CHOVER-TYPE LAWS OF THE ITERATED LOGARITHM FOR WEIGHTED SUMS OF NA SEQUENCES

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Abstract. To derive a Baum-Katz type result, a Chover-type law of the iterated logarithm is established for weighted sums of negatively associated (NA) and identically distributed random variables with a distribution in the domain of a stable law in this paper.

Keywords: negatively associated sequence, laws of the iterated logarithm, weighted sum, stable law, Rosentall maximal inequality

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1. Introduction

Let \{X_j, j \geq 1\} be independently identically distributed (i.i.d.) with symmetric stable distributions. And let these distributions belong to the domain of normal attraction and non-degeneration. So, their characteristic functions are of the forms:

$$E \exp(itX_j) = \exp(-|t|^\alpha), \ t \in \mathbb{R}, \ j \geq 1.$$  

Chover (1966) has obtained that

$$(1.1) \limsup_{n \to \infty} \left( n^{-1/\alpha} \left| \sum_{j=1}^{n} X_j \right| \right)^{1/\log \log n} = e^{1/\alpha} \ a.s.$$  

We call it Chover-type LIL (Laws of the iterated logarithm). This type of LIL has been shown by Vasudeva and Divanji [11], Zinchenko [13] for delayed sums, by Chen and Huang [2] for geometric weighted sums, and by Chen [1] for weighted sums. Note that Qi and Cheng [9] extended the Chover-type law of the iterated logarithm for the partial sums to the case when the underlying distribution is in the domain of attraction of a non-symmetric stable distribution (see below for details).
Let $L_\alpha$ denote a stable distribution with exponent $\alpha \in (0, 2)$. Recall that the distribution of $X$ is said to be in the domain of attraction of $L_\alpha$ if there exist constants $A_n \in \mathbb{R}$ and $B_n > 0$ such that

$$
\frac{\sum_{j=1}^{n} X_j - A_n}{B_n} \xrightarrow{d} L_\alpha.
$$

Assuming (1.2), Qi and Cheng (1996) and Peng and Qi (2003) showed that

$$
\lim \sup_{n \to \infty} \left( B_n^{-1} \sum_{j=1}^{n} X_j - A_n \right)^{1/\log \log n} = e^{1/\alpha} \text{ a.s.}
$$

It is well known that (1.2) holds if and only if

$$
1 - F(x) = \frac{C_1(x)}{x^\alpha}, \quad F(-x) = \frac{C_2(x)}{x^\alpha}, \quad x > 0,
$$

where $F(x)$ denotes a stable distribution with exponent $\alpha \in (0, 2)$ for $x > 0$, $C_i(x) \geq 0$, $\lim_{x \to \infty} C_i(x) = C_i, i = 1, 2$, $C_1 + C_2 > 0$, and $l(x) \geq 0$ is a slowly varying in the sense of Karamata function, i.e.,

$$
\lim_{t \to \infty} \frac{l(tx)}{l(t)} = 1 \text{ for } x > 0.
$$

According to Lin (1999, page 76, Exercise 21), we have $B_n = (nl(n))^{1/\alpha}$.

As for negatively associated (NA) random variables, Joag (1983) gave the following definition.

**Definition** (Joag, 1983). A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets $T_1$ and $T_2$ of $\{1, 2, \ldots, n\}$, we have

$$
\text{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,
$$

whenever $f_1$ and $f_2$ are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

To derive a Baum-Katz type result, the main purpose of this paper is to establish a Chover-type law of the iterated logarithm for weighted sums of NA and indentically distributed random variables with a distribution in the domain of a stable law.

Throughout this paper, let $h \in B[0, 1]$ denote that a function $h$ is bounded on $[0, 1]$. Further, $C$ will represent a positive constant though its value may change from one appearance to another, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$. 

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In order to prove our results, we need the following lemma and definition.

**Lemma 2.1** (Shao, 2000). Let \( \{X_i, i \geq 1\} \) be a sequence of NA random variables, \( \mathbb{E}X_i = 0, \mathbb{E}|X_i|^p < \infty \) for some \( p \geq 2 \) and for every \( i \geq 1 \). Then there exists \( C = C(p) \), such that

\[
\mathbb{E} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right|^p \leq C \left\{ \sum_{i=1}^{n} \mathbb{E}|X_i|^p + \left( \sum_{i=1}^{n} \mathbb{E}X_i^2 \right)^{p/2} \right\}.
\]

**Definition** (Lin and Lu, 1997). A function \( f(x) > 0 \) (\( x > 0 \)) is said to be quasi-monotone non-decreasing, if

\[
\limsup_{x \to \infty} \sup_{0 \leq t \leq x} \frac{f(t)}{f(x)} < \infty.
\]

Now we state the main results and their proofs.

