DOMINATION WITH RESPECT TO NONDEGENERATE AND HEREDITARY PROPERTIES

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Abstract. For a graphical property $\mathcal{P}$ and a graph $G$, a subset $S$ of vertices of $G$ is a $\mathcal{P}$-set if the subgraph induced by $S$ has the property $\mathcal{P}$. The domination number with respect to the property $\mathcal{P}$, is the minimum cardinality of a dominating $\mathcal{P}$-set. In this paper we present results on changing and unchanging of the domination number with respect to the nondegenerate and hereditary properties when a graph is modified by adding an edge or deleting a vertex.

Keywords: domination, independent domination, acyclic domination, good vertex, bad vertex, fixed vertex, free vertex, hereditary graph property, induced-hereditary graph property, nondegenerate graph property, additive graph property

MSC 2000: 05C69

1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [8]. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. The complement of a graph $G$ is denoted by $\overline{G}$. For a vertex $x$ of $G$, $N(x, G)$ denotes the set of all neighbors of $x$ in $G$ and $N[x, G] = N(x, G) \cup \{x\}$. The complete graph on $m$ vertices is denoted by $K_m$.

For a graph $G$, let $x \in X \subseteq V(G)$. A vertex $y$ is a private neighbor of $x$ with respect to $X$ if $N[y, G] \cap X = \{x\}$. The private neighbor set of $x$ with respect to $X$ is $\text{pn}_G[x, X] = \{y : N[y, G] \cap X = \{x\}\}$.

Let $\mathcal{G}$ denote the set of all mutually nonisomorphic graphs. A graph property is any non-empty subset of $\mathcal{G}$. We say that a graph $G$ has the property $\mathcal{P}$ whenever
there exists a graph $H \in \mathcal{P}$ which is isomorphic to $G$. For example, we list some graph properties:

- $\mathcal{I} = \{ H \in \mathcal{G} : H \text{ is totally disconnected}\}$;
- $\mathcal{C} = \{ H \in \mathcal{G} : H \text{ is connected}\}$;
- $\mathcal{T} = \{ H \in \mathcal{G} : H \text{ is without isolates}\}$;
- $\mathcal{F} = \{ H \in \mathcal{G} : H \text{ is a forest}\}$;
- $\mathcal{UK} = \{ H \in \mathcal{G} : \text{each component of } H \text{ is complete}\}$.

A graph property $\mathcal{P}$ is called hereditary (induced-hereditary), if from the fact that a graph $G$ has the property $\mathcal{P}$, it follows that all subgraphs (induced subgraphs) of $G$ also belong to $\mathcal{P}$. A property is called additive if it is closed under taking disjoint unions of graphs. A property $\mathcal{P}$ is called nondegenerate if $I \subseteq \mathcal{P}$. Note that: (a) $\mathcal{I}$ and $\mathcal{F}$ are nondegenerate, additive and hereditary properties; (b) $\mathcal{UK}$ is nondegenerate, additive, induced-hereditary and is not hereditary; (c) $\mathcal{C}$ is neither additive nor induced-hereditary nor nondegenerate; (d) $\mathcal{T}$ is additive but neither induced-hereditary nor nondegenerate. Further, an additive and induced-hereditary property is always nondegenerate.

A dominating set for a graph $G$ is a set of vertices $D \subseteq V(G)$ such that every vertex of $G$ is either in $D$ or is adjacent to an element of $D$. A dominating set $D$ is a minimal dominating set if no set $D' \subsetneq D$ is a dominating set. The set of all minimal dominating sets of a graph $G$ is denoted by $\text{MDS}(G)$. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality taken over all dominating sets of $G$. The upper domination number $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of $G$.

Any set $S \subseteq V(G)$ such that the subgraph $(S, G)$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set. The concept of domination with respect to any property $\mathcal{P}$ was introduced by Goddard et al. [7]. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_\mathcal{P}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of $G$. Note that there may be no dominating $\mathcal{P}$-set of $G$ at all. For example, all graphs having at least two isolated vertices are without dominating $\mathcal{P}$-sets, where $\mathcal{P} \in \{\mathcal{C}, \mathcal{T}\}$. On the other hand, if a property $\mathcal{P}$ is nondegenerate then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_\mathcal{P}(G)$ exists. Let $S$ be a dominating $\mathcal{P}$-set of a graph $G$. Then $S$ is a minimal dominating $\mathcal{P}$-set if no set $S' \subsetneq S$ is a dominating $\mathcal{P}$-set. The set of all minimal dominating $\mathcal{P}$-sets of a graph $G$ is denoted by $\text{MD}_\mathcal{P}S(G)$. The upper domination number with respect to the property $\mathcal{P}$, denoted by $\Gamma_\mathcal{P}(G)$, is the maximum cardinality of a minimal dominating $\mathcal{P}$-set of $G$. Michalak [12] has considered these parameters when the property is additive and induced-hereditary. Note that:

(a) in the case $\mathcal{P} = \mathcal{G}$ we have $\text{MD}_\mathcal{G}S(G) = \text{MDS}(G)$, $\gamma_\mathcal{G}(G) = \gamma(G)$ and $\Gamma_\mathcal{G}(G) = \Gamma(G)$;
(b) in the case $\mathcal{P} = \mathcal{I}$, every element of $\text{MD}_\mathcal{I} S(G)$ is an independent dominating set and the numbers $\gamma_\mathcal{I}(G)$ and $\Gamma_\mathcal{I}(G)$ are well known as the independent domination number $\gamma(G)$ and the independence number $\beta_0(G)$;

(c) in the case $\mathcal{P} = \mathcal{C}$, every element of $\text{MD}_\mathcal{C} S(G)$ is a connected dominating set of $G$, $\gamma_\mathcal{C}(G)$ ($\Gamma_\mathcal{C}(G)$) is denoted by $\gamma_c(G)$ ($\Gamma_c(G)$) and is called the connected (upper connected) domination number;

