CONTINUITY IN THE ALEXIEWICZ NORM

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. If $f$ is a Henstock-Kurzweil integrable function on the real line, the Alexiewicz norm of $f$ is $\|f\| = \sup_I \int_I |f|$ where the supremum is taken over all intervals $I \subset \mathbb{R}$. Define the translation $\tau_x$ by $\tau_x f(y) = f(y - x)$. Then $\|\tau_x f - f\|$ tends to 0 as $x$ tends to 0, i.e., $f$ is continuous in the Alexiewicz norm. For particular functions, $\|\tau_x f - f\|$ can tend to 0 arbitrarily slowly. In general, $\|\tau_x f - f\| \geq \text{osc} f|_{\mathbb{R}}$ as $x \to 0$, where $\text{osc} f$ is the oscillation of $f$. It is shown that if $F$ is a primitive of $f$ then $\|\tau_x F - F\| \leq \|f\|_{\mathbb{R}}$. An example shows that the function $y \to \tau_x F(y) - F(y)$ need not be in $L^1$. However, if $f \in L^1$ then $\|\tau_x F - F\|_1 \leq \|f\|_1|_{\mathbb{R}}$. For a positive weight function $w$ on the real line, necessary and sufficient conditions on $w$ are given so that $\|\tau_x f|w\| \to 0$ as $x \to 0$ whenever $f w$ is Henstock-Kurzweil integrable. Applications are made to the Poisson integral on the disc and half-plane. All of the results also hold with the distributional Denjoy integral, which arises from the completion of the space of Henstock-Kurzweil integrable functions as a subspace of Schwartz distributions.

Keywords: Henstock-Kurzweil integral, Alexiewicz norm, distributional Denjoy integral, Poisson integral

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1. Introduction

For $f: \mathbb{R} \to \mathbb{R}$ define the translation by $\tau_x f(y) = f(y - x)$ for $x, y \in \mathbb{R}$. If $f \in L^p (1 \leq p < \infty)$ then it is a well known result of Lebesgue integration that $f$ is continuous in the $p$-norm, i.e., $\lim_{x \to 0} \|\tau_x f - f\|_p = 0$. For example, see [4, Lemma 6.3.5]. In this paper we consider continuity of Henstock-Kurzweil integrable functions in

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Alexiewicz and weighted Alexiewicz norms on the real line. Let $\mathcal{HK}$ be the set of functions $f: \mathbb{R} \to \mathbb{R}$ that are Henstock-Kurzweil integrable. The Alexiewicz norm of $f \in \mathcal{HK}$ is defined \( \|f\| = \sup_I \|f_I\| \) where the supremum is over all intervals $I \subseteq \mathbb{R}$. Identifying functions almost everywhere, $\mathcal{HK}$ becomes a normed linear space under $\|\cdot\|$ that is barrelled but not complete. See [1] and [5] for a discussion of the Henstock-Kurzweil integral and the Alexiewicz norm. It is shown below that translations are continuous in norm and that for $f \in \mathcal{HK}$ we have $\|\tau_x f - f\| \geq \text{osc } f [x]$ where $\text{osc } f$ is the oscillation of $f$. For particular $f \in \mathcal{HK}$ the quantity $\|\tau_x f - f\|$ can tend to 0 arbitrarily slowly. If $F$ is a primitive of $f$ then $\|\tau_x F - F\| \leq \|f\|\|x\|$. An example shows that if $f \in \mathcal{HK}$ then the function defined by $y \mapsto \tau_x F(y) - F(y)$ need not be in $L^1$ but if $f \in L^1$ then $\|\tau_x F - F\|_1 \leq \|f\|\|x\|$. For a positive weight function $w$ on the real line, necessary and sufficient conditions on $w$ are given so that $\|\tau_x f - f w\| \to 0$ as $x \to 0$ whenever $f w$ is Henstock-Kurzweil integrable. The necessary and sufficient conditions involve properties of the function $g_x(y) = w(y + x)/w(y)$. Sufficient conditions are given on $w$ for $\|\tau_x f - f w\| \to 0$. Applications to the Dirichlet problem in the disc and half-plane are given.

First we prove continuity in the Alexiewicz norm.

**Theorem 1.** Let $f \in \mathcal{HK}$. For $x, y \in \mathbb{R}$ define $\tau_x f(y) = f(y - x)$. Then $\|\tau_x f - f\| \to 0$ as $x \to 0$.