**Theorem 1.** Let \( \{X, X_i, i \geq 1\} \) be an NA sequence of identically distributed random variables with distribution \( F(x) \), where \( F(x) \) denotes a stable distribution with exponent \( \alpha \in (0, 2) \). Let \( h \) be a bounded function on \([0, 1]\), \( S_n = \sum_{i=1}^{n} h(i/n)X_i \). We have \( \mathbb{E}X = 0, \alpha > 1 \). Let \( f(x) > 0 \) be quasi-monotone non-decreasing and \( \int_{1}^{\infty} 1/(xf(x)) \, dx < \infty \). \( l(x) > 0 \) is a slowly varying in the sense of Karamata function, \( \sup l(a_n)/l(n) < \infty \), where \( a_n = (nf(n)l(n))^{1/\alpha} \). Then under condition (1.2), for any \( \varepsilon > 0 \), we have

\[
\sum_{n=1}^{\infty} n^{-1} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon (nf(n)l(n))^{1/\alpha} \right) < \infty.
\]

**Proof of Theorem 1.** For any \( i \geq 1 \), define \( X_i^{(n)} = X_i I(|X_i| \leq a_n) \),

\[
S_j^{(n)} = \sum_{i=1}^{j} (h(i/n)X_i^{(n)} - \mathbb{E}h(i/n)X_i^{(n)}),
\]

where \( a_n = (nf(n)l(n))^{1/\alpha} \). Then for any \( \varepsilon > 0 \), we have

\[
P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon a_n \right) \leq P \left( \max_{1 \leq j \leq n} |X_j| > a_n \right)
\]

\[
+ P \left( \max_{1 \leq j \leq n} |S_j^{(n)}| > \varepsilon a_n - \max_{1 \leq j \leq n} \sum_{i=1}^{j} \mathbb{E}h(i/n)X_i^{(n)} \right).
\]
First we show that

\[(2.3) \quad \frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E} h(i/n) X_i^{(n)} \right| \to 0, \text{ as } n \to \infty.\]

Let us consider two cases, (i) when $0 < \alpha \leq 1$, notice that $h \in B[0,1]$. Then for any positive integers $n, N$,

\[
\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E} h(i/n) X_i^{(n)} \right| \leq \frac{1}{a_n} \sum_{i=1}^{n} \mathbb{E} |h(i/n) X_i^{(n)}| \\
\leq C \frac{a_n}{a_n} \int_{|x| \leq a_n} |x| dF(x) \leq \frac{C_n}{a_n} a_N + \frac{C_n}{a_n} \int_{a_N < |x| \leq a_n} |x| dF(x) \\
= : C(A + B).
\]

Notice that $f(x) > 0$ is quasi-monotone non-decreasing and (1.3) holds. We have for $n \geq N$, $N$ large enough,

\[
B = \frac{n}{a_n} \sum_{k=N+1}^{n} \int_{a_{k-1} < |x| \leq a_k} |x| dF(x) \leq \frac{n}{a_n} \sum_{k=N+1}^{n} a_k P(a_{k-1} < |X| \leq a_k) \\
\leq C \sum_{k=N+1}^{\infty} k P(a_{k-1} < |X| \leq a_k) \leq C P(|X| \geq a_N) + C \sum_{k=N}^{\infty} P(|X| \geq a_k) \\
\leq C \frac{1}{f(N)} + C \frac{1}{k f(k)} \leq C \frac{1}{f(N)} + C \int_{N}^{\infty} \frac{dx}{k f(k)} \leq \frac{\epsilon}{4}.
\]

It is obvious that for each given $N$,

\[
A \leq C \frac{a_N}{(f(n))^{1/\alpha}} \to 0, \quad n \to \infty.
\]

So, for $0 < \alpha \leq 1$, we have (2.3).

(ii) When $1 < \alpha < 2$, using $EX_i = 0$, $h \in B[0,1]$ and (1.3), when $n \to \infty$, then

\[
\frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E} h(i/n) X_i^{(n)} \right| = \frac{1}{a_n} \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \mathbb{E} h(i/n) X_i I(|X_i| > a_n) \right| \\
\leq \frac{1}{a_n} \sum_{i=1}^{n} \mathbb{E} |h(i/n) X_i| I(|X_i| > a_n) \leq \frac{C_n}{a_n} \mathbb{E} |X| I(|X| > a_n) \\
= \frac{C_n}{a_n} \int_{a_n}^{\infty} P(|X| \geq x) dx = \frac{C_n}{a_n} \int_{a_n}^{\infty} \frac{cl(x)}{x^{\alpha}} dx \\
= \frac{n}{a_n} C a_n^{1-\alpha} ((a_n) \leq \frac{C}{f(n)} < \frac{\epsilon}{2}.
\]
So, for $1 < \alpha < 2$, we also have (2.3). Further, (i) and (ii) imply (2.3).

