THE HENSTOCK-KURZWEIL APPROACH TO YOUNG INTEGRALS WITH INTEGRATORS IN BV_φ

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. In 1938, L.C. Young proved that the Moore-Pollard-Stieltjes integral \( \int_a^b f \, dg \) exists if \( f \in BV_{\varphi}[a,b] \), \( g \in BV_{\psi}[a,b] \) and \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \). In this note we use the Henstock-Kurzweil approach to handle the above integral defined by Young.

Keywords: Henstock integral, Stieltjes integral, Young integral, \( \varphi \)-variation

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1. Introduction

In 1936, L.C. Young proved that the Riemann-Stieltjes integral \( \int_a^b f \, dg \) exists, if \( f \in BV_p[a,b] \), \( g \in BV_q[a,b] \), \( 1/p + 1/q > 1 \) and \( f, g \) do not have common discontinuous points, see [7], [11]. Two years later, he was able to drop the condition on common discontinuity for his new integral (called Young integral), see [12]. The Young integral is defined by the Moore-Pollard approach, see [2, pp. 23–27, pp. 113–138] and [3], [8], [9]. In other words, the integral is defined by way of refinements of partitions and the integral is the Moore-Smith limit of the Riemann-Stieltjes sums using the directed set of partitions. However, modified Riemann-Stieltjes sums involving \( g(x+) \) and \( g(x-) \) are used in Young integrals. Furthermore, he generalized his result and proved that the Young integral \( \int_a^b f \, dg \) exists if the following Young’s condition holds:

\[ f \in BV_{\varphi}[a,b], \quad g \in BV_{\psi}[a,b] \]
and

$$\sum_{n=1}^{\infty} \varphi^{-1}\left(\frac{1}{n}\right) \psi^{-1}\left(\frac{1}{n}\right) < \infty,$$

where $\text{BV}_\varphi [a, b]$ is the space of functions of bounded $\varphi$-variation on $[a, b]$.

The Young integral with an integrator in $\text{BV}_p$ using the Henstock-Kurzweil approach is given in [1]. In this note we will again use the Henstock-Kurzweil approach to handle the Young integral with an integrator in $\text{BV}_\varphi$.

Now we shall introduce Henstock-Kurzweil integrals, see [4].

Let $P = \{[u_i, v_i]\}_{i=1}^n$ be a finite collection of non-overlapping subintervals of $[a, b]$, then $P$ is said to be a partial partition of $[a, b]$. If, in addition, $\bigcup_{i=1}^n [u_i, v_i] = [a, b]$, then $P$ is said to be a partition of $[a, b]$.

Let $\delta$ be a positive function on $[a, b]$, $[u, v] \subseteq [a, b]$ and $\xi \in [a, b]$. Then an interval-point pair $(\xi, [u, v])$ is said to be $\delta$-fine if $\xi \in [u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$. Let $D = \{((\xi_i, [u_i, v_i]))\}_{i=1}^n$ be a finite collection of interval-point pairs. Then $D$ is said to be a $\delta$-fine partial division of $[a, b]$ if $\{[u_i, v_i]\}_{i=1}^n$ is a partial partition of $[a, b]$ and for each $i$, $(\xi_i, [u_i, v_i])$ is $\delta$-fine. In addition, if $\{[u_i, v_i]\}_{i=1}^n$ is a partition of $[a, b]$, then $D$ is said to be a $\delta$-fine division of $[a, b]$.

In this note, $\mathbb{R}$ denotes the set of real numbers.

Now, we shall define integrals of Stieltjes type by the Henstock-Kurzweil approach.

**Definition 1.1.** Let $f, g: [a, b] \to \mathbb{R}$. Then $f$ is said to be Henstock-Kurzweil integrable (or HK-integrable) to a real number $A$ on $[a, b]$ with respect to $g$ if for every $\varepsilon > 0$ there exists a positive function $\delta$ defined on $[a, b]$ such that for every $\delta$-fine division $D = \{((\xi_i, [t_i, t_{i+1}]))\}_{i=1}^n$ of $[a, b]$, we have

$$|S(f, \delta, D) - A| \leq \varepsilon,$$

where

$$S(f, \delta, D) = \sum_{i=1}^n f(\xi_i)(g(t_{i+1}) - g(t_i)).$$

$A$ is denoted by $\int_a^b f \, dg$.

It is known that if $f \in \text{BV}_p[a, b], g \in \text{BV}_q[a, b], 1/p + 1/q > 1$, then $f$ is HK-integrable with respect to $g$ on $[a, b]$, see [1].

In this note we follow ideas of Young to show that if $f \in \text{BV}_\varphi[a, b], g \in \text{BV}_\psi[a, b]$ and $\sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty$, then $f$ is HK-integrable with respect to $g$ on $[a, b]$. 

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2. Young’s series

The above series is called Young’s series. We shall present some properties of Young’s series. Results and proofs are known, see [5], [6], [12]. We give proofs here for easy reference.

In this section, let \( \lambda, \mu \) be strictly increasing continuous non-negative functions on \([0, \infty)\) with \( \lambda(0) = \mu(0) = 0 \) and let \( \omega, \kappa \) be increasing functions on \([\alpha, \beta]\) with

\[
\omega(\beta) - \omega(\alpha) \leq A \quad \text{and} \quad \kappa(\beta) - \kappa(\alpha) \leq B.
\]

**Lemma 2.1.** For \( p = 0, 1, 2, \ldots \), there exists \( E_p = \{x_1, x_2, \ldots, x_{n_p}\} \subset [\alpha, \beta] \) such that for any \( \xi, \eta \in (x_i, x_{i+1}) \), \( i = 1, 2, \ldots, n_p - 1 \), we have

\[
|\omega(\eta) - \omega(\xi)| \leq A2^{-p}
\]

and

\[
|\kappa(\eta) - \kappa(\xi)| \leq B2^{-p}.
\]

Furthermore, \( E_q \supseteq E_p \) if \( p \leq q \), \( \#(E_p) \leq 2^{p+1} \) and \( \#(E_{p+1} \setminus E_p) \leq 2^{p+1} \), where \( \#(E_p) \) denotes the number of elements in the set \( E_p \).

**Proof.** Denote \( |\omega(\xi) - \omega(\eta)|, |\kappa(\xi) - \kappa(\eta)| \) by \( \omega(\xi, \eta), \kappa(\xi, \eta) \) respectively.

Let \( E_0^\omega = \{x_1^{(0)}, x_2^{(0)}\} \), where \( x_1^{(0)} = \alpha, x_2^{(0)} = \beta \). Then for any \( \xi, \eta \in (x_1^{(0)}, x_2^{(0)}) \), we can see that

\[
\omega(\xi, \eta) \leq A.
\]

Let \( x_{1'}^{(0)} = \sup\{x \in [x_1^{(0)}, x_2^{(0)}] : \omega(\xi, \eta) \leq A2^{-1} \text{ for any } \xi, \eta \in (x_1^{(0)}, x)\} \) and let

\[
E_1^\omega = \{x_1^{(0)}, x_1', x_2^{(0)}\}.
\]

It is possible that \( x_1' = x_2^{(0)} \), i.e., \( E_1^\omega = E_0^\omega \). We may assume that the above supremum is well-defined, otherwise we use \( (x, x_2^{(0)}) \) instead of \( (x_1^{(0)}, x) \). We will rename points in \( E_1^\omega \) according to their order using the notation

\[
E_1^\omega = \{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\}.
\]

We claim that for any \( \xi, \eta \in (x_2^{(1)}, x_3^{(1)}) \), \( \omega(\xi, \eta) \leq A2^{-1} \).

Suppose that there exist \( \xi, \eta \in (x_2^{(1)}, x_3^{(1)}) \) such that \( \omega(\xi, \eta) > A2^{-1} \). Since \( \xi > x_2^{(1)} \), there exists a point \( \beta \in (x_2^{(1)}, x_3^{(1)}) \) such that \( \omega(\beta, \xi) > A2^{-1} \). Since \( \xi > x_2^{(1)} \),

\[
\omega(\beta, \eta) = \omega(\beta, \xi) + \omega(\xi, \eta) > A2^{-1} + A2^{-1} = A.
\]
It contradicts the definition of $E_0^\omega$. Hence, for any $\xi, \eta \in (x_2^{(1)}, x_3^{(1)})$, $$\omega(\xi, \eta) \leq A2^{-1}.$$ That is, for any $\xi, \eta \in (x_i^{(1)}, x_{i+1}^{(1)})$, $i = 1, 2$, we have $$\omega(\xi, \eta) \leq A2^{-1}.$$

Let $x_1^{(1)} = \sup\{x \in [x_1^{(1)}, x_2^{(1)}] : \omega(\xi, \eta) \leq A2^{-2} \text{ for every } \xi, \eta \in (x_1^{(1)}, x)\}$, $x_2^{(1)} = \sup\{x \in [x_2^{(1)}, x_3^{(1)}] : \omega(\xi, \eta) \leq A2^{-2} \text{ for every } \xi, \eta \in (x_2^{(1)}, x)\}$ and $$E_2' = \{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\}.$$ It is still possible that $x_1^{(1)} = x_2^{(1)}$ or $x_2^{(1)} = x_3^{(1)}$. We again rename $E_2'$ according to their order by $$E_2' = \{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\}.$$ Using the same argument as above, we also have for any $\xi, \eta \in (x_i^{(2)}, x_{i+1}^{(2)})$ and for every $i = 1, 2, 3, 4$, $$\omega(\xi, \eta) \leq A2^{-2}.$$ Using this method, we can have $E_p^\omega = \{x_1^{(p)}, x_2^{(p)}, \ldots, x_n^{(p)}\}$, $p = 0, 1, 2, \ldots$, such that for any $\xi, \eta \in (x_i^{(p)}, x_{i+1}^{(p)})$, $i = 1, 2, \ldots, n_p - 1$, we have $$\omega(\xi, \eta) \leq A2^{-p}.$$ We can also see that $E_q^\omega \subseteq E_p^\omega$ whenever $q \leq p$, the number of elements in $E_p^\omega$ is at most $2^p + 1$ and the number of elements in $E_{p+1}^\omega \setminus E_p^\omega$ is at most $2^p$.