**Proof.** Let $x, \alpha, \beta \in \mathbb{R}$. Then $\int_{\alpha}^{\beta} (\tau_x f - f) = f_{\alpha - x}^{\beta - x} f = f_{\alpha}^{\beta} f$. Write $F(x) = \int_{-\infty}^{x} f$. Taking the supremum over $\alpha$ and $\beta$,

\[
\|\tau_x f - f\| \leq \sup_{\beta \in \mathbb{R}} |F(\beta - x) - F(\beta)| + \sup_{\alpha \in \mathbb{R}} |F(\alpha - x) - F(\alpha)|
\]

\to 0 as $x \to 0$ since $F$ is uniformly continuous on $\mathbb{R}$.

Notice that for each $x \in \mathbb{R}$, the translation $\tau_x$ is an isometry on $\mathcal{HK}$, i.e., it is a homeomorphism such that $\|\tau_x f\| = \|f\|$. It is also clear that we have continuity at each point: for each $x_0 \in \mathbb{R}$, $\|\tau_x f - \tau_{x_0} f\| \to 0$ as $x \to x_0$.

The theorem also applies on any interval $I \subseteq \mathbb{R}$. Restrict $\alpha$ and $\beta$ to lie in $I$ and extend $f$ to be 0 outside $I$. Or, one could use a periodic extension. The same results also hold for the equivalent norm $\|f\| = \sup_{x \in \mathbb{R}} |f_{-\infty}^{x} f|$.
Under the Alexiewicz norm, the space of Henstock-Kurzweil integrable functions is not complete. Its completion with respect to the norm \( \|f\| = \sup_{x \in \mathbb{R}} |f'_x f| \) is the subspace of distributions that are the distributional derivative of a function in \( \tilde{C} := \{ F: \mathbb{R} \to \mathbb{R}; F' \in C^0(\mathbb{R}), \lim_{x \to -\infty} F(x) = 0, \lim_{x \to \infty} F(x) \in \mathbb{R} \} \), i.e., they are distributions of order 1. See [6], where the completion is denoted \( \mathcal{A} \). Thus, if \( f \in \mathcal{A} \) then \( f \in \mathcal{D}' \) (Schwartz distributions) and there is a function \( F \in \tilde{C} \) such that \( \langle F', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_0^{\infty} F \varphi' = \langle f, \varphi \rangle \) for all test functions \( \varphi \in \mathcal{D} = C_c^\infty(\mathbb{R}) \).

The distributional integral of \( f \) is then \( f_0^b f = F(b) - F(a) \) for all \( -\infty \leq a \leq b \leq \infty \). We can compute the Alexiewicz norm of \( f \) via \( \|f\| = \sup_{x \in \mathbb{R}} |F(x)| = \|F\|_\infty \). If \( f \in \mathcal{D}' \) then \( \tau_x f \) is defined by \( \langle \tau_x f, \varphi \rangle := \langle f, \tau_{-x} \varphi \rangle = \langle f', \tau_{-x} \varphi' \rangle = -(\langle f, \tau_{-x} \varphi' \rangle)' = -(-\langle f, \tau_{-x} \varphi' \rangle)' = \langle \tau_x F, \varphi' \rangle = \langle \tau_x F', \varphi \rangle \). Of course we have \( L^1 \subset HK \subset \mathcal{A} \) and each inclusion is strict.

The theorem only depends on uniform continuity of the primitive and not on its pointwise differentiability properties so it also holds in \( \mathcal{A} \). The same is true for the other theorems in this paper.

**Corollary 2.** Let \( f \in \mathcal{A} \). Then \( \|\tau_x f - f\| \to 0 \) as \( x \to 0 \).

The following theorem gives us more precise information on the decay rate of \( \|\tau_x f - f\| \).

**Theorem 3.** (a) Let \( \psi: (0,1] \to (0,\infty) \) such that \( \lim_{x \to 0} \psi(x) = 0 \). Then there is \( f \in L^1 \) such that \( \|\tau_x f - f\| \geq \psi(x) \) for all sufficiently small \( x > 0 \). (b) If \( f \in HK \) and \( f \neq 0 \) a.e. then the most rapid decay is \( \|\tau_x f - f\| = O(x) \) as \( x \to 0 \) and this is the best estimate in the sense that if \( \|\tau_x f - f\|/x \to 0 \) as \( x \to 0 \) then \( f = 0 \) a.e. The implied constant in the order relation is the oscillation of \( f \).