(d) in the case $\mathcal{P} = \mathcal{T}$, every element of $\text{MD}_\mathcal{T} S(G)$ is a total dominating set of $G$, $\gamma_\mathcal{T}(G)$ ($\Gamma_\mathcal{T}(G)$) is denoted by $\gamma_t(G)$ ($\Gamma_t(G)$) and is called the total (upper total) domination number;

(e) in the case $\mathcal{P} = \mathcal{F}$, every element of $\text{MD}_\mathcal{F} S(G)$ is an acyclic and dominating set of $G$, $\gamma_\mathcal{F}(G)$ ($\Gamma_\mathcal{F}(G)$) is denoted by $\gamma_a(G)$ ($\Gamma_a(G)$) and is called the acyclic (upper acyclic) domination number. The concept of acyclic domination in graphs was introduced by Hedetniemi et al. [10].

From the above definitions we immediately have

**Observation 1.1.** Let $\mathcal{I} \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1 \subseteq \mathcal{G}$ and let $G$ be a graph. Then

1. $[7] \quad \gamma(G) \leq \gamma_{\mathcal{P}_1}(G) \leq \gamma_{\mathcal{P}_2}(G) \leq \gamma(G)$;
2. $[7] \quad \Gamma(G) \geq \Gamma_{\mathcal{P}_1}(G) \geq \Gamma_{\mathcal{P}_2}(G) \geq \beta_0(G)$.

**Observation 1.2.** Let $G$ be a graph, $\mathcal{P} \subseteq \mathcal{G}$ and $\text{MD}_\mathcal{P} S(G) \neq \emptyset$. A dominating $\mathcal{P}$-set $S \subseteq V(G)$ is a minimal dominating $\mathcal{P}$-set if and only if for each nonempty subset $U \subseteq S$ at least one of the following holds:

(a) there is a vertex $v \in (V(G) - S) \cup U$ with $\emptyset \neq N[v, G] \cap S \subseteq U$;

(b) $S - U$ is no $\mathcal{P}$-set.

**Proof.** Assume first that $S \in \text{MD}_\mathcal{P} S(G)$, $\emptyset \neq U \subsetneq S$ and $S_U = S - U$ is a $\mathcal{P}$-set of $G$. Hence some vertex $v$ in $V(G) - S_U$ has no neighbors in $S_U$. If $v \in U$ then $\emptyset \neq N[v, G] \cap S \subseteq U$. Let $v \in V(G) - S$. Since $v$ is not dominated by $S_U$ but is dominated by $S$ it follows that $\emptyset \neq N[v, G] \cap S \subseteq U$. In both cases, condition (a) holds.

For the converse, suppose $S$ is a dominating $\mathcal{P}$-set of $G$ and for each $U$, $\emptyset \neq U \subsetneq S$ one of the two above stated conditions holds. Suppose to the contrary that $S \not\in \text{MD}_\mathcal{P} S(G)$. Then there exists a set $U$, $\emptyset \neq U \subsetneq S$ such that $S_U = S - U$ is a dominating $\mathcal{P}$-set. Since $S_U$ is a $\mathcal{P}$-set, condition (b) does not hold. Since $S_U$ is a dominating set it follows that every vertex of $V(G) - S_U$ has at least one neighbor in $S_U$, that is, condition (a) does not hold. Thus in all cases we have a contradiction.

\[ \Box \]
Corollary 1.3. Let $G$ be a graph, $\mathcal{P} \subseteq \mathcal{G}$ be an induced-hereditary property and $\text{MD}_\mathcal{P} S(G) \neq \emptyset$. A dominating $\mathcal{P}$-set $S \subseteq V(G)$ is a minimal dominating $\mathcal{P}$-set if and only if $\text{pn}_G[u, S] \neq \emptyset$ for each vertex $u \in S$.

This result when $\mathcal{P} = \mathcal{G}$ was proved by Ore [13].

We shall use the term $\gamma_{\mathcal{P}}$-set for a minimal dominating $\mathcal{P}$-set of cardinality $\gamma_{\mathcal{P}}(G)$.

Let $G$ be a graph and $v \in V(G)$. Fricke et al. [5] defined a vertex $v$ to be

(f) $\gamma_{\mathcal{P}}$-good, if $v$ belongs to some $\gamma_{\mathcal{P}}$-set of $G$;

(g) $\gamma_{\mathcal{P}}$-bad, if $v$ belongs to no $\gamma_{\mathcal{P}}$-set of $G$;

Sampathkumar and Neerlagi [16] defined a $\gamma_{\mathcal{P}}$-good vertex $v$ to be

(h) $\gamma_{\mathcal{P}}$-fixed if $v$ belongs to every $\gamma_{\mathcal{P}}$-set;

(i) $\gamma_{\mathcal{P}}$-free if $v$ belongs to some $\gamma_{\mathcal{P}}$-set but not to all $\gamma_{\mathcal{P}}$-sets.

For a graph $G$ and a property $\mathcal{P} \subseteq \mathcal{G}$ such that $\text{MD}_\mathcal{P} S(G) \neq \emptyset$ we define:

$\mathcal{G}_\mathcal{P}(G) = \{x \in V(G): x \text{ is } \gamma_{\mathcal{P}}\text{-good}\}$;

$\mathcal{B}_\mathcal{P}(G) = \{x \in V(G): x \text{ is } \gamma_{\mathcal{P}}\text{-bad}\}$;

$\mathcal{F}_\mathcal{P}(G) = \{x \in V(G): x \text{ is } \gamma_{\mathcal{P}}\text{-fixed}\}$;

$\mathcal{Fr}_\mathcal{P}(G) = \{x \in V(G): x \text{ is } \gamma_{\mathcal{P}}\text{-free}\}$.