Using the same argument, we also can define $E_p^\kappa = \{y_1^{(p)}, y_2^{(p)}, \ldots, y_{m_p}^{(p)}\}$ for $p = 0, 1, 2, \ldots$, so that for any $\xi, \eta \in (y_i^{(p)}, y_{i+1}^{(p)})$, $i = 1, 2, \ldots, m_p - 1$, we have $$\kappa(\xi, \eta) \leq B2^{-p}.$$ Furthermore, $E_q^\kappa \subseteq E_p^\kappa$ whenever $q \leq p$, the number of elements in $E_p^\kappa$ is at most $2^p + 1$ and the number of element in $E_{p+1}^\kappa \setminus E_p^\kappa$ is at most $2^p$.

Now, let $E_p = E_p^\omega \cup E_p^\kappa = \{z_1^{(p)}, z_2^{(p)}, \ldots, z_{r_p}^{(p)}\}$. Then for every $\xi, \eta \in (z_i^{(p)}, z_{i+1}^{(p)})$, $i = 1, 2, \ldots, r_p - 1$, $$\omega(\xi, \eta) \leq A2^{-p} \text{ and } \kappa(\xi, \eta) \leq B2^{-p}.$$ Furthermore, $E_q \subseteq E_p$ whenever $q \leq p$, the number of elements in $E_{p+1} \setminus E_p$ is at most $2 \cdot 2^p = 2^{p+1}$ and number of elements in $E_p$ is at most $2(2^p + 1) - 2 = 2^{p+1}$, since $\alpha, \beta \in E_p^\omega \cap E_p^\kappa$.

\[\Box\]
Lemma 2.2. (i) For any positive integer \( v \), the following inequalities hold:

\[
\sum_{n=0}^{\infty} 2^{n+v} \lambda(A^{2^{-(n+v)}}) \mu(B^{2^{-(n+v)}}) \leq 2 \sum_{n=2^{v-1}}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right)
\]

and

\[
\sum_{n=1}^{\infty} \lambda\left(\frac{A^{2^{-v}}}{n}\right) \mu\left(\frac{B^{2^{-v}}}{n}\right) \leq \frac{1}{2^{v-1}} \sum_{n=2^{v-1}}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right),
\]

(ii) \[
\sum_{n=0}^{\infty} 2^n \lambda(A^{2^{-n}}) \mu(B^{2^{-n}}) \leq 3 \sum_{n=1}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right).
\]

Proof. (i)

\[
\sum_{n=0}^{\infty} 2^{n+v} \lambda(A^{2^{-(n+v)}}) \mu(B^{2^{-(n+v)}}) = \sum_{k=v}^{\infty} 2^k \lambda(A^{2^{-k}}) \mu(B^{2^{-k}})
\]

\[
\leq 2 \sum_{k=v}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) = 2 \sum_{n=2^{v-1}+1}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right)
\]

\[
\leq 2 \sum_{n=2^{v-1}}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right).
\]

Similarly, we have

\[
\sum_{n=1}^{\infty} \lambda\left(\frac{A^{2^{-v}}}{n}\right) \mu\left(\frac{B^{2^{-v}}}{n}\right) = \sum_{k=2^v}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \lambda\left(\frac{A^{2^{-v}}}{k}\right) \mu\left(\frac{B^{2^{-v}}}{k}\right)
\]

\[
\leq \sum_{n=0}^{\infty} 2^n \lambda(A^{2^{-(v+n)}}) \mu(B^{2^{-(v+n)}}) = \frac{1}{2^v} \sum_{n=0}^{\infty} 2^{n+v} \lambda(A^{2^{-(n+v)}}) \mu(B^{2^{-(n+v)}})
\]

\[
\leq \frac{1}{2^{v-1}} \sum_{n=2^{v-1}}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right).
\]

(ii) As in the first part of (i),

\[
\sum_{n=0}^{\infty} 2^n \lambda(A^{2^{-n}}) \mu(B^{2^{-n}}) = \lambda(A) \mu(B) + \sum_{k=1}^{\infty} 2^k \lambda(A^{2^{-k}}) \mu(B^{2^{-k}})
\]

\[
\leq \lambda(A) \mu(B) + 2 \sum_{n=1}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) \leq 3 \sum_{n=1}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right).
\]

\[\square\]
Lemma 2.3.
\[ \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) \mu\left(\frac{1}{n}\right) < \infty \text{ if and only if } \sum_{n=1}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) < \infty. \]

Proof. Suppose \( \sum_{n=1}^{\infty} \lambda(1/n)\mu(1/n) < \infty \). Let \( m \) be a positive integer such that \( A \leq m \) and \( B \leq m \). Then
\[
\sum_{n=1}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) = \sum_{n=1}^{m-1} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) + \sum_{n=m}^{\infty} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) \\
= \sum_{n=1}^{m-1} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) + \sum_{k=1}^{\infty} \sum_{n=km}^{(k+1)m} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) \\
\leq \sum_{n=1}^{m-1} \lambda\left(\frac{A}{n}\right) \mu\left(\frac{B}{n}\right) + m \sum_{k=1}^{\infty} \lambda\left(\frac{1}{k}\right) \mu\left(\frac{1}{k}\right) < \infty.
\]

Conversely, suppose \( \sum_{n=1}^{\infty} \lambda(A/n)\mu(B/n) < \infty \). Let \( \lambda'(x) = \lambda(Ax), \mu'(x) = \mu(Bx) \). Then \( \sum_{n=1}^{\infty} \lambda'(1/n)\mu'(1/n) < \infty \). Therefore, \( \sum_{n=1}^{\infty} \lambda'(1/(An))\mu'(1/(Bn)) < \infty \). Consequently, \( \sum_{n=1}^{\infty} \lambda(1/n)\mu(1/n) < \infty. \)

\[ \square \]

3. Integrals of step functions

In this section we shall present Young’s results on integrals of step functions, see [12]. Let \( g \) be a regulated function on \([\alpha, \beta]\) and \( s \) a step function on \([\alpha, \beta]\) with
\[
s(x) = \sum_{i=1}^{q} c_i \chi_{(t_i, t_{i+1})}(x) + \sum_{i=1}^{q+1} d_i \chi_{(t_i)}(x),
\]
where \( \chi_{G} \) is the characteristic function of \( G \), and \( \alpha = t_1 < t_2 < \ldots < t_{q+1} = \beta \).

It is known, see [1], that
\[
\int_{\alpha}^{\beta} s \, dg = \sum_{i=1}^{q} c_i (g(t_{i+1}) - g(t_i)) + \sum_{i=1}^{q+1} d_i (g(t_i) - g(t_i)).
\]

Furthermore, we always assume that the following conditions hold:

\[
\begin{align*}
|s(\xi) - s(\eta)| &\leq \lambda(\omega(\xi) - \omega(\eta)), \\
|g(\xi) - g(\eta)| &\leq \mu(\kappa(\xi) - \kappa(\eta))
\end{align*}
\]

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for any $\xi, \eta \in [\alpha, \beta]$ with $\xi > \eta$, where $\lambda, \mu, \omega, \kappa$ are given in Section 1. In this section we always assume that $\sum_{n=1}^{\infty} \lambda(A/n)\mu(B/n) < \infty$. Recall that $\omega(\beta) - \omega(\alpha) \leq A$ and $\kappa(\beta) - \kappa(\alpha) \leq B$.

**Definition 3.1.** Let $s$ be a step function defined on $[\alpha, \beta]$ and $E_p$ the finite set as defined in Lemma 2.1. Let $E = \{x_i : i = 1, 2, \ldots, m + 1\}$ be any fixed finite set containing $E_0$. We define $s_E$ to be the step function induced by $s$ and $E$ as follows:

$$s_E(x) = \sum_{i=1}^{m} s(x_i)\chi_{(x_i, x_{i+1})}(x) + \sum_{i=1}^{m+1} s(x_i)\chi_{[x_i]}(x).$$

We have, by the formula for the value of the integral of a step function with respect to $g$ presented above,

$$\int_{\alpha}^{\beta} s_E \, dg = \sum_{i=1}^{m} s(x_i)(g(x_{i+1}) - g(x_i)) + \sum_{i=1}^{m+1} s(x_i)(g(x_i) - g(x_{i-1})).$$

We remark that if $E$ contains all points of discontinuity of $s$, then $s_E = s$ and $\int_{\alpha}^{\beta} s_E \, dg = \int_{\alpha}^{\beta} s \, dg$.

