]}^\circ) = \bigwedge_{a \in L} (a \vee A_{[a]}^\circ)$.

**Theorem 5.5** [17]. Let $(X, \omega_L(\tau))$ be the $L$-space topologically generated by $(X, \tau)$ and $A \in L^X$. Then the following facts hold.

1. $\text{cl}(A) = \bigvee_{a \in L} (a \wedge (A_{[a]})^-) = \bigvee_{a \in L} (a \wedge (A_{[a]})^-)$.
2. $\text{cl}(A_{[a]}^\circ) \subseteq (A_{[a]})^- \subseteq \text{cl}(A_{[a]}^\circ)$.
3. $\text{cl}(A) = \bigwedge_{a \in L} (a \vee (A_{[a]})^-) = \bigwedge_{a \in L} (a \vee (A_{[a]})^-)$.
4. $\text{int}(A_{[a]}^\circ) \subseteq (A_{[a]})^- \subseteq \text{int}(A_{[a]}^\circ)$.
5. $\text{int}(A) = \bigvee_{a \in L} (a \wedge (A_{[a]})^\circ) = \bigvee_{a \in L} (a \wedge (A_{[a]})^\circ)$.
6. $\text{int}(A_{[a]}^\circ) \subseteq (A_{[a]})^\circ \subseteq \text{int}(A_{[a]}^\circ)$.
7. $\text{int}(A) = \bigwedge_{a \in L} (a \vee (A_{[a]})^\circ) = \bigwedge_{a \in L} (a \vee (A_{[a]})^\circ)$.
8. $\text{int}(A_{[a]}^\circ) \subseteq (A_{[a]})^\circ \subseteq \text{int}(A_{[a]}^\circ)$, where $(A_{[a]})^-$ and $(A_{[a]})^\circ$ denote respectively the closure and the interior of $A_{[a]}$ in $(X, \tau)$ and so on.

**Lemma 5.6.** Let $(X, \omega_L(\tau))$ be generated topologically by $(X, \tau)$. If $A$ is an $\alpha$-open set in $(X, \tau)$, then $\chi_A$ is an $\alpha$-open $L$-set in $(X, \omega_L(\tau))$. If $B$ is an $\alpha$-open $L$-set in $(X, \omega_L(\tau))$, then $B_{(a)}$ is an $\alpha$-open set in $(X, \tau)$ for every $a \in L$.

**Proof.** If $A$ is an $\alpha$-open set in $(X, \tau)$, then $A \subseteq ((A^\circ)^-)$. Thus we have

$$\chi_A \subseteq \chi_{((A^\circ)^-)} = \text{int}(\chi_{A^\circ}) = \text{int}(\text{cl}(\chi_A)) = \text{int}(\text{cl}(\chi_{A^\circ})).$$

This shows that $\chi_A$ is $\alpha$-open in $(X, \omega_L(\tau))$.

If $B$ is an $\alpha$-open $L$-set in $(X, \omega_L(\tau))$, then $B \subseteq \text{int}(\text{cl}(B)))$. From Theorem 5.5 we have

$$B_{(a)} \subseteq \text{int}(\text{cl}(B)))_{(a)} \subseteq (\text{cl}(B)))_{(a)}^\circ \subseteq ((B_{(a)})^-)^\circ \subseteq (((B_{(a)})^\circ)^-)^\circ.$$

This shows that $B_{(a)}$ is an $\alpha$-open set in $(X, \tau)$.

The next two theorems show that $\alpha$-compactness, countable $\alpha$-compactness and the $\alpha$-Lindelöf property are good extensions.
**Theorem 5.7.** Let \((X, \omega_L(\tau))\) be generated topologically by \((X, \tau)\). Then \((X, \omega_L(\tau))\) is (countably) \(\alpha\)-compact if and only if \((X, \tau)\) is (countably) \(\alpha\)-compact.