Clearly $\{\mathcal{G}_\mathcal{P}(G), \mathcal{B}_\mathcal{P}(G)\}$ is a partition of $V(G)$, and $\{\mathcal{F}_\mathcal{P}(G), \mathcal{Fr}_\mathcal{P}(G)\}$ is a partition of $\mathcal{G}_\mathcal{P}(G)$. If additionally $\text{MD}_\mathcal{P} S(G - v) \neq \emptyset$ for each vertex $v \in V(G)$, then we define:

$\mathcal{V}^0_{\mathcal{P}}(G) = \{x \in V(G): \gamma_\mathcal{P}(G - x) = \gamma_\mathcal{P}(G)\}$;

$\mathcal{V}^-_{\mathcal{P}}(G) = \{x \in V(G): \gamma_\mathcal{P}(G - x) < \gamma_\mathcal{P}(G)\}$;

$\mathcal{V}^+_{\mathcal{P}}(G) = \{x \in V(G): \gamma_\mathcal{P}(G - x) > \gamma_\mathcal{P}(G)\}$.

In this case $\{\mathcal{V}^-_{\mathcal{P}}(G), \mathcal{V}^0_{\mathcal{P}}(G), \mathcal{V}^+_{\mathcal{P}}(G)\}$ is a partition of $V(G)$.

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In this connection, in this paper we consider this question in the case $\gamma_{\mathcal{P}}(G)$ when a vertex is deleted from $G$ or an edge from $\overline{G}$ is added to $G$.

2. Vertex deletion

In this section we examine the effects on $\gamma_{\mathcal{P}}$ when a graph is modified by deleting a vertex.

Theorem 2.1. Let $G$ be a graph, $u, v \in V(G)$, $u \neq v$ and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under union with $K_1$.

(i) Let $v \in \mathcal{V}^-_{\mathcal{H}}(G)$.

(i.1) If $uv \in E(G)$ then $u$ is a $\gamma_{\mathcal{H}}$-bad vertex of $G - v$;
(i.2) if $M$ is a $\gamma_\mathcal{H}$-set of $G - v$ then $M \cup \{v\}$ is a $\gamma_\mathcal{H}$-set of $G$ and $\{v\} = p_n_G[v, M \cup \{v\}]$;

(i.3) $\gamma_\mathcal{H}(G - v) = \gamma_\mathcal{H}(G) - 1$;

(ii) let $v \in V^+_\mathcal{H}(G)$. Then $v$ is a $\gamma_\mathcal{H}$-fixed vertex of $G$;

(iii) if $v \in V^-_\mathcal{H}(G)$ and $u$ is a $\gamma_\mathcal{H}$-fixed vertex of $G$ then $uv \notin E(G)$;

(iv) if $v$ is a $\gamma_\mathcal{H}$-bad vertex of $G$ then $\gamma_\mathcal{H}(G - v) = \gamma_\mathcal{H}(G)$;

(v) if $v \in V^-_\mathcal{H}(G)$ and $uv \in E(G)$ then $\gamma_\mathcal{H}(G - \{u, v\}) = \gamma_\mathcal{H}(G) - 1$.

Proof. (i.1): Let $uv \in E(G)$ and let $M$ be a $\gamma_\mathcal{H}$-set of $G - v$. If $u \in M$ then $M$ is a dominating $\mathcal{H}$-set of $G$ with $|M| < \gamma_\mathcal{H}(G)$—a contradiction.

(i.2) and (i.3): Let $M$ be a $\gamma_\mathcal{H}$-set of $G - v$. By (i.1), $M_1 = M \cup \{v\}$ is a dominating set of $G$. Any vertex $u \in V(G) - M_1$ has a neighbor in $M$, hence $v$ is isolated in $M_1$ (otherwise $M$ would dominate $G$) and $\{v\} = p_n_G[v, M \cup \{v\}]$. Since $\mathcal{H}$ is closed under union with $K_1$ it follows that $M_1$ is a dominating $\mathcal{H}$-set of $G$ and $|M_1| = \gamma_\mathcal{H}(G - v) + 1 \leq \gamma_\mathcal{H}(G)$. Hence $M_1$ is a $\gamma_\mathcal{H}$-set of $G$ and $\gamma_\mathcal{H}(G - v) = \gamma_\mathcal{H}(G) - 1$.

(ii): If $M$ is a $\gamma_\mathcal{H}$-set of $G$ and $v \notin M$ then $M$ is a dominating $\mathcal{H}$-set of $G - v$. But then $\gamma_\mathcal{H}(G) = |M| \geq \gamma_\mathcal{H}(G - v) > \gamma_\mathcal{H}(G)$ and the result follows.

(iii): Let $\gamma_\mathcal{H}(G - v) < \gamma_\mathcal{H}(G)$ and let $M$ be a $\gamma_\mathcal{H}$-set of $G - v$. Then by (i.2), $M \cup \{v\}$ is a $\gamma_\mathcal{H}$-set of $G$. This implies that $u \in M$ and by (i.1) we have $uv \notin E(G)$.

(iv): By (ii), $\gamma_\mathcal{H}(G - v) \leq \gamma_\mathcal{H}(G)$ and by (i.2), $\gamma_\mathcal{H}(G - v) \geq \gamma_\mathcal{H}(G)$.