**Lemma 3.2.** Let $E \supseteq E_0$. Then

$$\left| \int_{\alpha}^{\beta} (s_{E \cup E_p} - s_{E_p}) \, dg \right| \leq N_p\lambda\left(\frac{A}{2^p}\right)\mu\left(\frac{B}{2^p}\right),$$

where $N_p = \#(E \setminus E_p)$. Furthermore,

$$\lim_{p \to \infty} \left| \int_{\alpha}^{\beta} (s_{E \cup E_p} - s_{E_p}) \, dg \right| = 0.$$

**Proof.** Let $N_p$ denote $\#(E \setminus E_p)$. Let $s'$ denote the step function $s_{E \cup E_p} - s_{E_p}$. Suppose $s'$ is induced by a partition $\{[y_i, y_{i+1}]\}_{i=1}^{m}$ of $[\alpha, \beta]$. If $y_i \in E_p$, then $s'$ has zero values over a half-open subinterval $[y_i, y_{i+1}).$ Therefore, the number of subintervals where $s'$ has nonzero value is at most $N_p$. Then

$$\left| \int_{\alpha}^{\beta} (s_{E \cup E_p} - s_{E_p}) \, dg \right| \leq N_p\lambda(A2^{-p})\mu(B2^{-p}) \leq N\lambda(A2^{-p})\mu(B2^{-p}).$$

Hence, for any fixed finite set $E$,

$$\lim_{p \to \infty} \left| \int_{\alpha}^{\beta} (s_{E \cup E_p} - s_{E_p}) \, dg \right| \leq \lim_{p \to \infty} N\lambda(A2^{-p})\mu(B2^{-p}) = 0.$$

In the above, we use the fact that $\lambda, \mu$ are continuous at 0 and $\lambda(0) = \mu(0) = 0$. \(\square\)
Theorem 3.3. Let $s$ be a step function and $E_0$ as above. Then

$$\left| \int_{\alpha}^\beta (s - s_{E_0}) \, dg \right| \leq 6 \sum_{n=1}^\infty \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right).$$

Proof. From Lemma 3.2, $E_{p+1} = E_{p+1} \cup E_p$ and $\#(E_{p+1} - E_p) \leq 2^{p+1}$, we have

$$\left| \int_{\alpha}^\beta (s_{E_{p+1}} - s_{E_p}) \, dg \right| \leq 2^{p+1} \lambda (A2^{-p}) \mu (A2^{-p}) = 2^{p+1} \lambda (A2^{-p}) \mu (A2^{-p}).$$

Now, let $E^*$ be a finite set containing $E_0$ and all points of discontinuity of $s$, then $\int_{\alpha}^\beta s \, dg = \int_{\alpha}^\beta s_{E^*} \, dg = \int_{\alpha}^\beta s_{E^*} \cup E_0 \, dg$ for all $v = 0, 1, 2, \ldots$. Hence we have

$$\left| \int_{\alpha}^\beta (s - s_{E_0}) \, dg \right| \leq \lim_{q \to \infty} \left( \left| \int_{\alpha}^\beta (s_{E^*} - s_{E_{p+q}}) \, dg \right| + \left| \int_{\alpha}^\beta (s_{E_{p+q}} - s_{E_{p+q-1}}) \, dg \right| \\
+ \ldots + \left| \int_{\alpha}^\beta (s_{E_{p+1}} - s_{E_p}) \, dg \right| \right)$$

$$= \lim_{q \to \infty} \left( \left| \int_{\alpha}^\beta (s_{E^*} \cup E_{p+q} - s_{E_{p+q}}) \, dg \right| + \left| \int_{\alpha}^\beta (s_{E_{p+q}} - s_{E_{p+q-1}}) \, dg \right| \\
+ \ldots + \left| \int_{\alpha}^\beta (s_{E_{p+1}} - s_{E_p}) \, dg \right| \right) \leq 0 + \lim_{q \to \infty} \sum_{m=0}^{q-1} 2 \cdot 2^{p+m} \lambda (A2^{-(p+m)}) \mu (A2^{-(p+m)})$$

$$\leq 4 \sum_{n=2^{p-1}}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right) \text{ for } p = 1, 2, \ldots.$$ 

The last inequality holds by Lemma 2.2 (i).

When $p = 0$, by Lemma 2.2 (ii) we get

$$\left| \int_{\alpha}^\beta (s - s_{E_0}) \, dg \right| \leq 6 \sum_{n=1}^\infty \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right).$$

$\square$
Corollary 3.4. Suppose that \( s_1(x) = \sum_{i=1}^{n} d_i \chi_{[u_i, u_{i+1})}(x) + d_n \chi_{[u_{n+1}, \infty)}(x) \), \( s_2(x) = \sum_{i=1}^{m} e_i \chi_{[v_i, v_{i+1})}(x) + e_m \chi_{[v_{m+1}, \infty)}(x) \) are step functions defined on \([\alpha, \beta]\). Let (1) hold with \( s = s_1, s_2 \) and \( |d_1 - e_1| \leq \lambda(A) \). Then

\[
\left| \sum_{i=1}^{n} d_i (g(u_{i+1}) - g(u_i)) - \sum_{i=1}^{m} e_i (g(v_{i+1}) - g(v_i)) \right| \leq 13 \sum_{n=1}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right).
\]

**Proof.** First we shall prove that the following inequality holds:

(2) \[
\left| \sum_{i=1}^{n} d_i (g(u_{i+1}) - g(u_i)) - d_1 (g(\beta) - g(\alpha)) \right| \leq 6 \sum_{n=1}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right).
\]

Let \( g^*(u_i) = g(u_i) \), \( g^*(t) = g(u_i) \) for those \( t \) close to \( u_i \) from the left, otherwise, let \( g^*(t) = g(t) \). Then \( g^*(u_i) = g^*(u_i) \) and \( g^* \) also satisfies \( |g^*(\xi) - g^*(\eta)| \leq \mu(|\kappa(\xi) - \kappa(\eta)|) \) for any \( \xi, \eta \in [\alpha, \beta] \). Then

\[
\int_{\alpha}^{\beta} s_1 \, dg^* = \sum_{i=1}^{n} d_i (g^*(u_{i+1}) - g^*(u_i)) + d_n (g^*(u_{n+1}) - g^*(u_{n+1} -))
\]

\[
= \sum_{i=1}^{n} d_i (g^*(u_{i+1}) - g^*(u_i)) = \sum_{i=1}^{n} d_i (g(u_{i+1}) - g(u_i))
\]

Applying Theorem 3.3 to \( s = s_1 \) and \( g = g^* \), \( \int_{\alpha}^{\beta} s \, dg^* = d_1 (g^*(\beta) - g^*(\alpha)) + d_n (g^*(\beta) - g^*(\beta +)) = d_1 (g(\beta) - g(\alpha)) \), we get the inequality (2).

Thus

\[
\left| \sum_{i=1}^{n} d_i (g(u_{i+1}) - g(u_i)) - \sum_{i=1}^{m} e_i (g(v_{i+1}) - g(v_i)) \right|
\]

\[
\leq 12 \sum_{n=1}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right) + |d_1 (g(\beta) - g(\alpha)) - e_1 (g(\beta) - g(\alpha))|
\]

\[
\leq 12 \sum_{n=1}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right) + |d_1 - e_1||g(\beta) - g(\alpha)|
\]

\[
\leq 12 \sum_{n=1}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right) + \lambda(A) \mu(B) = 13 \sum_{n=1}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right).
\]
4. Integrable functions

Now we shall introduce $\text{BV}'_{\varphi}[a, b]$, which is a generalization of $\text{BV}[a, b]$, the space of functions of bounded variation on $[a, b]$, and prove an existence theorem (Theorem 4.6) in the Henstock-Kurzweil setting.

**Definition 4.1.** A function $\varphi: [0, \infty) \to \mathbb{R}$ is said to be an $N$-function if
1. $\varphi(0) = 0$;
2. $\varphi$ is continuous on $[0, \infty)$;
3. $\varphi$ is strictly increasing and
4. $\varphi(u) \to \infty$ as $u \to \infty$.

Examples of $N$-functions are $\varphi_1(u) = u^p$, $p \geq 1$ and $\varphi_2(u) = e^u - 1$.

**Definition 4.2.** Let $\varphi: [0, \infty) \to \mathbb{R}$ be an $N$-function and $f: [a, b] \to \mathbb{R}$. We define
$$V_\varphi(f; [a, b]) = \sup_n \sum_{i=1}^n \varphi(|f(x_{i+1}) - f(x_i)|),$$
where supremum is taken over all partitions $\{[x_i, x_{i+1}]\}_{i=1}^n$ of $[a, b]$. The number $V_\varphi(f; [a, b])$ is called the $\varphi$-variation of $f$ on $[a, b]$. Let $\text{BV}'_{\varphi}[a, b]$ denote the collection of all functions $f: [a, b] \to \mathbb{R}$ satisfying $V_\varphi(f; [a, b]) < \infty$, see [5], [6], [12]. Such functions are said to be of bounded $\varphi$-variation. When it is clear that we are considering the interval $[a, b]$, we shall denote $V_\varphi(f; [a, b])$ by $V_\varphi(f)$.

For example, where $\varphi(u) = u^p$, $p \geq 1$, $\text{BV}'_{\varphi}[a, b]$ is the space of functions of bounded $p$-variation on $[a, b]$.

The following lemma and its proof are known.

**Lemma 4.3.** If $f \in \text{BV}'_{\varphi}[a, b]$, then $f$ is bounded on $[a, b]$ and $f$ is a regulated function.

**Proof.** Suppose $f$ is unbounded. Let $M$ be any positive real number. Then there exists $x \in [a, b]$ such that $M \leq |f(x) - f(a)|$. Hence
$$M \leq |f(x) - f(a)| = \varphi^{-1}(\varphi(|f(x) - f(a)|))$$
$$\leq \varphi^{-1}(\varphi(|f(x) - f(a)|) + \varphi(|f(b) - f(x)|)) \leq \varphi^{-1}(V_\varphi(f; [a, b])).$$

Therefore
$$\varphi(M) \leq V_\varphi(f; [a, b])$$
for all $M > 0$.

Since $\varphi(M) \to \infty$ as $M \to \infty$, we have $V_\varphi(f; [a, b]) = \infty$. This leads to a contradiction.

The proof that $f$ is regulated is standard. \hfill $\Box$
Lemma 4.4. Let $g \in BV_\psi[a, b]$, $E = \{x_1, x_2, \ldots, x_n\} \supseteq E_0$, and $\varepsilon > 0$, where $E_0$ is given in Lemma 2.1. Then there exists a constant $\delta > 0$ such that for any finite collection of disjoint subintervals $\{(u_i, v_i)\}_{i=1}^n$ with $[u_i, v_i] \subset (x_i, x_i + \delta)$ or $[u_i, v_i] \subset (x_i - \delta, x_i)$ for each $i$, we have

$$\sum_{i=1}^n |g(v_i) - g(u_i)| \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. First, since $g$ is a regulated function, there exists a constant $\delta > 0$ such that

$$|g(t) - g(x_i)| \leq \frac{\varepsilon}{2n} \quad \text{whenever} \quad 0 < x_i - t < \delta$$

and

$$|g(x_i) - g(t)| \leq \frac{\varepsilon}{2n} \quad \text{whenever} \quad 0 < t - x_i < \delta$$

for each $i$. Therefore, we get the required result.

