TIETZE EXTENSION THEOREM FOR PAIRWISE ORDERED FUZZY EXTREMALLY DISCONNECTED SPACES

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Abstract. In this paper a new class of fuzzy topological spaces called pairwise ordered fuzzy extremally disconnected spaces is introduced. Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces has been discussed as in the paper of Kubiak (1987) besides proving several other propositions and lemmas.

Keywords: pairwise ordered fuzzy extremally disconnected space, ordered $T_1$-fuzzy continuous function, lower (upper) $T_1$-fuzzy continuous functions

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1. Introduction and preliminaries

The fuzzy concept has invaded almost all branches of Mathematics since the introduction of the concept by Zadeh [11]. Fuzzy sets have applications in many fields such as information [8] and control [9]. The theory of fuzzy topological spaces was introduced and developed by Chang [5] and since then various notions in classical topology have been extended to fuzzy topological spaces [2], [3], [4]. A new class of fuzzy topological spaces called pairwise ordered fuzzy extremally disconnected spaces is introduced in this paper by using the concepts of fuzzy extremally disconnected space [1], fuzzy open sets [4], ordered fuzzy topology [7] and fuzzy bitopology [6]. Some interesting properties and characterizations are studied. Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces has been discussed as in [10] besides proving several other propositions and lemmas.

Definition 1 [5]. Let $f : (X, T) \to (Y, S)$ be a mapping from a fuzzy topological space $X$ to another fuzzy topological space $Y$. A mapping $f$ is called
(a) a fuzzy continuous mapping [1] if \( f^{-1}(\lambda) \in T \) for each \( \lambda \in S \) or equivalently \( f^{-1}(\mu) \) is a fuzzy closed subset of \( X \) for each fuzzy closed set \( \mu \) of \( Y \).

(b) a fuzzy open (fuzzy closed) mapping [1] if \( f(\lambda) \) is a fuzzy open (fuzzy closed) subset of \( X \) for each fuzzy open (fuzzy closed) set \( \lambda \) of \( X \).

**Definition 2** [7]. A fuzzy set \( \lambda \) in \( (X, T) \) is called increasing (decreasing) if \( \lambda(x) \leq \lambda(y) (\lambda(x) \geq \lambda(y)) \) whenever \( x \leq y \) in \( (X, T) \) and \( x, y \in X \).

**Definition 3** [7]. An ordered set on which a fuzzy topology is given is called an ordered fuzzy topological space.

**Definition 4** [6]. A fuzzy bitopological space is a triple \( (X, T_1, T_2) \) where \( X \) is a set and \( T_1, T_2 \) are any two fuzzy topologies on \( X \).

## 2. Pairwise Ordered Fuzzy Extremally Disconnected Spaces

In this section the concept of a pairwise ordered fuzzy extremally disconnected space is introduced. Its characterizations and properties are studied.

**Definition 5.** Let \( \lambda \) be any fuzzy set in the ordered fuzzy topological space \( (X, T, \leq) \). Then we define

\[
I(\lambda) = \text{increasing fuzzy closure of } \lambda = \bigwedge \{ \mu : \mu \text{ is a fuzzy closed increasing set and } \mu \geq \lambda \};
\]

\[
D(\lambda) = \text{decreasing fuzzy closure of } \lambda = \bigwedge \{ \mu : \mu \text{ is a fuzzy closed decreasing set and } \mu \geq \lambda \};
\]

\[
I^0(\lambda) = \text{increasing fuzzy interior of } \lambda = \bigvee \{ \mu : \mu \text{ is a fuzzy open increasing set and } \mu \leq \lambda \};
\]

\[
D^0(\lambda) = \text{decreasing fuzzy interior of } \lambda = \bigvee \{ \mu : \mu \text{ is a fuzzy open decreasing set and } \mu \leq \lambda \}.
\]

Clearly, \( I(\lambda)(D(\lambda)) \) is the smallest fuzzy closed increasing (decreasing) set containing \( \lambda \) and \( I^0(\lambda)(D^0(\lambda)) \) is the largest fuzzy open increasing (decreasing) set contained in \( \lambda \).