**Proof.** (a) Given \( \psi \), define \( \psi_1(x) = \sup_{0 < t \leq x} \psi(t) \). Then \( \psi_1 \geq \psi \) and \( \psi_1(x) \) decreases to 0 as \( x \) decreases to 0. Define \( \psi_2(x) = \psi_1(1/n) \) when \( x \in (1/(n+1), 1/n] \) for some \( n \in \mathbb{N} \). Then \( \psi_2 \geq \psi \) and \( \psi_2 \) is a step function that decreases to 0 as \( x \) decreases to 0. Now let

\[
\psi_3(x) = |\psi_2(1/(n-1)) - \psi_2(1/n)| n(n+1) \left( x - \frac{1}{n+1} \right) + \psi_2(1/n)
\]

when \( x \in [1/(n+1), 1/n] \) for some \( n \geq 2 \). Define \( \psi_3 = \psi_2 \) on \( (1/2, 1] \). Then \( \psi_3 \geq \psi \) and \( \psi_3 \) is a piecewise linear continuous function that decreases to 0 as \( x \) decreases to 0. Define \( f(x) = \psi_3(x) \) for \( x \in (0,1] \) and \( f(x) = 0 \), otherwise. For \( 0 < x < 1 \),

\[
\|\tau_x f - f\| \geq |\int_0^x \left( f(y-x) - f(y) \right) dy| = \int_0^x f = \psi_3(x) \geq \psi(x).
\]

Since \( \psi_3 \) is absolutely continuous, \( f \in L^1 \).
(b) Test functions are dense in \( \mathcal{H} \mathcal{K} \), i.e., for each \( f \in \mathcal{H} \mathcal{K} \) and \( \varepsilon > 0 \) there is \( \varphi \in \mathcal{D} \) such that \( \| f - \varphi \| < \varepsilon \). Let \( x \in \mathbb{R} \). Then, since \( \tau_x \) is a linear isometry, 
\[
\| (\tau_x f - f) - (\tau_x \varphi - \varphi) \| = \| \tau_x (f - \varphi) - (f - \varphi) \| < 2\varepsilon \quad \text{and} \quad \| \tau_x \varphi - \varphi \| < 2\varepsilon.
\]
It therefore suffices to prove the theorem in \( \mathcal{D} \). Hence, let \( f \in \mathcal{D} \) and let \( a, b \in \mathbb{R} \). Write \( F(y) = \int_a^y f \). Then, since \( F \in C^2(\mathbb{R}) \),
\[
\int_a^b (\tau_x f - f) = [F(b - x) - F(b)] - [F(a - x) - F(a)]
= -F'(b)x + F''(\xi)x^2/2 + F'(a)x - F''(\eta)x^2/2,
\]
for some \( \xi, \eta \) in the support of \( f \). Now,
\[
\| \tau_x f - f \| \geq \sup_{a, b \in \mathbb{R}} |f(a) - f(b)| |x| - \| f' \| \| x \| x^2 = \text{osc } f |x| - \| f' \| \| x \| x^2.
\]
The oscillation of \( f \in \mathcal{D} \) is positive unless \( f \) is constant, but there are no constant functions in \( \mathcal{D} \) except 0. The proof is completed by noting that \( \| \tau_x f - f \| \leq \text{osc } f |x| + \| f' \| \| x \| x^2 \) so that \( \| \tau_x f - f \| = \mathcal{O}(x) \) as \( x \to 0 \). \( \square \)

Part (b) is proven in [3, Proposition 1.2.3] for \( f \in L^1 \).

It is interesting to note that if \( f \in \mathcal{H} \mathcal{K} \) and \( F \) is its primitive then the function \( \tau_x F - F \) is in \( \mathcal{H} \mathcal{K} \) for each \( x \in \mathbb{R} \), even though \( F \) need not be in \( \mathcal{H} \mathcal{K} \).

**Theorem 4.** Let \( f \in \mathcal{H} \mathcal{K} \), let \( F \) be one of its primitives and let \( x \in \mathbb{R} \). Then the function \( y \mapsto \tau_x F(y) - F(y) \) is in \( \mathcal{H} \mathcal{K} \) even though none of the primitives of \( f \) need be in \( \mathcal{H} \mathcal{K} \). We have the estimate \( \| \tau_x F - F \| \leq \| f \||x| \|. \) In general, \( \tau_x F - F \) need not be in \( L^1 \). However, if \( f \in L^1 \) then \( \tau_x F - F \in L^1 \) and \( \| \tau_x F - F \|_1 \leq \| f \|_1 |x| \).