Next, we shall prove Lemma 4.5 using Lemma 2.2 and Corollary 3.4. We need the following notation.

Let $A \geq V_\varphi(f)$ and $B \geq V_\varphi(g)$. Define $\omega(x) = V_\varphi(f; [a, x])$ and $\kappa(x) = V_\varphi(g; [a, x])$. Let $\lambda = \varphi^{-1}$, $\mu = \psi^{-1}$. Hence, for any $\xi, \eta \in [a, b]$ with $\eta > \xi$,

$$\lambda(\omega(\eta) - \omega(\xi)) = \varphi^{-1}(\omega(\eta) - \omega(\xi)) \geq |f(\eta) - f(\xi)|.$$

Similarly, $\mu(\kappa(\eta) - \kappa(\xi)) \geq |g(\eta) - g(\xi)|$.

Let $E_0 = \{x_1, x_2, \ldots, x_n\}$ be given as in Lemma 2.1 with $v \geq 1$ and $[\alpha, \beta] = [a, b]$. Then $\#(E_0) \leq 2^{v+1}$. Furthermore,

$$|f(\eta) - f(\xi)| \leq \lambda(\omega(\eta) - \omega(\xi)) \leq \lambda(A2^{-v}) = \varphi^{-1}(A2^{-v})$$

and

$$|g(\eta) - g(\xi)| \leq \mu(\kappa(\eta) - \kappa(\xi)) \leq \mu(B2^{-v}) = \psi^{-1}(B2^{-v})$$

for any $\eta, \xi \in (x_k, x_{k+1})$ with $\eta > \xi$, $k = 1, 2, \ldots, n_v - 1$. The above is equivalent to (1) mentioned before Definition 3.1.

From now onwards, a division $D = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ is always denoted by $D = \{(\xi, [u, v])\}$.
Lemma 4.5. Let \( f \in BV_\varphi[a, b] \) and \( g \in BV_\psi[a, b] \) with \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \). Let \( v \geq 1 \) be fixed and \( E_v = \{x_1, x_2, \ldots, x_n\} \) given as above. Suppose \( D = \{(\xi, [u, v])\} \) and \( D' = \{([\xi', [u', v'])\} \) are two partial divisions of \([a, b]\) such that \( \bigcup[u, v] = \bigcup[u', v'] \) and \([u, v] \subset (x_k, x_{k+1}) \), \([u', v'] \subset (x_k, x_{k+1}) \). Then for any \( \xi \in [u, v], \xi' \in [u', v'] \), we have
\[
\left| (D) \sum f(\xi)(g(v) - g(u)) - (D') \sum f(\xi')(g(v') - g(u')) \right| \\
\leq 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1} \left( \frac{A}{n} \right) \psi^{-1} \left( \frac{B}{n} \right),
\]
where \( v \geq 1 \).

Proof. Let \( D_k = \{(\xi, [u, v]) \in D; [u, v] \subset (x_k, x_{k+1})\} \) and \( D'_k = \{([\xi', [u', v']) \in D'; [u', v'] \subset (x_k, x_{k+1})\}, k = 1, 2, \ldots, n_v - 1 \). It is clear that \( |f(\xi) - f(\xi')| < \varphi^{-1}(A2^{-v}) \). Note that \( \bigcup[u, v]; [u, v] \subset (x_k, x_{k+1}) \) = \( \bigcup[u', v']; [u', v'] \subset (x_k, x_{k+1}) \) = \([a, \beta]\). Applying Corollary 3.4, for any \( \xi, \xi' \) we have
\[
\left| (D_k) \sum f(\xi)(g(v) - g(u)) - (D'_k) \sum f(\xi')(g(v') - g(u')) \right| \\
\leq 13 \sum_{n=1}^{\infty} \varphi^{-1} \left( \frac{A2^{-v}}{n} \right) \psi^{-1} \left( \frac{B2^{-v}}{n} \right)
\]
for \( k = 1, 2, \ldots, n_v - 1 \).

Note that \( D = \bigcup_{k=1}^{n_v} D_k \) and \( n_v \leq 2^{v+1} \). Hence, by Lemma 2.2 (i),
\[
\left| (D) \sum f(\xi)(g(v) - g(u)) - (D') \sum f(\xi')(g(v') - g(u')) \right| \\
\leq 13(2^{v+1} - 1) \sum_{n=1}^{\infty} \varphi^{-1} \left( \frac{A2^{-v}}{n} \right) \psi^{-1} \left( \frac{B2^{-v}}{n} \right)
\]
\[
\leq 13(2^{v+1} - 1) \frac{1}{2^{v-1}} \sum_{n=2^{v-1}}^{\infty} \varphi^{-1} \left( \frac{A}{n} \right) \psi^{-1} \left( \frac{B}{n} \right)
\]
\[
\leq 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1} \left( \frac{A}{n} \right) \psi^{-1} \left( \frac{B}{n} \right).
\]

The following existence theorem is proved in [12] by the Moore-Pollard approach. Now we will prove it by the Henstock-Kurzweil approach.
**Theorem 4.6 (Existence Theorem).** Let \( f \in BV_\varphi[a,b] \) and \( g \in BV_\psi[a,b] \). Suppose that \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \). Then \( \int_a^b f \, dg \) exists.

**Proof.** First, let \( A \supseteq V_\varphi(f;[a,b]) \) and \( B \supseteq V_\psi(f;[a,b]) \). Then by Lemma 2.3, \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \).

Let \( \varepsilon > 0 \), choose \( v \) such that \( \sum_{n=2v-1}^{\infty} \lambda(A/n)\mu(B/n) \leq \varepsilon/52 \).

Let \( E_v = \{x_1, x_2, \ldots, x_n\} \) be given as in Lemma 2.1. Let \( \delta' \) be given as in Lemma 4.4 with \( E = E_v \). Let \( \delta \) be a positive function defined on \([a,b]\) with \( \delta(x) < \delta' \) for all \( x \in [a,b] \) such that if \( D = \{(\xi, [u,v])\} \) is a \( \delta \)-fine division of \([a,b]\), then \([u,v] \subseteq (\xi - \delta', \xi + \delta') \) and \( \xi \in E_v, [u,v] \subset (x_k, x_{k+1}) \) and \( \xi \in (x_k, x_{k+1}), k = 1, 2, \ldots, n_0 - 1 \).

Now let \( D = \{(\xi, [u,v])\} \) and \( D' = \{(\xi', [u', v'])\} \) be two \( \delta \)-fine divisions of \([a,b]\). Let \( D = D_1 \cup D_2, \ D' = D'_1 \cup D'_2 \) where \( D_1 = \{(\xi, [u,v]) \in D; \xi \in E_v\}, \ D'_1 = \{(\xi', [u', v']) \in D; \xi' \in E_v\}, \ D_2 = D \setminus D_1 \) and \( D'_2 = D' \setminus D'_1 \). Suppose \( (\xi, [u,v]) \in D_2 \) and \( x_i + \delta(x_i) \in [u,v] \) (or \( x_i - \delta(x_i) \in [u,v] \)). Then we divide \([u,v]\) into two parts \([u, x_i + \delta(x_i)], [x_i + \delta(x_i), v]\) \(([u, x_i - \delta(x_i)], [x_i - \delta(x_i), v], \) respectively).

Let \( \overline{D}_1 \) be the union of \( D_1 \) and \( (\xi, [u, x_i + \delta(x_i)]), (\xi, [x_i - \delta(x_i), v]). \) Let \( \overline{D}_2 = D \setminus \overline{D}_1 \). Similarly, we construct \( \overline{D}'_1 \) and \( \overline{D}'_2 = D' \setminus \overline{D}'_1 \).

Then, by Lemmas 4.4 and 4.5, we get

\[
\left| (D) \sum f(\xi)(g(v) - g(u)) - (D') \sum f(\xi')(g(v') - g(u')) \right|
\leq \left| (\overline{D}_1) \sum f(\xi)(g(v) - g(u)) - (\overline{D}'_1) \sum f(\xi)(g(v) - g(u)) \right|
\hfill
\]
\[
+ \left| (\overline{D}_2) \sum f(\xi)(g(v) - g(u)) - (\overline{D}'_2) \sum f(\xi')(g(v') - g(u')) \right|
\leq 8\|f\|_\infty \varepsilon + \varepsilon,
\]

where \( \|f\|_\infty = \sup\{f(x); x \in [a,b]\} \). Thus \( \int_a^b f \, dg \) exists. \( \square \)

**5. Approximation**

In this section we show that \( \int_a^b f \, dg \) can be approximated by \( \int_a^b s \, dg \), where \( s \) is a step function. This approximation theorem can be found in [12].

**Theorem 5.1.** Let \( f \in BV_\varphi[a,b], g \in BV_\psi[a,b] \) and \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \). Then, given any \( \varepsilon > 0 \), there exists a step function \( s \) on \([a,b]\) such that \( \left| \int_a^b (f - s) \, dg \right| \leq \varepsilon \).
Proof. First, let \( A \geq V_\varphi(f; [a, b]) \) and \( B \geq V_\psi(f; [a, b]) \). Then, by Lemma 2.3, \( \sum_{n=1}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) < \infty \).