**Proposition 1.** For any fuzzy subset \( \lambda \) of an ordered fuzzy topological space \( (X, T, \leq) \), the following identities hold:

(a) \( 1 - I(\lambda) = D^0(1 - \lambda) \),
(b) \( 1 - D(\lambda) = I^0(1 - \lambda) \),
(c) \( 1 - I^0(\lambda) = D(1 - \lambda) \),
(d) \( 1 - D^0(\lambda) = I(1 - \lambda) \).
Definition 6. Let \((X, T_1, T_2, \leq)\) be an ordered fuzzy bitopological space. Let \(\lambda\) be any \(T_1\)-fuzzy open increasing (decreasing) set. If \(I_{T_2}(\lambda)(D_{T_2}(\lambda))\) is \(T_2\)-fuzzy open increasing (decreasing) set then \((X, T_1, T_2, \leq)\) is said to be \(T_1\)-upper (\(T_1\)-lower) fuzzy extremally disconnected. Similarly we can define the \(T_2\)-upper (\(T_2\)-lower) fuzzy extremally disconnected space. A fuzzy topological space \((X, T_1, T_2, \leq)\) is said to be pairwise upper fuzzy extremally disconnected if it is both \(T_1\)-upper fuzzy extremally disconnected and \(T_2\)-upper fuzzy extremally disconnected. Similarly we can define the pairwise lower fuzzy extremally disconnected space. A fuzzy topological space \((X, T_1, T_2, \leq)\) is said to be pairwise ordered fuzzy extremally disconnected if it is both pairwise upper fuzzy extremally disconnected and pairwise lower fuzzy extremally disconnected.

Example 1. Let \(X = \{a, b, c\}, T_1 = \{0, 1, \lambda_1, \lambda_2\}\) and \(T_2 = \{0, 1, \mu_1, \mu_2\}\), where \(\lambda_1: X \to [0, 1]\) is such that

\[
\begin{align*}
\lambda_1(a) &= 3/4, & \lambda_1(b) &= 1/2, & \lambda_1(c) &= 1/4, \\
\lambda_2(a) &= 0, & \lambda_2(b) &= 1/4, & \lambda_2(c) &= 1/4,
\end{align*}
\]

and \(\mu_1: X \to [0, 1]\) is such that

\[
\begin{align*}
\mu_1(a) &= 1, & \mu_1(b) &= 9/10, & \mu_1(c) &= 8/10, \\
\mu_2(a) &= 0, & \mu_2(b) &= 7/10, & \mu_2(c) &= 8/10.
\end{align*}
\]

Clearly, \(\lambda_1\) is a \(T_1\)-fuzzy open decreasing set and \(D_{T_2}(\lambda_1) = 1\) is a \(T_2\)-fuzzy open decreasing set, \(\mu_2\) is a \(T_2\)-fuzzy open decreasing set and \(D_{T_1}(\mu_2) = 1\) is a \(T_1\)-fuzzy open decreasing set. Hence \((X, T_1, T_2, \leq)\) is a pairwise lower fuzzy extremely disconnected space. Similarly we can prove that \((X, T_1, T_2, \leq)\) is a pairwise upper fuzzy extremely disconnected space. Therefore \((X, T_1, T_2, \leq)\) is a pairwise ordered fuzzy extremely disconnected space.

Proposition 2. For an ordered fuzzy bitopological space \((X, T_1, T_2, \leq)\), the following assertions are equivalent.