(v): Immediately follows by (i) and (iv). □

Let $\mathcal{P} \subseteq \mathcal{G}$ be nondegenerate and closed under union with $K_1$. Since $\gamma_\mathcal{P}(G - v) \leq |V(G)| - 1$ for every $v \in V(G)$ and because of Theorem 2.1 we have $\gamma_\mathcal{P}(G - v) = \gamma_\mathcal{P}(G) + p$, where $p \in \{-1, 0, 1, \ldots, |V(G)| - 2\}$. This motivated us to define for a nontrivial graph $G$:

$\textbf{Fr}^-_\mathcal{P}(G) = \{x \in \textbf{Fr}_\mathcal{P}(G) : \gamma_\mathcal{P}(G - x) = \gamma_\mathcal{P}(G) - 1\}$;

$\textbf{Fr}^0_\mathcal{P}(G) = \{x \in \textbf{Fr}_\mathcal{P}(G) : \gamma_\mathcal{P}(G - x) = \gamma_\mathcal{P}(G)\}$;

$\textbf{Fi}^-_\mathcal{P}(G) = \{x \in \textbf{Fi}_\mathcal{P}(G) : \gamma_\mathcal{P}(G - x) = \gamma_\mathcal{P}(G) + p\}, \ p \in \{-1, 0, 1, \ldots, |V(G)| - 2\}$.

We will refine the definitions of the $\gamma_\mathcal{P}$-free vertex and the $\gamma_\mathcal{P}$-fixed vertex. Let $G$ be a graph and let $\mathcal{P} \subseteq \mathcal{G}$ be nondegenerate and closed under union with $K_1$. A vertex $x \in V(G)$ is called

(j) $\gamma^0_\mathcal{P}$-free if $x \in \textbf{Fr}^0_\mathcal{P}(G)$;

(k) $\gamma^-_\mathcal{P}$-free if $x \in \textbf{Fr}^-_\mathcal{P}(G)$;

(l) $\gamma^q_\mathcal{P}$-fixed if $x \in \textbf{Fi}^q_\mathcal{P}(G)$, where $q \in \{-1, 0, 1, \ldots, |V(G)| - 2\}$.

Now, by Theorem 2.1 we have
Corollary 2.2. Let $G$ be a graph of order $n \geq 2$ and let $\mathcal{H} \subseteq G$ be nondegenerate and closed under union with $K_1$. Then

1. $\{\text{Fr}^-_{\mathcal{H}}(G), \text{Fr}^0_{\mathcal{H}}(G)\}$ is a partition of $\text{Fr}_{\mathcal{H}}(G)$;
2. $\{\text{Fr}^1_{\mathcal{H}}(G), \text{Fr}^2_{\mathcal{H}}(G), \ldots, \text{Fr}^{n-2}_{\mathcal{H}}(G)\}$ is a partition of $\text{Fr}_{\mathcal{H}}(G)$;
3. $\{\text{Fr}^1_{\mathcal{H}}(G), \text{Fr}^2_{\mathcal{H}}(G)\}$ is a partition of $\text{Fr}^-_{\mathcal{H}}(G)$;
4. $\{\text{Fr}^1_{\mathcal{H}}(G), \text{Fr}^2_{\mathcal{H}}(G), B_{\mathcal{H}}(G)\}$ is a partition of $\text{Fr}^+_\mathcal{H}(G)$;
5. $\{\text{Fr}^1_{\mathcal{H}}(G), \text{Fr}^2_{\mathcal{H}}(G), \ldots, \text{Fr}^{n-2}_{\mathcal{H}}(G)\}$ is a partition of $\text{Fr}^+_\mathcal{H}(G)$.

A vertex $v$ of a graph $G$ is $\mathcal{P}$-critical if $\gamma_{\mathcal{P}}(G - v) \neq \gamma_{\mathcal{P}}(G)$. The graph $G$ is vertex-$\mathcal{P}$-critical if all its vertices are $\mathcal{P}$-critical.

Theorem 2.3. Let $G$ be a graph of order $n \geq 2$ and let $\mathcal{H} \subseteq G$ be additive and induced-hereditary. Then $G$ is a vertex-$\mathcal{H}$-critical graph if and only if $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$ for all $v \in V(G)$.

Proof. Necessity is obvious. Sufficiency: Let $G$ be a vertex-$\mathcal{H}$-critical graph. Clearly, $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$ for every isolated vertex $v \in V(G)$. Hence if $G$ is isomorphic to $K_n$ then $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$ for all $v \in V(G)$. So, let $G$ have a component of order at least two, say $Q$. Because of Theorem 2.1 (ii), (iii) and (i.3), either $\gamma_{\mathcal{H}}(Q - v) > \gamma_{\mathcal{H}}(Q)$ for all $v \in V(Q)$, or $\gamma_{\mathcal{H}}(Q - v) = \gamma_{\mathcal{H}}(Q) - 1$ for all $v \in V(Q)$. Suppose that $\gamma_{\mathcal{H}}(Q - v) > \gamma_{\mathcal{H}}(Q)$ for all $v \in V(Q)$. But then Theorem 2.1 (ii) implies that $V(Q)$ is a $\mathcal{H}$-set of $Q$. This is a contradiction with $\gamma_{\mathcal{H}}(Q - v) > \gamma_{\mathcal{H}}(Q)$. □

Theorem 2.3 when $\mathcal{H} \in \{G, I, F\}$ is due to Carrington et al. [2], Ao and MacGillivray (see [9, Chapter 16]) and the present author [15], respectively. Further properties of these graphs can be found in [1], [6], [8, Chapter 5], [9, Chapter 16], [11], [14].

Now we concentrate on graphs having cut-vertices. Observe that domination and some of its variants in graphs having cut-vertices have been the topic of several studies—see for example [1], [18], [14] and [9, Chapter 16].

Let $G_1$ and $G_2$ be connected graphs, both of order at least two, and let them have a unique vertex in common, say $x$. Then a coalescence $G_1 \hat{\circ} G_2$ is the graph $G_1 \cup G_2$. Clearly, $x$ is a cut-vertex of $G_1 \hat{\circ} G_2$.

Theorem 2.4. Let $G = G_1 \hat{\circ} G_2$ and let $\mathcal{H} \subseteq G$ be induced-hereditary and closed under union with $K_1$. Then $\gamma_{\mathcal{H}}(G) \geq \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$.