Let \( E_v = \{x_1, x_2, \ldots, x_{n_v}\} \) be given as in Lemma 4.5 with \( v \geq 1 \). Define

\[
 s(x) = \sum_{k=1}^{n_v} f(x_k)\chi_{[x_k)}(x) + \sum_{k=1}^{n_v-1} f(x_{k+})\chi_{(x_k, x_{k+1})}(x).
\]

Then \( (f - s)(x_k) = 0 \) for all \( x_k \in E_v \). By Theorem 4.6, \( \int_a^b f \, dg - \int_a^b s \, dg = \int_a^b (f - s) \, dg \) exists. Let \( \varepsilon > 0 \); there exists a positive function \( \delta \) on \([a, b]\) such that whenever \( D = \{(\xi, [u, v])\} \) is a \( \delta \)-fine division of \([a, b]\), \( x_i \) is a tag for every \( i = 1, 2, \ldots, n_v \), and

\[
 \left| \int_a^b (f - s) \, dg - (D) \sum_{\xi \notin E_v} (f - s)(\xi)(g(v) - g(u)) \right| \leq \frac{\varepsilon}{2}.
\]

By Lemma 4.5, and all \( x_i \) being tags, we have

\[
 \left| (D) \sum_{\xi \notin E_v} (f - s)(\xi)(g(v) - g(u)) \right| = \left| (D) \sum_{\xi \notin E_v} (f - s)(\xi)(g(v) - g(u)) \right| \\
\leq 52 \sum_{n=2^{v-1}}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right) \text{ for } v \geq 1.
\]

Therefore

\[
 \left| \int_a^b (f - s) \, dg \right| \leq \frac{\varepsilon}{2} + 52 \sum_{n=2^{v-1}}^{\infty} \lambda \left( \frac{A}{n} \right) \mu \left( \frac{B}{n} \right).
\]

Choosing \( v \) big enough, we get the required result. \( \square \)

**Corollary 5.2.** Let \( f \in BV_\varphi[a, b], g \in BV_\psi[a, b] \) and \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \). Then

\[
 \left| \int_a^b f \, dg \right| \leq 6 \sum_{n=1}^{\infty} \varphi^{-1} \left( \frac{V_\varphi(f)}{n} \right) \psi^{-1} \left( \frac{V_\psi(g)}{n} \right) + f(a)(g(a+ - g(a)) + f(a+)(g(b) - g(a+)) + f(b)(g(b) - g(b-)).
\]

**Proof.** Let \( \varepsilon > 0 \). By Theorem 5.1 there exists a step function \( s \) on \([a, b]\) such that \( \left| \int_a^b (f - s) \, dg \right| \leq \varepsilon \). Hence

\[
 \left| \int_a^b f \, dg \right| \leq \varepsilon + \left| \int_a^b s \, dg \right|.
\]
By Theorem 3.3, 
\[ \left| \int_a^b s \, dg \right| \leq 6 \sum_{n=1}^{\infty} \lambda \left( \frac{V_s(f)}{n} \right) \mu \left( \frac{V_s(g)}{n} \right) + \left| \int_a^b s_{E_0} \, dg \right| . \]

Note that \( \int_a^b s_{E_0} \, dg = f(a)(g(a+)-g(a)) + f(a+)(g(b)-g(a+)) + f(b)(g(b)-g(b-)) \).
Hence we get the required result. \( \square \)

6. Integration by parts

A general result for integration by parts in the setting of Henstock-Kurzweil integrals of Stieltjes type can be found in [10]. In this section, we will prove this result in more concrete forms.

For any partial division \( D = \{ (\xi_i, [u_i, v_i]) \} \) on \([a, b] \), define 
\[
S_-(f, g, D) = (D) \sum (f(\xi) - f(u))(g(\xi) - g(u)), \\
S_+(f, g, D) = (D) \sum (f(v) - f(\xi))(g(v) - g(\xi))
\]
and 
\[
S(f, g, D) = S_-(f, g, D) - S_+(f, g, D).
\]

We say that \( S_-(f, g) \) exists if there exists \( S^{(1)} \) such that for every \( \varepsilon > 0 \) there exists a positive function \( \delta \) on \([a, b] \) such that when \( D \) is a \( \delta \)-fine division of \([a, b] \), we have 
\[
|S_-(f, g, D) - S^{(1)}| \leq \varepsilon.
\]
We then denote \( S^{(1)} \) by \( S_-(f, g) \). Similarly, we can define \( S_+(f, g) \) and \( S(f, g) \). Clearly, if two of \( S_-(f, g), S_+(f, g) \) and \( S(f, g) \) exist, then the third exists and 
\[
S(f, g) = S_-(f, g) - S_+(f, g).
\]

Lemma 6.1. Let \( f \in BV_{\varphi}[a, b] \) and \( g \in BV_{\psi}[a, b] \) with \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \). Then 
\[
S_+(f, g) = \sum_{i=1}^{\infty} (f(t_i+) - f(t_i))(g(t_i+) - g(t_i))
\]
and 
\[
S_-(f, g) = \sum_{i=1}^{\infty} (f(t_i) - f(t_i-))(g(t_i) - g(t_i-)),
\]

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where \( t_i \) are the common points of discontinuity of \( f \) and \( g \) and the above series converge absolutely.

**Proof.** Let \( \varepsilon > 0 \), let \( v \) be a positive integer such that \( \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) \lesssim \varepsilon/52 \). Let \( E_v = \{x_1, x_2, \ldots, x_n\} \) be given as in Lemma 2.1. \( E_v \) may contain some points of \( \{t_i\}_{i=1}^\infty \). We may assume that there exists a positive integer \( N \) such that \( t_j \notin E_v \) whenever \( j \geq N \). Now take any two positive integers \( m, n \geq N \) with \( m < n \). Let \( \delta \) be a positive number such that for every \( i = m, m+1, \ldots, n \), if \( t_i \in (x_j, x_{j+1}) \) for some \( j \), then \( (t_i, t_i + \delta) \subset (x_j, x_{j+1}) \) and \( \{(t_i, t_i + \delta)\}_{i=m}^n \) are non-overlapping intervals. Let \( \eta_i \in (t_i, t_i + \delta) \) for all \( i = m, m+1, \ldots, n \), and \( D = \{t_i, \eta_i\}_{i=m}^n \) and \( D' = \{\eta_i, [t_i, \eta_i]\}_{i=m}^n \). From the definition of \( D \) and \( D' \) it is clear that \( D \) and \( D' \) are partial divisions of \([a, b]\) and satisfy the condition of Theorem 4.5. Hence, we have

\[
\begin{align*}
&\sum_{i=m}^{n} (f(t_i) - f(t_i))(g(\eta_i) - g(t_i)) \\
&= \left| (D') \sum_{i=1}^{n} f(\eta_i)(g(\eta_i) - g(t_i)) - (D) \sum_{i=1}^{n} f(t_i)(g(\eta_i) - g(t_i)) \right| \\
&\lesssim 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) \lesssim \frac{52 \varepsilon}{52} = \varepsilon.
\end{align*}
\]

Then

\[
\sum_{i=m}^{n} \left| (f(t_i+1) - f(t_i))(g(t_i+1) - g(t_i)) \right| \leq \varepsilon.
\]

Observe that \( D \) and \( D' \) are partial divisions. Therefore

\[
\sum_{i=m}^{n} \left| (f(t_i+1) - f(t_i))(g(t_i+1) - g(t_i)) \right| \leq 2\varepsilon,
\]

where \( m, n \geq N \).

Hence \( S_+(f, g) \) converges absolutely. Similarly, \( S_-(f, g) \) converges absolutely. \( \Box \)

**Lemma 6.2.** Let \( f \in BV_\varphi[a, b] \) and \( g \in BV_\psi[a, b] \), \( E = \{x_1, x_2, \ldots, x_n\} \supseteq E_0 \), and \( \varepsilon > 0 \). Then there exists a constant \( \delta' > 0 \) such that for any finite collection of disjoint subintervals \( \{[u_i, v_i]\}_{i=1}^n \) with \([u_i, v_i] \subset (x_i, x_{i} + \delta') \) for each \( i \) or \([u_i, v_i] \subset (x_i - \delta', x_i) \) for each \( i \), we have

\[
\sum_{i=1}^{n} |f(v_i) - f(u_i)| \lesssim \frac{\varepsilon}{V_\psi(B)}.
\]

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and
\[ \sum_{i=1}^{n} |g(v_i) - g(u_i)| \leq \frac{\varepsilon}{V_\psi(A)}. \]

**Proof.** The proof is similar to that of Lemma 4.4. Let \( \varepsilon > 0 \) be given. First, observe that \( f \) and \( g \) are regulated functions. Therefore, there exists a constant \( \delta' > 0 \) such that
\[
|g(t) - g(x_i -)| \leq \min \left\{ \frac{\varepsilon}{2n}, \frac{\varepsilon}{2nV_\psi(A)} \right\} \quad \text{whenever } 0 < x_i - t < \delta',
\]
\[
|g(x_i +) - g(t)| \leq \min \left\{ \frac{\varepsilon}{2n}, \frac{\varepsilon}{2nV_\psi(A)} \right\} \quad \text{whenever } 0 < t - x_i < \delta',
\]
\[
|f(t) - f(x_i -)| \leq \min \left\{ \frac{\varepsilon}{2n}, \frac{\varepsilon}{2nV_\psi(B)} \right\} \quad \text{whenever } 0 < x_i - t < \delta'
\]
and
\[
|f(x_i +) - f(t)| \leq \min \left\{ \frac{\varepsilon}{2n}, \frac{\varepsilon}{2nV_\psi(B)} \right\} \quad \text{whenever } 0 < t - x_i < \delta'
\]
for each \( i \). Therefore, the required result follows. \( \Box \)

**Lemma 6.3.** Let \( f \in BV_\varphi[a,b] \) and \( g \in BV_\psi[a,b] \) with \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n)\psi^{-1}(1/n) < \infty \). Let \( \varepsilon > 0 \). If \( E_\varphi = \{x_1, x_2, \ldots, x_{n_0}\} \) is the set given in Lemma 2.1 and \( \{t_j\}_{j=1}^{m} \subseteq E_\varphi \), where \( \{t_j\}_{j=1}^{m} \) are all common points of discontinuity of \( f \) and \( g \) such that
\[
\sum_{n=2^{i-1}}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) < \varepsilon/312 \quad \text{and} \quad \sum_{j=m+1}^{\infty} |(f(t_j) - f(t_j-))(g(t_j) - g(t_j-))| \leq \varepsilon/6,
\]
then there exists a positive real number \( \delta' \) such that for any \( \delta' \)-fine partial division \( D = \{(x_i, [u_i, v_i])\}_{i=1}^{n_0} \) of \( [a,b] \) we have
\[
|S(f,g,D) - (S_+(f,g) - S_-(f,g))| \leq \frac{2}{3} \varepsilon.
\]