(a) \((X, T_1, T_2, \leq)\) is pairwise upper fuzzy extremely disconnected.
(b) For each \(T_1\)-fuzzy closed decreasing set \(\lambda\), \(D_{T_2}^0(\lambda)\) is \(T_2\)-fuzzy closed decreasing. Similarly, for each \(T_2\)-fuzzy closed decreasing set \(\lambda\), \(D_{T_1}^0(\lambda)\) is \(T_1\)-fuzzy closed decreasing.
(c) For each \(T_1\)-fuzzy open increasing set \(\lambda\) we have \(I_{T_2}(\lambda) + D_{T_2}(1 - I_{T_2}(\lambda)) = 1\). Similarly, for each \(T_2\)-fuzzy open increasing set \(\lambda\) we have \(I_{T_1}(\lambda) + D_{T_1}(1 - I_{T_1}(\lambda)) = 1\).
(d) For each pair of a $T_1$-fuzzy open increasing set $\lambda$ and a $T_1$-fuzzy open decreasing set $\mu$ with $I_{T_2}(\lambda) + \mu = 1$ we have $I_{T_2}(\lambda) + D_{T_2}(\mu) = 1$. Similarly, for each pair of a $T_2$-fuzzy open increasing set $\lambda$ and a $T_2$-fuzzy open decreasing set $\mu$ with $I_{T_1}(\lambda) + \mu = 1$ we have $I_{T_1}(\lambda) + D_{T_1}(\mu) = 1$.

Proof. (a) $\Rightarrow$ (b). Let $\lambda$ be any $T_1$-fuzzy closed decreasing set. We claim that $D^0_{T_2}(\lambda)$ is a $T_2$-fuzzy closed decreasing set. Now, $1 - \lambda$ is $T_1$-fuzzy open increasing and so by assumption (a), $I_{T_2}(1 - \lambda)$ is $T_2$-fuzzy open increasing. That is, $D^0_{T_2}(\lambda)$ is $T_2$-fuzzy closed decreasing.

(b) $\Rightarrow$ (c). Let $\lambda$ be any $T_1$-fuzzy open increasing set. Then

\begin{equation}
1 - I_{T_2}(\lambda) = D^0_{T_2}(1 - \lambda).
\end{equation}

Consider $I_{T_2}(\lambda) + D_{T_2}(1 - I_{T_2}(\lambda)) = I_{T_2}(\lambda) + D_{T_2}(D^0_{T_2}(1 - \lambda))$. As $\lambda$ is $T_1$-fuzzy open increasing, $1 - \lambda$ is $T_1$-fuzzy closed decreasing and by assumption (b), $D^0_{T_2}(1 - \lambda)$ is $T_2$-fuzzy closed decreasing. Therefore, $D_{T_2}(D^0_{T_2}(1 - \lambda)) = D^0_{T_2}(1 - \lambda)$. Now,

\begin{align*}
I_{T_2}(\lambda) + D_{T_2}(D^0_{T_2}(1 - \lambda)) &= I_{T_2}(\lambda) + D^0_{T_2}(1 - \lambda) = I_{T_2}(\lambda) + 1 - I_{T_2}(\lambda) = 1.
\end{align*}

That is, $I_{T_2}(\lambda) + D_{T_2}(1 - I_{T_2}(\lambda)) = 1$.

(c) $\Rightarrow$ (d). Let $\lambda$ be any $T_1$-fuzzy open increasing set and $\mu$ any $T_2$-fuzzy open decreasing set such that

\begin{equation}
I_{T_2}(\lambda) + \mu = 1.
\end{equation}

By assumption (c),

\begin{equation}
I_{T_2}(\lambda) + D_{T_2}(1 - I_{T_2}(\lambda)) = 1 = I_{T_2}(\lambda) + \mu.
\end{equation}

That is, $\mu = D_{T_2}(1 - I_{T_2}(\lambda))$. Since $\mu = 1 - I_{T_2}(\lambda)$ (by (2)), we have

\begin{equation}
D_{T_2}(\mu) = D_{T_2}(1 - I_{T_2}(\lambda)).
\end{equation}

From (3) and (4) we obtain $I_{T_2}(\lambda) + D_{T_2}(\mu) = 1$.