Proof. Since $\mathcal{H}$ is induced-hereditary and closed under union with $K_1$ it follows that $\mathcal{H}$ is nondegenerate. Let $M$ be a $\gamma_{\mathcal{H}}$-set of $G$ and $M_i = M \cap V(G_i)$, $i = 1, 2$. Since $\mathcal{H}$ is induced-hereditary it follows that $M_1$ and $M_2$ are $\mathcal{H}$-sets of $G_1$ and $G_2$, respectively. Hence there exist three possibilities:
(a) $x \notin M$ and $M_i$ is a dominating $\mathcal{H}$-set of $G_i$, $i = 1, 2$;
(b) $x \notin M$ and there are $i, j$ such that $\{i, j\} = \{1, 2\}$, $M_i$ is a dominating $\mathcal{H}$-set of $G_i$ and $M_j$ is a dominating $\mathcal{H}$-set of $G_j - x$;
(c) $x \in M$ and $M_i$ is a dominating $\mathcal{H}$-set of $G_i$, $i = 1, 2$.

If (a) holds, then $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| \geq \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2)$. If (c) holds then $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| - 1 \geq \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$. Finally, let (b) hold. Then $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| \geq \gamma_{\mathcal{H}}(G_i) + \gamma_{\mathcal{H}}(G_j - x)$. Now by Theorem 2.1 (i),

$$\gamma_{\mathcal{H}}(G) \geq \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1.$$  

Thus, in all cases, $\gamma_{\mathcal{H}}(G) \geq \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$.  

\[\Box\]

**Theorem 2.5.** Let $G = G_1 \circ G_2$, let $\mathcal{H} \subseteq \mathcal{G}$ be additive and induced-hereditary, and $\gamma_{\mathcal{H}}(G_1 - x) < \gamma_{\mathcal{H}}(G_1)$. Then

(a) $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$;
(b) if $\gamma_{\mathcal{H}}(G_2 - x) < \gamma_{\mathcal{H}}(G_2)$ then $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G) - 1$;
(c) if $\gamma_{\mathcal{H}}(G_2 - x) > \gamma_{\mathcal{H}}(G_2)$ then $x$ is a $\gamma_{\mathcal{H}}$-fixed vertex of $G$;
(d) if $x$ is a $\gamma_{\mathcal{H}}$-bad vertex of $G_2$ then $x$ is a $\gamma_{\mathcal{H}}$-bad vertex of $G$.

**Proof.** Since $\mathcal{H}$ is additive and induced-hereditary it follows that $\mathcal{H}$ is nondegenerate and closed under union with $K_1$.

(a): Let $U_1$ be a $\gamma_{\mathcal{H}}$-set of $G_1 - x$ and let $U_2$ be a $\gamma_{\mathcal{H}}$-set of $G_2$. Then $U = U_1 \cup U_2$ is a dominating set of $G$. It follows by Theorem 2.1(i.2) that $(U, G)$ has two components, namely $(U_1, G)$ and $(U_2, G)$. Since $\mathcal{H}$ is additive, $U$ is an $\mathcal{H}$-set of $G$. Thus $U$ is a dominating $\mathcal{H}$-set of $G$. Hence $\gamma_{\mathcal{H}}(G) \leq |U_1 \cup U_2| = \gamma_{\mathcal{H}}(G_1 - x) + \gamma_{\mathcal{H}}(G_2) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$. Now the result follows by Theorem 2.4.

(b): By Theorem 2.1 (i.3) we have $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G_1 - x) + \gamma_{\mathcal{H}}(G_2 - x) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 2$. Hence by (a), $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G) - 1$.

(c): $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G_1 - x) + \gamma_{\mathcal{H}}(G_2 - x) = \gamma_{\mathcal{H}}(G_1) - 1 + \gamma_{\mathcal{H}}(G_2 - x) = \gamma_{\mathcal{H}}(G) + \gamma_{\mathcal{H}}(G_2 - x) - \gamma_{\mathcal{H}}(G_2) > \gamma_{\mathcal{H}}(G)$. The result now follows by Theorem 2.1 (ii).

(d): Let $M$ be a $\gamma_{\mathcal{H}}$-set of $G$ and $M_i = M \cap V(G_i)$, $i = 1, 2$. Suppose $x \in M$. Hence $M_i$ is a dominating $\mathcal{H}$-set of $G_i$, $i = 1, 2$ and then $\gamma_{\mathcal{H}}(G_i) \leq |M_i|$. Since $x$ belongs to no $\gamma_{\mathcal{H}}$-set of $G_2$ we have $|M_2| > \gamma_{\mathcal{H}}(G_2)$. Hence $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| - 1 \geq \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2)$—a contradiction with (a). 

\[\Box\]

**Theorem 2.6.** Let $\mathcal{H} \subseteq \mathcal{G}$ be additive and induced-hereditary and let $G = G_1 \circ G_2$, where $G_1$, $G_2$ are both vertex-$\gamma_{\mathcal{H}}$-critical. Then $G$ is vertex-$\gamma_{\mathcal{H}}$-critical and $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$.

**Proof.** By Theorem 2.5(b) it follows that $\gamma_{\mathcal{H}}(G) - 1 = \gamma_{\mathcal{H}}(G - x)$. Let without loss of generality $y \in V(G_2 - x)$. If $G_2 - y$ is connected then $G - y = G_1 \circ (G_2 - y)$ and
by Theorem 2.5(a), \( \gamma_H(G - y) = \gamma_H(G_1) + \gamma_H(G_2 - y) - 1 = \gamma_H(G_1) + \gamma_H(G_2) - 2 = \gamma_H(G) - 1. \)

So, assume \( G_2 - y \) is not connected and let \( Q \) be the component of \( G_2 - y \) which contains \( x \). By Theorem 2.1 (i), \( V(Q) \neq \{x\} \). Now, by Theorem 2.5 (a), \( \gamma_H(G \cup Q) = \gamma_H(G_1) + \gamma_H(Q) - 1 \) and then \( \gamma_H(G - y) = \gamma_H(G_1 \cup Q) + \gamma_H(G_2 - (V(Q) \cup \{y\})) = \gamma_H(G_1) + \gamma_H(G_2 - y) - 1 = \gamma_H(G_1) + \gamma_H(G_2) - 2 = \gamma_H(G) - 1. \)

\[ \Box \]

3. Edge Addition

Here we present results on changing and unchanging of \( \gamma_P(G) \) when an edge from \( G \) is added to \( \bar{G} \). Recall that if a property \( \mathcal{P} \) is hereditary and closed under union with \( K_1 \) then \( \mathcal{P} \) is nondegenerate and hence all graphs have a domination number with respect to \( \mathcal{P} \).