**Proof.** Applying Lemma 6.2 to \( \varepsilon/18 \) and \( E = E_\varphi \), we get a positive constant \( \delta' \). Let \( D = \{(x_i, [u_i, v_i])\}_{i=1}^{n_0} \), then
\[
S_+(f,g,D) = \sum_{i=1}^{n_0} (f(x_i) - f(u_i))(g(x_i) - g(u_i))
\]
\[
= \sum_{i=1}^{n_0} [(f(x_i) - f(x_i-)) + (f(x_i-) - f(u_i))][(g(x_i) - g(x_i-)) + (g(x_i-) - g(u_i))]
\]
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\[
= \sum_{i=1}^{n_u} (f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) + \sum_{i=1}^{n_v} (f(x_i) - f(x_i-))(g(x_i) - g(u_i)) \\
+ \sum_{i=1}^{n_v} (f(x_i- - f(u_i))(g(x_i) - g(x_i-)) + \sum_{i=1}^{n_v} (f(x_i- - f(u_i))(g(x_i) - g(u_i)).
\]

Let \( F = E_v \setminus \{ t_j \}_{j=1}^m \), then \( F \) is the set of points in \( E_v \) which are not common points of discontinuity of \( f \) and \( g \). Hence
\[
\sum_{x_i \in F} (f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) = 0.
\]

Consider
\[
\sum_{i=1}^{n_u} (f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) \\
= \sum_{x_i \in F} (f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) + \sum_{x_i \notin F} (f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) \\
= 0 + \sum_{x_i \notin F} (f(x_i) - f(x_i-))(g(x_i) - g(x_i-)) \\
= \sum_{j=1}^{m} (f(t_j) - f(t_j-))(g(t_j) - g(t_j-)).
\]

Then
\[
S_-(f, g, D) = \sum_{i=1}^{n_u} (f(x_i) - f(u_i))(g(x_i) - g(u_i)) \\
= \sum_{j=1}^{m} (f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) + \sum_{i=1}^{n_v} (f(x_i) - f(x_i-))(g(x_i) - g(u_i)) \\
+ \sum_{i=1}^{n_v} (f(x_i- - f(u_i))(g(x_i) - g(x_i-)) + \sum_{i=1}^{n_v} (f(x_i- - f(u_i))(g(x_i) - g(u_i)).
\]

Therefore
\[
|S_-(f, g, D) - S_-(f, g)| \\
= \left| \sum_{i=1}^{n_v} (f(x_i) - f(u_i))(g(x_i) - g(u_i)) - \sum_{j=1}^{\infty} (f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) \right| \\
\leq \left| \sum_{i=1}^{n_v} (f(x_i) - f(u_i))(g(x_i) - g(u_i)) - \sum_{j=1}^{m} (f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) \right| \\
+ \left| \sum_{j=m+1}^{\infty} (f(t_j) - f(t_j-))(g(t_j) - g(t_j-)) \right|
\]

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\[
= \left| \sum_{i=1}^{n_v} (f(x_i) - f(x_i-))(g(x_i) - g(u_i)) \right| + \left| \sum_{i=1}^{n_v} (f(x_i-))g(x_i) - g(x_i-)) \right| \\
\quad + \left| \sum_{i=1}^{n_v} (f(x_i)-f(u_i))(g(x_i)-g(u_i)) \right| + \varepsilon/6.
\]

By Lemma 6.2 we have
\[
\left| \sum_{i=1}^{n_v} (f(x_i) - f(u_i))(g(x_i) - g(u_i)) \right| \leq \frac{\varepsilon}{18},
\]
\[
\left| \sum_{i=1}^{n_v} (f(x_i)-f(x_i-))(g(x_i)-g(x_i-)) \right| \leq \frac{\varepsilon}{18} V_\varphi(A) = \frac{\varepsilon}{18}
\]
and
\[
\left| \sum_{i=1}^{n_v} (f(x_i) - f(x_i-))(g(x_i)-g(u_i)) \right| \leq \frac{\varepsilon}{18} V_\varphi(B) = \frac{\varepsilon}{18}.
\]

Thus
\[
|S_-(f,g,D) - S_-(f,g)| \leq \frac{\varepsilon}{18} + \frac{\varepsilon}{18} + \frac{\varepsilon}{6} \leq \frac{\varepsilon}{3}.
\]

Similarly,
\[
|S_+(f,g,D) - S_+(f,g)| \leq \frac{\varepsilon}{3}.
\]

Hence
\[
|S(f,g,D) - (S_+(f,g) - S_-(f,g))| \\
\leq |S_-(f,g,D) - S_-(f,g)| + |S_+(f,g,D) - S_+(f,g)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2}{3} \varepsilon.
\]

Lemma 6.4. Let \( f \in \text{BV}_\varphi[a,b] \) and \( g \in \text{BV}_\psi[a,b] \) with \( \sum_{n=1}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) < \infty \). Then for any given \( \varepsilon > 0 \) there exists a positive function \( \delta \) such that for any \( \delta \)-fine division \( D \) of \([a,b]\) we have
\[
|S(f,g,D) - (S_+(f,g) - S_-(f,g))| \leq \varepsilon.
\]

Proof. Let \( \varepsilon > 0 \), choose \( v \) such that \( \sum_{n=1}^{\infty} \varphi^{-1}(A/n)\psi^{-1}(B/n) \leq \varepsilon/312 \.
Let \( E_v = \{x_1, x_2, \ldots, x_{n_v}\} \) be given as in Lemma 2.1. Applying Lemma 6.3 to \( E = E_v \), we get a positive constant \( \delta' \). Let \( \delta \) be a positive function defined on \([a,b]\) with \( \delta(x) < \delta' \) for all \( x \in [a,b] \) such that if \( D = \{\{\xi, [u,v]\}\} \) is a \( \delta \)-fine division
of \([a, b]\), then \([u, v] \subset (\xi - \delta', \xi + \delta')\) when \(\xi \in E_v\) and \([u, v] \subset (x_k, x_{k+1})\) when \(\xi \in (x_k, x_{k+1}), k = 1, 2, \ldots, n_v - 1\). Now let \(D = \{(\xi, [u, v])\}\) be a \(\delta\)-fine division of \([a, b]\). Let \(D = D_1 \cup D_2\), where \(D_1 = \{(\xi, [u, v]) \in D: \xi \in E_v\}\), \(D_2 = D \setminus D_1\). Hence, by Lemma 6.3,

\[
|S(f, g, D) - (S_+(f, g) - S_-(f, g))| \\
\leq |S_-(f, g, D_2)| + |S_+(f, g, D_2)| + |S(f, g, D_1) - (S_+(f, g) - S_-(f, g))| \\
\leq |S_-(f, g, D_2)| + |S_+(f, g, D_2)| + \frac{2\varepsilon}{3}.
\]

By Lemma 4.5, we have

\[
|S_-(f, g, D_2)| = \left| (D_2) \sum (f(\xi) - f(u))(g(\xi) - g(u)) \right| \\
= \left| (D_2) \sum f(\xi)(g(\xi) - g(u)) - (D_2) \sum f(u)(g(\xi) - g(u)) \right| \\
\leq 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1} \left( \frac{A}{n} \right) \psi^{-1} \left( \frac{B}{n} \right) \leq \frac{\varepsilon}{6}
\]

and

\[
|S_+(f, g, D_2)| = \left| (D_2) \sum (f(v) - f(\xi))(g(v) - g(\xi)) \right| \\
\leq 52 \sum_{n=2^{v-1}}^{\infty} \varphi^{-1} \left( \frac{A}{n} \right) \psi^{-1} \left( \frac{B}{n} \right) \leq \frac{\varepsilon}{6}
\]

Hence,

\[
|S(f, g, D) - (S_+(f, g) - S_-(f, g))| \leq |S_-(f, g, D_2)| + |S_+(f, g, D_2)| + \frac{2\varepsilon}{3} \\
\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{2\varepsilon}{3} = \varepsilon.
\]

We can verify that \(S_-(f, g), S_+(f, g)\) exist and \(S_-(f, g) = \sum_{i=1}^{\infty} (f(t_i) - f(t_i))\)

\((g(t_i) - g(t_i), S_+(f, g) = \sum_{i=1}^{\infty} (f(t_i) - f(t_i)) (g(t_i) - g(t_i))\).

**Theorem 6.5.** Let \(f \in \text{BV}_\varphi[a, b]\) and \(g \in \text{BV}_\psi[a, b]\) with \(\sum_{m=1}^{\infty} \varphi^{-1}(1/m) \psi^{-1}(1/m) < \infty\). Then

\[
\int_a^b f \, dg + \int_a^b g \, df = f(b)g(b) - f(a)g(a) + S(f, g),
\]

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where \( S(f, g) = \sum_{i=1}^{\infty} (f(t_i) - f(t_{i-1}))(g(t_i) - g(t_{i-1})) - \sum_{i=1}^{\infty} (f(t_i) - f(t_{i+1}))(g(t_i) - g(t_{i+1})) \) and \( \{t_i\} \) are all common points of discontinuity of \( f \) and \( g \).