By (2.2) and (2.3), we have that

$$P\left(\max_{1 \leq j \leq n} |S_j| > \varepsilon a_n\right) \leq \sum_{j=1}^{n} P(|X_j| > a_n) + P\left(\max_{1 \leq j \leq n} |S_j^{(n)}| > \frac{\varepsilon}{2} a_n\right),$$

for $n$ large enough. Hence we need only to prove

(2.4) \[ I := \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|X_j| > a_n) < \infty, \]

(2.5) \[ II := \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |S_j^{(n)}| > \frac{\varepsilon}{2} a_n\right) < \infty. \]

From (1.3), it is easily seen that

(2.6) \[ I = \sum_{n=1}^{\infty} P(|X| > a_n) \leq \sum_{n=1}^{\infty} \frac{C}{nf(n)} \leq C \int_{1}^{\infty} \frac{dx}{xf(x)} < \infty. \]

Lemma 2.1 and the fact that $h \in B[0, 1]$ imply that

(2.7) \[
II \leq C \sum_{n=1}^{\infty} n^{-1} E \max_{1 \leq j \leq n} |S_j^{(n)}|^2 a_n^2 \leq C \sum_{n=1}^{\infty} n^{-1} \frac{1}{a_n^2} \left(\sum_{i=1}^{n} E |h(i/n)X_i^{(n)}|^2\right)
\]
\[
\leq C \sum_{n=1}^{\infty} \frac{1}{a_n^2} E|X|^2 I(|X| \leq a_n) = C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \int_{0}^{a_n} x^2 dF(x)
\]
\[
= C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} x^2 dF(x) \leq C \sum_{k=1}^{\infty} a_k^2 P(a_{k-1} < |X| \leq a_k) \sum_{n=k}^{\infty} \frac{1}{a_n^2} 
\]
\[
\leq C \sum_{k=1}^{\infty} kP(a_{k-1} < |X| \leq a_k) \leq C \int_{1}^{\infty} \frac{dx}{xf(x)} < \infty.
\]

Now we complete the proof of Theorem 1.

**Corollary 1.** Under the conditions of Theorem 1, we have

(2.8) \[ \limsup_{n \to \infty} \left(\frac{|S_n|}{B_n}\right)^{1/\log \log n} \leq e^{1/\alpha} \text{ a.s.} \]

**Proof of Corollary 1.** Notice that for any positive integer $n$ there exists a non-negative integer $k$, such that $2^k \leq n < 2^{k+1}$. And there exists a $t \in [0, 1)$, such
that \( n = 2^{k+t} \). Using (2.1), we obtain

\[
\sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1} - 1} (2^{k+1} - 1)^{-1} P \left( \max_{1 \leq j \leq 2^{k+t}} |S_j| > \varepsilon (2^{k+1} f(2^{k+t}) l((2^{k+t}))^{1/\alpha}) \right) < \infty.
\]

Then

\[
\sum_{k=0}^{\infty} P \left( \max_{1 \leq j \leq 2^{k+t}} |S_j| > \varepsilon (2^{k+1} f(2^{k+t}) l((2^{k+t}))^{1/\alpha}) \right) < \infty,
\]

and consequently

\[
\frac{\max_{1 \leq j \leq 2^{k+t}} |S_j|}{(2^{k+1} f(2^{k+t}) l((2^{k+t}))^{1/\alpha})} \rightarrow 0 \text{ a.s.}
\]

Then

\[
\limsup_{n \to \infty} \frac{|S_n|}{(nf(n)l(n))^{1/\alpha}} = 0 \text{ a.s.}
\]

Given \( \varepsilon > 0 \), let \( f(x) = \log^{1+\varepsilon} x \). It is obvious that \( \int_1^{\infty} 1/(xf(x)) \, dx < \infty \). By (2.9), we have

\[
\limsup_{n \to \infty} \frac{|S_n|}{(n l(n) \log^{1+\varepsilon} n)^{1/\alpha}} = 0 \text{ a.s.}
\]

Then

\[
\limsup_{n \to \infty} \left( \frac{|S_n|}{B(n)} \right)^{1/\log \log n} \leq e^{(1+\varepsilon)/\alpha} \text{ a.s.}
\]

Therefore

\[
\limsup_{n \to \infty} \left( \frac{|S_n|}{B(n)} \right)^{1/\log \log n} \leq e^{1/\alpha} \text{ a.s.}
\]

Now we complete the proof of (2.8). \( \square \)

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References


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