**Theorem 3.1.** Let \( x \) and \( y \) be two different and nonadjacent vertices in a graph \( G \). Let \( \mathcal{H} \subseteq \mathcal{G} \) be hereditary and closed under union with \( K_1 \). If \( \gamma_H(G + xy) < \gamma_H(G) \) then \( \gamma_H(G + xy) = \gamma_H(G) - 1 \). Moreover, \( \gamma_H(G + xy) = \gamma_H(G) - 1 \) if and only if at least one of the following holds:

(i) \( x \in V_{\bar{H}}^{-}(G) \) and \( y \) is a \( H \)-good vertex of \( G - x \);

(ii) \( x \) is a \( H \)-good vertex of \( G - y \) and \( y \in V_{\bar{H}}^{-}(G) \).

**Proof.** Let \( \gamma_H(G + xy) < \gamma_H(G) \) and let \( M \) be a \( H \)-set of \( G + xy \). Since \( \mathcal{H} \) is hereditary, \( M \) is an \( H \)-set of \( G \). Further, \( \{x, y\} \cap M = 1 \), otherwise \( M \) would be a dominating \( H \)-set of \( G \), a contradiction. Let without loss of generality \( x \notin M \) and \( y \in M \). Since \( M \) is an \( H \)-set of \( G \) it follows that \( M \) is no dominating set of \( G \), which implies \( M \cap N(x, G) = \emptyset \). Hence \( M_1 = M \cup \{x\} \) is a dominating \( H \)-set of \( G \) with \( |M_1| = \gamma_H(G + xy) + 1 \), which implies \( \gamma_H(G) = \gamma_H(G + xy) + 1 \). Since \( M \) is a dominating \( H \)-set of \( G - x \) we have \( \gamma_H(G - x) \leq \gamma_H(G + xy) \). Hence \( \gamma_H(G) \geq \gamma_H(G - x) + 1 \) and Theorem 2.1 implies \( \gamma_H(G) = \gamma_H(G - x) + 1 \). Thus \( x \) is in \( V_{\bar{H}}^{-}(G) \) and \( M \) is a \( H \)-set of \( G - x \). Since \( y \in M \), \( y \) is a \( H \)-good vertex of \( G - x \).

For the converse let without loss of generality (i) hold. Then there is a \( \gamma_H \)-set \( M \) of \( G - x \) with \( y \in M \). Certainly \( M \) is a dominating \( H \)-set of \( G + xy \) and consequently \( \gamma_H(G + xy) \leq |M| = \gamma_H(G - x) = \gamma_H(G) - 1 \leq \gamma_H(G + xy) \). \[ \Box \]

**Corollary 3.2.** Let \( x \) and \( y \) be two different and nonadjacent vertices in a graph \( G \), let \( \mathcal{H} \subseteq \mathcal{G} \) be hereditary and closed under union with \( K_1 \), and let \( x \in V_{\bar{H}}^{-}(G) \). Then \( \gamma_H(G) - 1 \leq \gamma_H(G + xy) \leq \gamma_H(G) \).

**Proof.** Let \( M \) be a \( \gamma_H \)-set of \( G - x \). If \( y \in G_H(G - x) \) then Theorem 3.1 yields \( \gamma_H(G) - 1 = \gamma_H(G + xy) \). So, let \( y \in B_H(G - x) \). By Theorem 2.1, \( M_1 = M \cup \{x\} \)
is a $\gamma_H$-set of $G$ and $M_1 \cap N(x, G) = \emptyset$. Hence $M_1$ is a dominating $\mathcal{H}$-set of $G + xy$ and $\gamma_H(G + xy) \leq |M_1| = \gamma_H(G - x) + 1 = \gamma_H(G)$. \qed

We need the following lemma:

**Lemma 3.3.** Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under union with $K_1$ and let $x$ be a $\gamma_H^0$-fixed vertex of a graph $G$. Then $N(x, G) \subseteq B_H(G - x) \cap (V_H^0(G) \cup F^1_H(G))$ and for each $y \in N(x, G)$, $\gamma_H(G - \{x, y\}) = \gamma_H(G)$.

**Proof.** Let $M$ be a $\gamma_H$-set of $G - x$ and $y \in N(x, G)$. If $y \in M$ then $M$ is a dominating $\mathcal{H}$-set of $G$ of cardinality $|M| = \gamma_H(G - x) = \gamma_H(G)$—a contradiction with $x \in F^i_H(G)$. Thus $N(x, G) \subseteq B_H(G - x)$. Now by Theorem 2.1 (iv), $\gamma_H(G - \{x, y\}) = \gamma_H(G - x) = \gamma_H(G)$. Further, Theorem 2.1(iii) implies $y \notin V_H^0(G)$. If $y \notin V_H^0(G)$, from Corollary 2.2(5) it follows that $y \in F^i_H(G)$ for some $p \geq 1$. Assume $p \geq 2$. Since $M$ is a dominating $\mathcal{H}$-set of $G - x$ and $M \cap N(x, G) = \emptyset$ it follows that $M_2 = M \cup \{x\}$ is a dominating $\mathcal{H}$-set of $G$ and $y \notin M_2$. Hence $M_2$ is a dominating $\mathcal{H}$-set of $G - y$. This implies $\gamma_H(G) + p = \gamma_H(G - y) \leq |M_2| = |M| + 1 = \gamma_H(G - x) + 1 = \gamma_H(G) + 1$, a contradiction. \qed

It is a well known fact that $\gamma(G + e) \leq \gamma(G)$ for any edge $e \in \mathcal{G}$. In general, for $\gamma_p$ this is not valid.