**Proof.** Since \( S(f, g) = S_+(f, g) - S_-(f, g) \), by Lemma 6.1, \( S(f, g) \) exists. Let \( \varepsilon > 0 \) be given and let \( f, g: [a, b] \to \mathbb{R} \). Then there exists a positive function \( \delta_1 \) on \([a, b]\) such that for any \( \delta_1 \)-fine partial division \( D' = \{[u_i, v_i], \xi_i\} \) of \([a, b]\),

\[
|S(f, g, D') - S(f, g)| \leq \frac{\varepsilon}{2}.
\]

Since \( f \) is integrable with respect to \( g \), there exists a positive function \( \delta_2 \) on \([a, b]\) such for any \( \delta_2 \)-fine division \( D'' = \{[t_i, t_{i+1}], \xi_i\} \) of \([a, b]\), we have

\[
\left| \int_a^b f \, dg \right| \leq \frac{\varepsilon}{2}.
\]

Choose \( \delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\} \). Let \( D = \{[t_i, t_{i+1}], \xi_i\} \) be a \( \delta \)-fine partial division of \([a, b]\). We can see that

\[
\left| \left( \int D \sum g(\xi_i)(f(t_{i+1}) - f(t_i)) \right) - \left( f(b)g(b) - f(a)g(a) + S(f, g) - \int_a^b f \, dg \right) \right| = \left| \left( \int D \sum g(\xi_i)(f(t_{i+1}) - f(t_i)) - f(t_{i+1})g(t_{i+1}) + f(t_i)g(t_i) \right. \right.
\]

\[
+ \left. \int_{t_i}^{t_{i+1}} f \, dg \right) - S(f, g) \right| = \left| \left( \int D \sum f(\xi_i)(g(t_{i+1}) - g(t_i)) + (f(\xi_i) - f(t_i))(g(\xi_i) - g(t_i)) \right.ight.
\]

\[
- (f(t_{i+1}) - f(\xi_i))(g(t_{i+1}) - g(\xi_i)) + \int_{t_i}^{t_{i+1}} f \, dg - S(f, g) \right| \leq \left| \left( \int D \sum f(\xi_i)(g(t_{i+1}) - g(t_i)) - \int_{t_i}^{t_{i+1}} f \, dg \right) \right| + |S(f, g, D) - S(f, g)|
\]

\[
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus, we can conclude that \( g \) is integrable to \( f(b)g(b) - f(a)g(a) + S(f, g) - \int_a^b f \, dg \) on \([a, b]\) with respect to \( f \).

Hence, we have

\[
\int_a^b f \, dg + \int_a^b g \, df = f(b)g(b) - f(a)g(a) + S(f, g).
\]

\( \square \)
7. Convergence theorem

In this section we will use Young's idea, see [11], [12], to prove some convergence theorems for our setting.

**Definition 7.1** (Two-norm convergence). A sequence \( \{f^{(n)}\} \) of functions in \( BV_{\varphi}[a,b] \) is said to be two-norm convergent to \( f \) if

(i) \( f^{(n)} \) is uniformly convergent to \( f \) on \( [a,b] \), and

(ii) \( \varphi(f^{(n)}) \leq A \) for every \( n = 1, 2, \ldots \).

In symbols, we denote the two-norm convergence by \( f^{(n)} \to f \).

It is clear that \( BV_{\varphi}[a,b] \) is complete under two-norm convergence, i.e., if \( f^{(n)} \in BV_{\varphi}[a,b] \), \( n = 1, 2, \ldots \), and \( f^{(n)} \to f \), then \( f \in BV_{\varphi}[a,b] \).

We need the following two lemmas.

**Lemma 7.2.** Let \( \varphi \) be a strictly decreasing continuous function on \( (0, \infty) \) with \( \lim_{x \to 1} \varphi(x) = 0 \) and let \( \int_1^\infty \varphi(x) \, dx \) exist. Then there exists a strictly increasing continuous function \( \omega \) on \( [0, \infty) \) with \( \lim_{x \to \infty} \omega(x) = \infty \) such that

\[
\lim_{x \to \infty} \frac{\omega(x)}{x} = \infty \quad \text{and} \quad \int_1^\infty \varphi(x) \, d\omega(x) \text{ exists.}
\]

**Proof.** Since \( \int_1^\infty \varphi(x) \, dx \) exists, there exists a positive function on \( [0, \infty) \)
\( \iota(x) \geq 0 \) with \( \lim_{x \to \infty} \iota(x) = \infty \) and \( \iota(x) = 0 \) for \( x \leq 1 \), such that \( \int_1^\infty \varphi(x) \iota(x) \, dx \) and \( \int_0^x \iota(t) \, dt \) exist for every \( x \in (0, \infty) \). Let

\[
\omega(x) = x + \int_0^x \iota(t) \, dt.
\]

Then \( \omega \) is a strictly increasing function with \( \lim_{x \to \infty} \omega(x) = \infty \). Therefore,

\[
\int_1^\infty \varphi(x) \, d\omega(x) = \int_1^\infty \varphi(x)[1 + \iota(x)] \, dx < \infty.
\]

Now we shall prove that \( \lim_{x \to \infty} \omega(x)/x = \infty \). Let \( x > 2n \). Then \( (x - n)/x > \frac{1}{2} \). By Mean-Value Theorem for integral, there exists \( y \in (n, x) \) such that

\[
\frac{1}{x - n} \int_n^x \iota(x) \, dx = \iota(y).
\]
Hence
\[
\frac{\varrho(x)}{x} = 1 + \frac{1}{x} \int_0^x \vartheta(x) \, dx \geq \frac{x - n}{x} \left[ \frac{1}{x - n} \int_n^x \vartheta(x) \, dx \right] \geq \frac{1}{2} \vartheta(y) \geq \frac{1}{2} \vartheta(n).
\]

Since \( \vartheta(n) \to \infty \) as \( n \to \infty \), we have
\[
\lim_{x \to \infty} \frac{\varrho(x)}{x} = \infty.
\]

**Corollary 7.3.** Let \( \vartheta \) be a strictly decreasing continuous function on \((0, \infty)\) with \( \lim_{x \to \infty} \vartheta(x) = 0 \) and let \( \int_1^\infty \vartheta(x) \, dx \) exist. Then there exists a strictly increasing continuous function \( \varsigma \) on \((0, \infty)\) with \( \lim_{x \to \infty} \varsigma(x) = \infty \), such that
\[
\lim_{x \to \infty} \frac{\varsigma(x)}{x} = 0 \quad \text{and} \quad \int_1^\infty \vartheta(\varsigma(x)) \, dx \quad \text{exists}.
\]

**Proof.** Let \( \varsigma = \varrho^{-1} \), where \( \varrho \) is given in Lemma 7.2. Thus we get the required result. \( \square \)

**Lemma 7.4.** Suppose \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n) \psi^{-1}(1/n) < \infty \). Then there exist two \( N \)-functions \( \varphi^*, \psi^* \) such that \( \varphi^*(u) \leq \overline{\varphi}(u) \varphi(u) \) and \( \psi^*(u) \leq \overline{\psi}(u) \psi(u) \), where \( \overline{\varphi}, \overline{\psi} \) are increasing and \( \lim_{x \to 0} \overline{\varphi}(x) = \lim_{x \to 0} \overline{\psi}(x) = 0 \), with
\[
\sum_{n=1}^{\infty} \left( \varphi^* \right)^{-1} \left( \frac{1}{n} \right) \left( \psi^* \right)^{-1} \left( \frac{1}{n} \right) < \infty.
\]

**Proof.** Given \( \varphi, \psi \) and \( \sum_{n=1}^{\infty} \varphi^{-1}(1/n) \psi^{-1}(1/n) < \infty \), we want to construct \( \varphi^*, \psi^* \) such that \( \varphi^*(u) \leq \overline{\varphi}(u) \varphi(u) \), \( \psi^*(u) \leq \overline{\psi}(u) \psi(u) \), where \( \overline{\varphi}, \overline{\psi} \) are increasing functions with \( \lim_{x \to 0} \overline{\varphi}(x) = \lim_{x \to 0} \overline{\psi}(x) = 0 \) and
\[
\sum_{n=1}^{\infty} \left( \varphi^* \right)^{-1} \left( \frac{1}{n} \right) \left( \psi^* \right)^{-1} \left( \frac{1}{n} \right) < \infty.
\]

Let \( \vartheta(u) = \varphi^{-1}(1/u) \psi^{-1}(1/u) \) for \( u \in (0, \infty) \). Then \( \vartheta \) satisfies the conditions of Corollary 7.3. Hence there exists a strictly increasing continuous function \( \varsigma \) on \([0, \infty)\) with \( \lim_{x \to \infty} \varsigma(x) = \infty \), such that
\[
\lim_{x \to \infty} \frac{\varsigma(x)}{x} = 0 \quad \text{and} \quad \int_1^\infty \vartheta(\varsigma(x)) \, dx \quad \text{exists}.
\]
Let $\theta(u) = u \zeta(u^{-1})$ for $u \in (0, \infty)$ and $\theta(0) = 0$. Then $\lim_{u \to 0} \theta(u) = \lim \zeta(u^{-1})/u^{-1} = 0$ and $u/\theta(u) = 1/\zeta(u^{-1})$ is a strictly increasing continuous function on $(0, \infty)$.

Let $\Phi(u) = \varphi^{-1}(u/\theta(u))$, $\Psi(u) = \psi^{-1}(u/\theta(u))$, $\Phi(0) = 0$ and $\Psi(0) = 0$. Then $\Phi$ and $\Psi$ are strictly increasing continuous functions on $[0, \infty)$. Furthermore, let $\varphi^* = (\Phi)^{-1}$ and $\psi^* = (\Psi)^{-1}$. Then

$$\sum_{n=1}^{\infty} (\varphi^*)^{-1}\left(\frac{1}{n}\right)(\psi^*)^{-1}\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \varphi^{-1}\left(\frac{n^{-1}}{\theta(n^{-1})}\right) - \psi^{-1}\left(\frac{n^{-1}}{\theta(n^{-1})}\right)$$

$$= \sum_{n=1}^{\infty} \varphi^{-1}\left(\frac{1}{\zeta(n)}\right) - \psi^{-1}\left(\frac{1}{\zeta(n)}\right)$$

$$= \sum_{n=1}^{\infty} \vartheta(\zeta(n)) \leq \int_{1}^{\infty} \vartheta(\zeta(x)) \, dx < \infty,$$

since $\vartheta(\zeta(x))$ is non-negative.