**Proof.** Let \( f \in \mathcal{H} \mathcal{K} \) and let \( F \) be any primitive. Since \( F \) is continuous, to prove \( \tau_x F - F \in \mathcal{H} \mathcal{K} \) we need only show integrability at infinity. Let \( a, x \in \mathbb{R} \). Then
\[
\int_0^a (\tau_x F - F) = \int_{-x}^{a-x} F - \int_0^a F = \int_{-x}^0 F - \int_{a-x}^a F = \int_{-x}^0 F - F(\xi)x
\]
for some \( \xi \) between \( a - x \) and \( a \), due to continuity of \( F \). So, \( \lim_{a \to -\infty} \int_0^a (\tau_x F - F) = \int_{-x}^0 F - x \lim_{y \to -\infty} F(y) \). Since \( F \) has limits at \( \pm \infty \), Hake’s theorem shows \( \tau_x F - F \in \mathcal{H} \mathcal{K} \). Now let \( a, b \in \mathbb{R} \). Then \( \int_b^a (\tau_x F - F) = \int_{a-x}^a F - \int_{b-x}^b F \). Since \( F \) is continuous, there are \( \xi \) between \( a \) and \( a - x \) and \( \eta \) between \( b \) and \( b - x \) such that \( \int_{a-x}^a F(\xi)x - F(\eta)x = x \int_{a-x}^a F \). It follows that \( \| \tau_x F - F \| \leq \| f \|_1 |x| \).
The example $f = \chi_{[0,1]}$, for which
\[ F(y) = \int_{-\infty}^{y} f = \begin{cases} 0, & y \leq 0, \\ y, & 0 \leq y \leq 1, \\ 1, & y \geq 1 \end{cases} \]
shows that no primitives need not be in $\mathcal{HK}$. And, if we let $F(y) = \sin(y)/y$, $f = F'$, then for $x \neq 0$,
\[ \tau_x F(y) - F(y) = \frac{\sin(y - x)}{y - x} - \frac{\sin(y)}{y} \sim \frac{\cos(x) - 1}{y} \sin(y) - \sin(x) \cos(y) \] as $y \to \infty$.

Hence, $\tau_x F - F \in \mathcal{HK} \setminus L^1$.

Suppose $f \in L^1$ and $x \geq 0$. Then, $|f| \in \mathcal{HK}$ so the theorem gives $\|\tau_x F - F\|_1 \leq \int_{-\infty}^{\infty} \int_{y-x}^{y} |f(z)| \, dz \, dy \leq \|f\|_1 \|x\|_1$. Similarly, if $x < 0$.

Example 5. Let $f$ be $2\pi$-periodic and Henstock-Kurzweil integrable over one period. The Poisson integral of $f$ on the unit circle is
\[ u(re^{i\theta}) = u_r(\theta) = \frac{1 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(\varphi)}{1 - 2r \cos(\varphi - \theta) + r^2} \, d\varphi. \]
Differentiating under the integral sign shows that $u$ is harmonic in the disc. And, after interchanging the order of integration, it can be seen that $\|u_r - f\| \to 0$ as $r \to 1^-$.

The Poisson integral defines a harmonic function that takes on the boundary values $f$ in the Alexiewicz norm. For details on this Dirichlet problem see [7].

Now we consider continuity in weighted Alexiewicz norms. First we need the following lemma. Lebesgue measure is denoted $\lambda$.

**Lemma 6.** For each $n \in \mathbb{N}$, suppose $g_n: \mathbb{R} \to \mathbb{R}$ and $g_n \chi_E \to g \chi_E$ in measure for some set $E \subset \mathbb{R}$ of positive measure and function $g$ of bounded variation. If $V g_n \leq M$ for all $n$ then $g_n$ is uniformly bounded on $\mathbb{R}$.

**Proof.** Define $S_n = \{x \in E; |g_n(x) - g(x)| > 1\}$. Then $\lambda(S_n) \to 0$ as $n \to \infty$. There is $N \in \mathbb{N}$ such that whenever $n \geq N$ we have $\lambda(E \setminus S_n) > 0$. Since $g \in BV$, $g$ is bounded. Let $n \geq N$. There is $x_n \in E \setminus S_n$ such that $|g(x_n)| \leq \|g\|_\infty$. Therefore, $|g_n(x_n)| \leq 1 + \|g\|_\infty$. Let $x \in \mathbb{R}$. Then $|g_n(x) - g_n(x_n)| \leq V g_n \leq M$. So, $|g_n(x)| \leq M + 1 + \|g\|_\infty$. Hence, $\{g_n\}$ is uniformly bounded. \qed
Theorem 7. Let \( w: \mathbb{R} \to (0, \infty) \). Define \( g_x: \mathbb{R} \to (0, \infty) \) by \( g_x(y) = w(y + x)/w(y) \) for each \( x \in \mathbb{R} \). Then \( \|(\tau_x f - f)w\| \to 0 \) as \( x \to 0 \) for all \( f: \mathbb{R} \to \mathbb{R} \) such that \( fw \in \mathcal{H}\mathcal{K} \). If and only if \( g_x \) is essentially bounded and of essential bounded variation, uniformly as \( x \to 0 \), and \( g_x \to 1 \) in measure on compact intervals as \( x \to 0 \).