**Theorem 3.4.** Let $x$ and $y$ be two different and nonadjacent vertices in a graph $G$ and let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with $K_1$. Then $\gamma_H(G + xy) > \gamma_H(G)$ if and only if no $\gamma_H$-set of $G$ is an $\mathcal{H}$-set of $G + xy$ and one of the following holds:

1. $x$ is a $\gamma_H^0$-fixed vertex of $G$ and $y$ is a $\gamma_H^0$-fixed vertex of $G$ for some $p, q \geq 1$;
2. $x \in F^i_H(G)$ and $y \in F^i_H(G) \cap B_H(G - x)$;
3. $x \in F^i_H(G) \cap B_H(G - y)$ and $y \in F^i_H(G)$;
4. $x, y \in F^i_H(G)$, $x \in B_H(G - y)$ and $y \in B_H(G - x)$.

**Proof.** Let $\gamma_H(G + xy) > \gamma_H(G)$. By Corollary 3.2 we have $x, y \in V_H^0(G) \cup V_H^0(G)$. Assume to the contrary that (without loss of generality) $x \notin F^i_H(G)$. Hence there is a $\gamma_H$-set $M$ of $G$ with $x \notin M$. But then $M$ is a dominating $\mathcal{H}$-set of $G + xy$ and $|M| = \gamma_H(G) < \gamma_H(G + xy)$—a contradiction. Thus both $x$ and $y$ are $\gamma_H$-fixed vertices of $G$. This implies that each $\gamma_H$-set $M$ of $G$ is a dominating set of $G + xy$ but not an $\mathcal{H}$-set of $G + xy$.

Let $x$ be $\gamma_H^0$-fixed, let $y$ be $\gamma_H^0$-fixed and without loss of generality, $q \geq p \geq 0$. Assume (1) does not hold. Hence $p = 0$. Let $M_1$ be a $\gamma_H$-set of $G - x$. Then $|M_1| = \gamma_H(G - x) = \gamma_H(G) < \gamma_H(G + xy)$ and $y \notin M_1$, for otherwise $M_1$ would be a dominating $\mathcal{H}$-set of $G + xy$; thus $y$ is a $\gamma_H$-bad vertex of $G - x$. By Lemma 3.3,
$N(x, G) \cap M_1 = \emptyset$. Then $M_1 \cup \{x\}$ is a dominating $\mathcal{H}$-set of $G + xy$, which implies $\gamma_\mathcal{H}(G + xy) = \gamma_\mathcal{H}(G) + 1$. Since $y \notin M_1 \cup \{x\}$ it follows that $M_1 \cup \{x\}$ is a dominating $\mathcal{H}$-set of $G - y$ and then $\gamma_\mathcal{H}(G) + 1 = |M_1 \cup \{x\}| \geq \gamma_\mathcal{H}(G - y) = \gamma_\mathcal{H}(G) + q$. So, $q \in \{0, 1\}$. If $q = 1$ then (2) holds. If $q = 0$ then, by symmetry, it follows that $x$ is a $\gamma_\mathcal{H}$-bad vertex of $G - y$ and hence (4) holds.

For the converse, let no $\gamma_\mathcal{H}$-set of $G$ be an $\mathcal{H}$-set of $G + xy$ and let one of the conditions (1), (2), (3) and (4) hold. Assume to the contrary that $\gamma_\mathcal{H}(G + xy) \leq \gamma_\mathcal{H}(G)$. By Theorem 3.1, $\gamma_\mathcal{H}(G + xy) = \gamma_\mathcal{H}(G)$. Let $M_2$ be a $\gamma_\mathcal{H}$-set of $G + xy$. Hence $|M_2 \cap \{x, y\}| = 1$—otherwise $M_2$ would be a $\gamma_\mathcal{H}$-set of $G$. Let without loss of generality $x \notin M_2$. Then $M_2$ is a dominating $\mathcal{H}$-set of $G - x$, which implies $\gamma_\mathcal{H}(G - x) \leq |M_2| = \gamma_\mathcal{H}(G + xy) = \gamma_\mathcal{H}(G)$. Thus $\gamma_\mathcal{H}(G - x) = \gamma_\mathcal{H}(G + xy) = \gamma_\mathcal{H}(G)$ and then $M_2$ is a $\gamma_\mathcal{H}$-set of $G - x$. Hence $x$ is a $\gamma_\mathcal{H}$-good vertex of $G$ and $y$ is a $\gamma_\mathcal{H}$-good vertex of $G - x$, which is a contradiction with each of (1)–(4).