If $t = (\varphi^*)^{-1}(u^*) = \varphi^{-1}(u^*/\theta(u^*))$, then $\varphi^*(t) = u^*$. On the other hand, if $t = \varphi^{-1}(u)$, then $\varphi(t) = u$. Hence $u = u^*/\theta(u^*)$ and

$$\frac{\varphi^*(t)}{\varphi(t)} = \frac{u^*}{u} = \theta(u^*) = \theta(\varphi^*(t)) =: \pi(t);$$

clearly $\lim_{t \to 0} \pi(t) = \lim_{t \to 0} \theta(\varphi^*(t)) = 0$. Similarly, we have

$$\frac{\psi^*(t)}{\psi(t)} = \frac{u^*}{u} = \theta(u^*) = \theta(\psi^*(t)) =: \gamma(t),$$

and $\lim_{t \to 0} \pi(t) = \lim_{t \to 0} \theta(\varphi^*(t)) = 0$. Denoting by $\overline{\pi}(t), \overline{\gamma}(t)$ the upper bounds of $\pi(u), \gamma(u)$ for $0 < u \leq t$ we see that $\overline{\pi}, \overline{\gamma}$ are increasing functions. Then

$$\varphi^*(t) \leq \overline{\pi}(t)\varphi(t)$$

and

$$\psi^*(t) \leq \overline{\gamma}(t)\psi(t).$$

Let $D = \{[u, v]\}$ be a partition of an interval $[\alpha, \beta]$. By Lemma 7.4, we have

$$(D) \sum \varphi^*(|f(v) - f(u)|) = (D) \sum \overline{\pi}(||f(v) - f(u)||)\varphi(|f(v) - f(u)|)$$

$$\leq \overline{\pi}(2||f||_{\infty})(D) \sum \varphi(|f(v) - f(u)|).$$

Hence, if $A$ and $A^*$ are the $\varphi$-variation and $\varphi^*$-variation of $f$, respectively, on $[\alpha, \beta]$, we have

$$A^* \leq A\overline{\pi}(2||f||_{\infty}) \leq A\overline{\pi}(\varphi^{-1}(A)).$$
Theorem 7.5. If \( g \in \text{BV}_\varphi[a, b] \) and \( \{f^{(n)}\} \) is two-norm convergent to \( f \) in \( \text{BV}_\varphi[a, b] \) with \( \sum_{m=1}^{\infty} \varphi^{-1}(1/m)\psi^{-1}(1/m) < \infty \), then \( \int_a^b f \, dg \) exists and

\[
\lim_{n \to \infty} \int_a^b f^{(n)} \, dg = \int_a^b f \, dg.
\]

Proof. Let \( \varepsilon > 0 \) be given. Let \( \{f^{(n)}\} \) be two-norm convergent to \( f \) in \( \text{BV}_\varphi[a, b] \) and \( g \in \text{BV}_\varphi[a, b] \). By the convexity of \( \text{BV}_\varphi'[a, b] \), \( \frac{1}{2}(f^{(n)} - f) \in \text{BV}_\varphi[a, b] \). Hence, \( \int_a^b (f^{(n)} - f) \, dg \) exists. Thus, there is a positive function \( \delta_n \) such that for every \( \delta_n \)-fine division \( D = \{(t_i, t_{i+1}, \xi_i)\} \) of \( [a, b] \),

\[
\left| \int_a^b (f^{(n)} - f) \, dg \right| = (D) \sum (f^{(n)}(\xi_i) - f(\xi_i))(g(t_{i+1}) - g(t_i)) \leq \varepsilon.
\]

Let

\[
V_\varphi(\frac{1}{2}(f^{(n)} - f)) \leq V_\varphi(f^{(n)}) + V_\varphi(f) \leq A \text{ for every } n \text{ and } V_\varphi(g) = B.
\]

By Lemma 7.4, there exist two \( N \)-function \( \varphi^* \) and \( \psi^* \) such that \( \varphi^*(u) \leq \pi(u)\varphi(u) \) and \( \psi^*(u) \leq \tau(u)\psi(u) \), where \( \pi, \tau \) are increasing and \( \lim_{x \to 0} \pi(x) = \lim_{x \to 0} \tau(x) = 0 \), with

\[
\sum_{n=1}^{\infty} (\varphi^*)^{-1}(\frac{1}{n})(\psi^*)^{-1}(\frac{1}{n}) < \infty.
\]

By Lemma 2.3, there exists a positive integer \( v \) such that

\[
\sum_{n=v+1}^{\infty} (\varphi^*)^{-1}\left(\frac{A\pi^{-1}(A)}{n}\right)(\psi^*)^{-1}\left(\frac{B\tau^{-1}(B)}{n}\right) < \varepsilon.
\]

For this \( v \), choose \( \tau > 0 \) such that

\[
(\varphi^*)^{-1}(A\pi(\tau)) \leq \frac{\varepsilon}{v(\psi^*)^{-1}\left(\frac{B\tau^{-1}(B)}{v}\right)}.
\]

Hence for \( n = 1, 2, \ldots, v \),

\[
(\varphi^*)^{-1}\left(\frac{A\pi_1}{n}\right) \leq \frac{\varepsilon}{v(\psi^*)^{-1}\left(\frac{B\tau^{-1}(B)}{v}\right)}.
\]
Since \( f^{(n)} \) converge to \( f \) uniformly on \([a, b]\), there is a positive integer \( N \) such that for every \( n \geq N \), we have

\[
\sup_{t \in [a, b]} \frac{1}{n} (|f^{(n)}(t) - f(t)|) = \| \frac{1}{n} (f^{(n)} - f) \|_\infty < \min \{ \varepsilon, \frac{1}{2} \tau \}.
\]

We may assume that when \( n \geq N \), then \( |(f^{(n)} - f)(a)(g(a) - g(a)) + (f^{(n)} - f)(a)(g(b) - g(a)) + (f^{(n)} - f)(b)(g(b) - g(b))| \leq \varepsilon \).

Hence for \( n \geq N \), applying Corollary 5.2 to \( \frac{1}{n} (f^{(n)} - f) \), we get

\[
\left| \int_a^b f^{(n)} \, dg - \int_a^b f \, dg \right| = 2 \left| \int_a^b \frac{f^{(n)} - f}{2} \, dg \right|
\leq 2 \cdot 6 \sum_{n=1}^{\infty} (\varphi^*)^{-1} \left( \frac{V_{\varphi^*} \left( \frac{1}{n} (f^{(n)} - f) \right)}{V_{\psi^*} (g) n} \right)^{-1} + \varepsilon
\leq 12 \sum_{n=1}^{\infty} (\varphi^*)^{-1} \left( \frac{V_{\varphi^*} \left( \frac{1}{n} (f^{(n)} - f) \right)}{V_{\psi^*} (g) n} \right)^{-1}
+ 12 \sum_{n=1}^{\infty} (\varphi^*)^{-1} \left( \frac{V_{\varphi^*} \left( \frac{1}{n} (f^{(n)} - f) \right)}{V_{\psi^*} (g) n} \right)^{-1}
\leq 12 \sum_{n=1}^{\infty} (\varphi^*)^{-1} \left( \frac{A \pi (2 \| f^{(n)} - f \|_\infty)}{B \pi (\psi^{-1} (B)) n} \right)^{-1} + \varepsilon
+ 12 \sum_{n=1}^{\infty} (\varphi^*)^{-1} \left( \frac{A \pi (\phi^{-1} (A))}{B \pi (\psi^{-1} (B)) n} \right)^{-1}
\leq 12 \sum_{n=1}^{\infty} (\varphi^*)^{-1} \left( \frac{A \pi (\tau)}{B \pi (\psi^{-1} (B)) n} \right)^{-1} + 13 \varepsilon
\leq 12 \varepsilon + 13 \varepsilon = 25 \varepsilon.
\]

Hence, \( \lim_{n \to \infty} \int_a^b f^{(n)} \, dg = \int_a^b f \, dg \). \hfill \Box

**Theorem 7.6.** If \( f \in \text{BV}_\varphi [a, b] \) and \( \{ g^{(n)} \} \) is two-norm convergent to \( g \) in \( \text{BV}_\varphi [a, b] \) with \( \sum_{m=1}^{\infty} \phi^{-1} (1/m) \psi^{-1} (1/m) < \infty \), then \( \int_a^b f \, dg \) exists and

\[
\lim_{n \to \infty} \int_a^b f \, dg^{(n)} = \int_a^b f \, dg.
\]

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Proof. Since $g^{(n)}$ converge to $g$ uniformly, there exists a positive integer $N$ such that for every $n > N_1$ we have

$$|S(f,g^{(n)}) - S(f,g)| \leq \frac{\varepsilon}{4},$$

$$|(f^{(n)})(b) - f(b))g(b)| \leq \frac{\varepsilon}{4}$$

and

$$|(f^{(n)})(a) - f(a))g(a)| \leq \frac{\varepsilon}{4}.$$  

By Theorem 7.5 there exists a positive integer $N > N_1$ such that for any $n > N$ we have

$$\left| \int_a^b g^{(n)}df - \int_a^b gdf \right| \leq \frac{\varepsilon}{4}.$$  

Hence

$$\left| \int_a^b f g^{(n)} - \int_a^b f g \right| \leq \left| \int_a^b g^{(n)}df - \int_a^b gdf \right| + |(f^{(n)})(b) - f(b))g(b)|$$

$$+ |(f^{(n)})(a) - f(a))g(a)| + |S(f,g^{(n)}) - S(f,g)| \leq \varepsilon.$$  

Hence, $\lim_{n \to \infty} \int_a^b f g^{(n)} = \int_a^b f g$. \qed

Hence, we also have the following theorem.

**Theorem 7.7.** If $\{f^{(n)}\}$ and $\{g^{(n)}\}$ are two-norm convergent to $f$ and $g$ in $BV_{\varphi}[a,b]$ and $BV_{\psi}[a,b]$, respectively, with $\sum_{m=1}^{\infty} \varphi^{-1}(1/m)\psi^{-1}(1/m) < \infty$, then $\int_a^b f g$ exists and

$$\lim_{n \to \infty} \int_a^b f^{(n)} g^{(n)} = \int_a^b f g.$$  

References


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