Proof. Let \( G(x) = \int_{-\infty}^{x} fw \). Let \( \alpha, \beta, \in \mathbb{R} \). Then
\[
\begin{align*}
\int_{\alpha}^{\beta} [f(y) - f(y)] w(y) dy \\
= \int_{\alpha-x}^{\beta-x} f(y) w(y) dy - \int_{\alpha}^{\beta} f(y) w(y) dy + \int_{\alpha-x}^{\beta-x} f(y) [w(y + x) - w(y)] dy \\
= [G(\beta) - G(\beta)] - [G(\alpha) - G(\alpha)] + \int_{\alpha-x}^{\beta-x} f(y) w(y) [g_x(y) - 1] dy.
\end{align*}
\]
Since \( G \) is uniformly continuous on \( \mathbb{R} \), we have \( \|(\tau_x f - f)w\| \to 0 \) if and only if the supremum of \( |\int_{\alpha}^{\beta} f(y) w(y) |g_x(y) - 1| dy| \) over \( a, b \in \mathbb{R} \) has limit 0 as \( x \to 0 \), i.e., \( \|fw(g_x - 1)\| \to 0 \). Given \( h \in \mathcal{H}\mathcal{K} \) we can always take \( f = h/w \). Hence, the theorem now follows from Lemma 6 (easily modified for the case of essential boundedness and essential variation) and the necessary and sufficient condition for convergence in norm given in [2, Theorem 6].

Corollary 8. Suppose that for each compact interval \( I \) there are real numbers \( 0 < m_I < M_I \) such that \( m_I < \|w\|_{\infty} < M_I \); \( w \) is continuous in measure on \( I \); \( w \in \mathcal{B}V_{loc} \). Then for all \( f: \mathbb{R} \to \mathbb{R} \) such that \( fw \in \mathcal{H}\mathcal{K} \) we have \( \|(\tau_x f - f)w\| \to 0 \) as \( x \to 0 \).

Proof. Fix \( \varepsilon > 0 \). Let \( I \) be a compact interval for which \( 0 < m_I < \|w\|_{\infty} < M_I \).

Define
\[
S_x := \{ y \in I ; |g_x(y) - 1| \geq \varepsilon \} = \{ y \in I ; |w(y + x) - w(y)| \geq \varepsilon w(y) \}
\]
except for a null set.

Since \( w \) is continuous in measure on \( I \) we have \( \lambda(S_x) \to 0 \) as \( x \to 0 \) and \( g_x \to 1 \) in measure on \( I \).

Using
\[
g_x(s_n) - g_x(t_n) = \frac{w(s_n + x) - w(t_n + x)}{w(t_n)} \to \frac{w(s_n + x) - w(t_n) + w(s_n) - w(t_n)}{w(s_n)w(t_n)}
\]
we see that \( V_I g_x \leq V_{I+x} w/m_I + M_I V_I w/m_I^2 \) where \( I + x = \{ y + x ; y \in I \} \) and \( V_I w \) is the variation of \( w \) over interval \( I \). Hence, \( g_x \) is of uniform bounded variation on \( I \).

With Lemma 6 this then gives the hypotheses of the theorem. \( \square \)
Of course, we are allowing \( w \) to be changed on a set of measure 0 so that \( w \) is of bounded variation rather than just equivalent to a function of bounded variation. This redundancy can be removed by replacing \( w \) with its limit from the right at each point so that \( w \) is right continuous.

As pointed out in [2], convergence of \( g_x \) to 1 in measure on compact intervals in the theorem can be replaced by convergence in \( L^1 \) norm: For each compact interval \( I, \| (g_x - 1) \chi_I \|_1 \to 0 \) as \( x \to 0 \). In the corollary, we can replace continuity in measure with the condition: As \( x \to 0, \int_I |\tau_x w - w| \to 0 \) for each compact interval \( I \).