(d) $\Rightarrow$ (a). Let $\lambda$ be any $T_1$-fuzzy open increasing set. Put $\mu = 1 - I_{T_2}(\lambda)$. Clearly, $\mu$ is $T_2$-fuzzy open decreasing and from the construction of $\mu$ it follows that $I_{T_2}(\lambda) + \mu = 1$. By assumption (d), we have $I_{T_2}(\lambda) + D_{T_2}(\mu) = 1$ and so $I_{T_2}(\lambda) = 1 - D_{T_2}(\mu)$ is $T_2$-fuzzy open increasing. Therefore $(X, T_1, T_2, \leq)$ is upper $T_1$-fuzzy extremally disconnected. Similarly we can prove that $(X, T_1, T_2, \leq)$ is upper $T_1$-fuzzy extremally disconnected. Hence the Proposition.
Proposition 3. Let \((X, T_1, T_2, \leq)\) be an ordered fuzzy bitopological space. Then \((X, T_1, T_2, \leq)\) is a pairwise ordered fuzzy extremely disconnected space if and only if for a \(T_1\)-fuzzy decreasing open set \(\lambda\) and a \(T_2\)-fuzzy decreasing closed set \(\mu\) such that \(\lambda \leq \mu\), we have \(D_{T_1}(\lambda) \leq D_{T_1}^0(\mu)\).

Proof. Suppose \((X, T_1, T_2, \leq)\) is an upper pairwise fuzzy extremely disconnected space. Let \(\lambda\) be any \(T_1\)-fuzzy open decreasing set, \(\mu\) be any \(T_2\)-fuzzy closed decreasing set such that \(\lambda \leq \mu\). Then by (b) of Proposition 2, \(D_{T_1}^0(\mu)\) is \(T_1\)-fuzzy closed decreasing. Also, since \(\lambda\) is \(T_1\)-fuzzy open decreasing and \(\lambda \leq \mu\), it follows that \(\lambda \leq D_{T_1}^0(\mu)\). Again, since \(D_{T_1}^0(\mu)\) is \(T_1\)-fuzzy closed decreasing, it follows that \(D_{T_1}(\lambda) \leq D_{T_1}^0(\mu)\). To prove the converse, let \(\mu\) be any \(T_2\)-fuzzy closed decreasing set. By Definition 5, \(D_{T_1}^0(\mu)\) is \(T_1\)-fuzzy open decreasing and it is also clear that \(D_{T_1}^0(\mu) \leq \mu\). Therefore it follows by assumption that \(D_{T_1}(D_{T_1}^0(\mu)) \leq D_{T_1}^0(\mu)\). This implies that \(D_{T_1}^0(\mu)\) is \(T_1\)-fuzzy closed decreasing. Hence, by (b) of Proposition 2, it follows that \((X, T_1, T_2, \leq)\) is upper \(T_1\)-fuzzy extremely disconnected. Similarly we can prove also the other cases.

Notation. An ordered fuzzy set which is both fuzzy decreasing (increasing) open and closed is called a fuzzy decreasing (increasing) clopen set.

Remark 1. Let \((X, T_1, T_2, \leq)\) be a pairwise upper fuzzy extremely disconnected space. Let \(\{\lambda_i, 1-\mu_i : i \in \mathbb{N}\}\) be a collection such that \(\lambda_i\)'s are \(T_1\)-fuzzy open decreasing sets, \(\mu_i\)'s are \(T_2\)-fuzzy closed decreasing sets and let \(\lambda, 1-\mu\) be \(T_1\)-fuzzy open decreasing and \(T_2\)-fuzzy open increasing sets, respectively. If \(\lambda_i \leq \lambda \leq \mu_j\) and \(\lambda_i \leq \mu \leq \mu_j\) for all \(i, j \in \mathbb{N}\), then there exists a \(T_1\)- and \(T_2\)-fuzzy clopen decreasing set \(\gamma\) such that \(D_{T_1}(\lambda_i) \leq \gamma \leq D_{T_1}^0(\mu_j)\) for all \(i, j \in \mathbb{N}\). By Proposition 3, \(D_{T_1}(\lambda_i) \leq D_{T_1}(\lambda) \land D_{T_1}^0(\mu_j) \leq D_{T_1}^0(\mu_j)\) (\(i, j \in \mathbb{N}\)). Put \(\gamma = D_{T_1}(\lambda) \land D_{T_1}^0(\mu)\). Now, \(\gamma\) satisfies the required condition.