**Proof.** (Necessity) Let \(\mathcal{A}\) be an \(\alpha\)-open cover (a countable \(\alpha\)-open cover) of \((X, \tau)\). Then \(\{\chi_A; A \in \mathcal{A}\}\) is a family of \(\alpha\)-open \(L\)-sets in \((X, \omega_L(\tau))\) with \(\bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{A}} \chi_A(x) \right) = 1\). From (countable) \(\alpha\)-compactness of \((X, \omega_L(\tau))\) we know that
\[
1 \geq \bigvee_{\mathcal{A} \subseteq 2^{(\mathcal{A})}} \bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{A}} \chi_A(x) \right) \geq \bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{A}} \chi_A(x) \right) = 1.
\]
This implies that there exists \(\mathcal{A}' \subseteq 2^{(\mathcal{A})}\) such that \(\bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{A}'} \chi_A(x) \right) = 1\). Hence \(\mathcal{A}'\) is a cover of \((X, \tau)\). Therefore \((X, \tau)\) is (countably) \(\alpha\)-compact.

(Sufficiency) Let \(\mathcal{V}\) be a (countable) family of \(\alpha\)-open \(L\)-sets in \((X, \omega_L(\tau))\) and let \(\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{V}} B(x) \right) = a\). If \(a = 0\), then we obviously have
\[
\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{V}} B(x) \right) \leq \bigvee_{\mathcal{A} \subseteq 2^{(\mathcal{V})}} \bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{A}} B(x) \right).
\]
Now we suppose that \(a \neq 0\). In this case, for any \(b \in \beta(a) \setminus \{0\}\) we have
\[
b \in \beta \left( \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{V}} B(x) \right) \right) \subseteq \bigcap_{x \in X} \beta \left( \bigvee_{B \in \mathcal{V}} B(x) \right) = \bigcap_{x \in X \setminus B \in \mathcal{V}} \beta(B(x)).
\]
By Lemma 5.6 this implies that \(\{B(b); B \in \mathcal{V}\}\) is an \(\alpha\)-open cover of \((X, \tau)\). From (countable) \(\alpha\)-compactness of \((X, \tau)\) we know that there exists \(\mathcal{A}' \subseteq 2^{(\mathcal{V})}\) such that \(\{B(b); B \in \mathcal{A}'\}\) is a cover of \((X, \tau)\). Hence \(b \leq \bigwedge_{x \in X \setminus B \in \mathcal{A}'} \left( \bigvee_{B \in \mathcal{A}'} B(x) \right)\). Further we have
\[
b \leq \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{A}'} B(x) \right) \leq \bigvee_{\mathcal{A} \subseteq 2^{(\mathcal{A}')}} \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{A}'} B(x) \right).
\]
This implies that
\[
\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{V}} B(x) \right) = a = \bigvee \{b; b \in \beta(a)\} \leq \bigvee_{\mathcal{A} \subseteq 2^{(\mathcal{V})}} \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{A}'} B(x) \right).
\]
Therefore \((X, \omega_L(\tau))\) is (countably) \(\alpha\)-compact. \(\square\)

Analogously we have the following result.
Theorem 5.8. Let \((X,\omega_L(\tau))\) be generated topologically by \((X,\tau)\). Then \((X,\omega_L(\tau))\) has the \(\alpha\)-Lindelöf property if and only if \((X,\tau)\) has the \(\alpha\)-Lindelöf property.

6. The relations of \(\alpha\)-compactness and other types of compactness

In this section we assume that \(L\) is again completely distributive.

Based on Kudri’s compactness in [7], Aygün presented a definition of \(\alpha\)-compactness in [1]. Since Kudri’s compactness is equivalent to strong compactness in [9], [19], we shall also refer to Aygün’s \(\alpha\)-compactness as \(\alpha\)-strong compactness. The following is its equivalent form.