The first two conditions in the corollary are necessary. Suppose \( \| (\tau_x f - f) w \| \to 0 \) whenever \( f w \in HK \). Then the essential infimum of \( w \) must be positive on each compact interval. If there is a sequence \( a_n \to a \in \mathbb{R} \) for which \( w(a_n) \to 0 \) then \( \text{esssup}_{|x| < \delta} g_x(a_n) = \infty \) for each \( \delta > 0 \) unless \( w = 0 \) a.e. in a neighbourhood of \( a \). Similarly, the essential supremum of \( w \) must be finite on compact intervals. This asserts the existence of \( m_I \) and \( M_I \) in the corollary. Also, let \( I \) be a compact interval on which \( 0 < m < \| w \|_{\infty} < M \). Let \( \varepsilon > 0 \) and define

\[
T_\varepsilon := \{ y \in I; \; |w(y + x) - w(y)| > \varepsilon \}
\]

\[
\subset \{ y \in I; |g_x(y) - 1| > \varepsilon / M \}
\text{except for a null set.}
\]

Hence, \( w \) is continuous in measure.

It is not known if \( \| (\tau_x f - f) w \| \to 0 \) for all \( f \) such that \( f w \in HK \) implies \( w \in BV_{loc} \). The example \( w(y) = e^y \) shows \( W \) need not be of bounded variation and can have its infimum zero and its supremum infinity. For, note that \( g_x(y) = \exp(y + x) / \exp(y) = e^x \) and so satisfies the conditions of the theorem. And, by the corollary, \( w(y) = 1 \) for \( y < 0 \) and \( w(y) = 2 \) for \( y \geq 0 \) is a valid weight function. Hence, \( w \) need not be continuous.

**Example 9.** Let \( w(y) = 1/(y^2 + 1) \). A calculation shows that the variation of \( \int_{-\infty}^{\infty} w(y + x)/w(y) \) is \( 2|x|/\sqrt{x^2 + 1} \) so \( w \) is a valid weight for Theorem 7. The half-plane Poisson kernel is \( \Phi_y(x) = w(x/y)/(\pi y) \). For \( f: \mathbb{R} \to \mathbb{R} \) the Poisson integral of \( f \) is

\[
u_y(x) = (\Phi_y * f)(x) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t) dt}{(x-t)^2 + y^2}.
\]

Define \( \Psi_z(t) = \Phi_y(x-t)/w(t) \) for \( z = x + iy \) in the upper half-plane, i.e., \( x \in \mathbb{R} \) and \( y > 0 \). For fixed \( z \) both \( \Psi_z \) and \( 1/\Psi_z \) are of bounded variation on \( \mathbb{R} \). Hence, necessary and sufficient for the existence of the Poisson integral on the upper half-plane \( f w \in HK \).

Define \( G(t) = \int_{-\infty}^{t} f(w) \). Integrate by parts to get \( u_y(x) = y G(x)/\pi - \int_{-\infty}^{\infty} G(t) \Psi_z(t) dt \). Since \( G \) is continuous on the extended real line (with \( G(\infty) := \lim_{t \to \infty} G(t) \)),

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dominated convergence now allows differentiation under the integral. This shows $u_y(x)$ is harmonic in the upper half-plane.

Neither $f$ nor $u_y$ need be in $\mathcal{HK}$. For example, the Poisson integral of 1 is 1. But, we have the boundary values taken on in the weighted norm: $\|(u_y - f)w\| \to 0$ as $y \to 0^+$. We sketch out the proof, leaving the technical detail of interchanging repeated integrals for publication elsewhere. For $a, b \in \mathbb{R}$ we then have

$$
\int_a^b [u_y(t) - f(t)] w(t) \, dt = \int_a^b \left\{ (f * \Phi_y)(t) - f(t) \int_{-\infty}^{\infty} \Phi_y(s) \, ds \right\} w(t) \, dt
$$

$$
= \int_{-\infty}^{\infty} \Phi_y(s) \int_a^b [f(t) - f(t)] w(t) \, dt \, ds.
$$

Therefore, $\|(u_y - f)w\| \leq \int_{-\infty}^{\infty} \Phi_y(s) \|(\tau_s f - f)w\| \, ds$. But, $s \mapsto \|(\tau_s f - f)w\|$ is continuous at $s = 0$. By the usual properties of the Poisson kernel (an approximate identity), we have $\|(\tau_s f - f)w\| \to 0$ as $y \to 0^+$.

**References**


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