By Theorem 3.1 and Theorem 3.4 we immediately obtain:

**Theorem 3.5.** Let $x$ and $y$ be two different and nonadjacent vertices in a graph $G$. Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with $K_1$. Then $\gamma_\mathcal{H}(G + xy) = \gamma_\mathcal{H}(G)$ if and only if at least one of the following holds:

1. $x \in V_{\mathcal{H}}^{-}(G) \cap B_{\mathcal{H}}(G - y)$ and $y \in V_{\mathcal{H}}^{-}(G) \cap B_{\mathcal{H}}(G - x)$;
2. $x \in V_{\mathcal{H}}^{-}(G)$ and $y \in B_{\mathcal{H}}(G - x) - V_{\mathcal{H}}^{-}(G)$;
3. $x \in B_{\mathcal{H}}(G - y) - V_{\mathcal{H}}^{-}(G)$ and $y \in V_{\mathcal{H}}^{-}(G)$;
4. $x, y \notin V_{\mathcal{H}}^{-}(G)$ and $|\{x, y\} \cap \Phi_{\mathcal{H}}(G)| \leq 1$;
5. $x \in F_{\mathcal{H}}^{s\gamma}(G)$ and $y \in F_{\mathcal{H}}^{s\gamma}(G) \cap G_{\mathcal{H}}(G - x)$ for some $s \in \{0, 1\}$;
6. $x \in F_{\mathcal{H}}^{s\gamma}(G) \cap G_{\mathcal{H}}(G - y)$ and $y \in F_{\mathcal{H}}^{s\gamma}(G)$ for some $s \in \{0, 1\}$;
7. $x \in F_{\mathcal{H}}^{s\gamma}(G)$ and $y \in F_{\mathcal{H}}^{s\gamma}(G)$ for some $q \geq 2$;
8. $x \in F_{\mathcal{H}}^{s\gamma}(G)$ and $y \in F_{\mathcal{H}}^{s\gamma}(G)$ for some $q \geq 2$;
9. there is a $\gamma_\mathcal{H}$-set of $G$ which is an $\mathcal{H}$-set of $G + xy$ and one of the conditions (1), (2), (3) and (4) stated in Theorem 3.4 holds.

**Corollary 3.6.** Let $x$ and $y$ be two different and nonadjacent vertices in a graph $G$. Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with $K_1$. If $x \in B_{\mathcal{H}}(G)$ then $\gamma_\mathcal{H}(G + xy) = \gamma_\mathcal{H}(G)$.

**Proof.** By Theorem 2.1 (iv), $x \notin V_{\mathcal{H}}^{-}(G)$. If $y \notin V_{\mathcal{H}}^{-}(G)$ then the result follows by Theorem 3.5(4). If $y \in V_{\mathcal{H}}^{-}(G)$ then by Theorem 2.1 (i,2) we have $x \in B_{\mathcal{H}}(G - y)$ and the result now follows by Theorem 3.5(3). □

Let $\mu \in \{\gamma, \gamma_c, i\}$. A graph $G$ is edge-$\mu$-critical if $\mu(G + e) < \mu(G)$ for every edge $e$ not belonging to $G$. These concepts were introduced by Sumner and Blitch [17], Xue-Gang Chen et al. [3] and Ao and MacGillivray [9, Chapter 16], respectively.
Here we define a graph $G$ to be edge-$\gamma_P$-critical if $\gamma_P(G + e) \neq \gamma_P(G)$ for every edge $e$ of $\mathcal{G}$, where $\mathcal{P} \subseteq \mathcal{G}$ is hereditary and closed under union with $K_1$. Relating edge addition and vertex removal, Sumner and Blitch [17] and Ao and MacGillivray showed that $V_+^P(G)$ is empty for $P \in \{G, T\}$, respectively. Furthermore, Favaron et al. [4] showed that if $V_0^G(G) \neq \emptyset$ then $\langle V_0^G(G), G \rangle$ is complete. In general, for edge-$\gamma_P$-critical graphs the following holds.

**Theorem 3.7.** Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with $K_1$ and let $G$ be an edge-$\gamma_H$-critical graph. Then

1. $V(G) = \text{Fi}_{\mathcal{H}}^{-1}(G) \cup \text{Fr}_{\mathcal{H}}(G)$ and if $\text{Fr}_{\mathcal{H}}^0(G) \neq \emptyset$ then $\langle \text{Fr}_{\mathcal{H}}^0(G), G \rangle$ is complete;
2. $\gamma_{\mathcal{H}}(G + e) < \gamma_{\mathcal{H}}(G)$ for every edge $e$ not belonging to $G$.

**Proof.** (1) If $x, y \in \text{Fr}_{\mathcal{H}}^0(G)$ and $xy \notin E(G)$ then Theorem 3.5(4) implies $\gamma_{\mathcal{H}}(G + xy) = \gamma_{\mathcal{H}}(G)$, a contradiction. If $x \in \mathcal{B}_{\mathcal{H}}(G)$ then Corollary 3.6 implies $N[x, G] = V(G)$ and hence $\{x\}$ is a $\gamma_{\mathcal{H}}$-set of $G$—a contradiction. Assume $x \in \text{Fi}_{\mathcal{H}}^{q-1}(G)$ for some $q \geq 0$. Let $M$ be any $\gamma_{\mathcal{H}}$-set of $G$. By Corollary 1.3, $\text{pn}_G[x, M] \neq \emptyset$. If $\text{pn}_G[x, M] = \{x\}$ then $M - \{x\}$ dominates $G - x$, so $x \in V_{-}^\mathcal{H}(G)$—a contradiction. Hence there is $y \in \text{pn}_G[x, M] - \{x\}$. Since $\text{pn}_G[x, M] \cap V_{-}^\mathcal{H}(G) = \emptyset$ (by Theorem 2.1 (iii)), $\mathcal{B}_{\mathcal{H}}(G) = \emptyset$ and $y \notin M$, it follows that $y \in \text{Fr}_{\mathcal{H}}^0(G)$. Let $M_1$ be a $\gamma_{\mathcal{H}}$-set of $G$ and $y \in M_1$. Then there is $z \in (\text{pn}_G[x, M_1] - \{x\}) \cap \text{Fr}_{\mathcal{H}}^0(G)$. Hence $y, z \in \text{Fr}_{\mathcal{H}}^0(G)$ and $yz \notin E(G)$—a contradiction. Thus $\text{Fi}_{\mathcal{H}}(G) = \text{Fi}_{\mathcal{H}}^{-1}(G)$ and the result follows.

(2) This immediately follows by (1) and Theorem 3.4.

**References**


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