Definition 6.1. Let \((X,\mathcal{F})\) be an \(L\)-space. \(G \in L^X\) is said to be \(\alpha\)-strongly compact if for any \(a \in P(L)\), each \(\alpha\)-open \(a\)-shading of \(G\) has a finite subfamily which is an \(a\)-shading of \(G\).

In [4], [8], Bai and Li et al. introduced the notions of SR-compactness and near SR-compactness by means of strongly semiopen \(L\)-sets. In fact, a strongly semiopen \(L\)-set is equivalent to an \(\alpha\)-open \(L\)-set. This implies that both SR-compactness and near SR-compactness are extensions of \(\alpha\)-compactness in general topology. Their equivalent forms can be stated as follows:

Definition 6.2 ([4]). Let \((X,\mathcal{F})\) be an \(L\)-space. \(G \in L^X\) is said to be SR-compact (we shall call it \(\alpha\)-N-compact) if for each \(a \in M(L)\), every \(\alpha\)-closed \(a\)-remote family of \(G\) has a finite subfamily which is a strong \(a\)-remote family of \(G\).

Definition 6.3 ([8]). Let \((X,\mathcal{F})\) be an \(L\)-space. \(G \in L^X\) is said to be near SR-compact if for each \(a \in M(L)\), every \(\alpha\)-closed \(a\)-remote family of \(G\) has a finite subfamily which is an \(a\)-remote family of \(G\).

It is obvious that Definition 6.3 is equivalent to Definition 6.1.

From Theorem 5.3 we easily obtain the following result.

Theorem 6.4. For an \(L\)-set \(G\) in an \(L\)-space, the following implications hold.

\[
\begin{align*}
\alpha\text{-N-compactness} & \Rightarrow \alpha\text{-strong compactness} \Rightarrow \alpha\text{-compactness} \\
\downarrow & \downarrow & \downarrow \\
N\text{-compactness} & \Rightarrow \text{strong compactness} & \Rightarrow \text{compactness}
\end{align*}
\]

Notice that none of the above implications is invertible. We only present a counterexample which is \(\alpha\)-compact but not \(\alpha\)-strongly compact. The other examples can be found in [8], [9], [19] and in general topology.
Example 6.5. Take $Y = \mathbb{N}$. For all $n \in Y$, define $B_n \in [0, 1]^Y$ as follows:

$$B_n(y) = \begin{cases} (n+1)^{-1}, & y = n; \\ 0, & y \neq n. \end{cases}$$

Let $\mathcal{T}$ be the $[0, 1]$-topology generated by the subbase $\mathcal{B} = \{B_n : n \in Y\}$. Obviously $\{B_n : n \in Y\}$ is an open 0-shading of $\chi_Y$, but $\{B_n : n \in Y\}$ has no finite subfamily which is an open 0-shading of $\chi_Y$. Therefore $(Y, \mathcal{T})$ is not strongly compact, of course it is not $\alpha$-strongly compact either.

Now we prove that $(Y, \mathcal{T})$ is $\alpha$-compact. It is easy to check that if $A$ is an $\alpha$-open $L$-set in $(Y, \mathcal{T})$ and $A \neq \chi_Y$, then $A \subseteq \bigvee_{n \in Y} B_n$.

For each $\alpha \in [0, 1)$, suppose that $\mathcal{W}$ is an $\alpha$-open strong $a$-shading of $\chi_Y$. If $\chi_Y \in \mathcal{W}$, then $\{\chi_Y\}$ is a strong $a$-shading of $\chi_Y$. Now we suppose that $\chi_Y \notin \mathcal{W}$. Then $\mathcal{W}$ is not a strong $a$-shading of $\chi_Y$ since $\bigwedge_{y \in Y} \left( \bigvee_{A \in \mathcal{W}} A(y) \right) \leq \bigwedge_{y \in Y} \left( \bigvee_{n \in Y} B_n(y) \right) = 0$.

This shows that $(Y, \mathcal{T})$ is $\alpha$-compact.

References


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