Proposition 4. Let \((X, T_1, T_2, \leq)\) be a pairwise ordered fuzzy extremely disconnected space. Let \(\{\lambda_q\}_{q \in Q}\) and \(\{\mu_q\}_{q \in Q}\) be monotone increasing collections of \(T_1\)-fuzzy open decreasing sets and \(T_2\)-fuzzy closed decreasing sets of \((X, T_1, T_2, \leq)\), respectively, and suppose that \(\lambda_{q_1} \leq \mu_{q_2}\) whenever \(q_1 < q_2\) (\(Q\) is the set of rational numbers). Then there exists a monotone increasing collection \(\{\gamma_q\}_{q \in Q}\) of \(T_1\)- and \(T_2\)-fuzzy clopen decreasing subsets of \((X, T_1, T_2, \leq)\) such that \(D_{T_1}(\lambda_{q_1}) \leq \gamma_{q_2}\) and \(\gamma_{q_1} \leq D_{T_1}^0(\mu_{q_2})\) whenever \(q_1 < q_2\).

Proof. Let us arrange the rational numbers into a sequence \(\{q_n\}\) without repetitions. For every \(n \geq 2\), we define inductively a collection \(\{\gamma_q : 1 \leq i < n\} \subset I^X\)
such that
\[
D_{T_i}(\lambda_q) \leq \gamma_q, \quad \text{if } q < q_i, \\
\gamma_q \leq D^0_{T_i}(\mu_q) \quad \text{if } q_i < q,
\]
for all \(i < n\).

By Proposition 3, the countable collections \(\{D_{T_i}(\lambda_q)\}\) and \(\{D^0_{T_i}(\mu_q)\}\) satisfy 
\(D_{T_1}(\lambda_{q_1}) \leq D^0_{T_1}(\mu_{q_2})\) if \(q_1 < q_2\). By Remark 1, there exists a \(T_1\)- and \(T_2\)-fuzzy clopen decreasing set \(\delta_1\) such that 
\(D_{T_1}(\lambda_{q_1}) \leq \delta_1 \leq D^0_{T_1}(\mu_{q_2})\). Setting \(\gamma_{q_1} = \delta_1\) we get \((S_2)\). Assume that \(T_1\)-fuzzy sets \(\gamma_q\) are already defined for \(i < n\) and satisfy \((S_n)\). Define \(\Sigma = \bigvee\{\gamma_{q_i} : i < n, q_i < q_n\} \lor \lambda_{q_n}\) and \(\Phi = \bigwedge\{\gamma_{q_j} : j < n, q_j > q_n\} \land \mu_{q_n}\). Then we have that 
\(D_{T_1}(\gamma_{q_i}) \leq D_{T_1}(\Sigma) \leq D^0_{T_1}(\gamma_{q_j})\) and 
\(D_{T_1}(\gamma_{q_i}) \leq D^0_{T_1}(\Phi) \leq D^0_{T_1}(\gamma_{q_j})\) whenever \(q_i < q_n < q_j\) \((i,j < n)\) as well as 
\(\lambda_q \leq D_{T_1}(\Sigma) \leq \mu_{q'}\) and 
\(\lambda_q \leq D^0_{T_1}(\Phi) \leq \mu_{q'}\) whenever \(q < q_n < q'\). This shows that the countable collections 
\(\{\gamma_{q_i} : i < n, q_i < q_n\}\) and \(\{\lambda_q : q < q_n\}\) and \(\{\mu_q : q > q_n\}\) together with \(\Sigma\) and \(\Phi\) fulfil all conditions of Remark 1. Hence, there exists a \(T_1\)- and \(T_2\)-fuzzy clopen decreasing set \(\delta_n\) such that 
\(D_{T_1}(\delta_n) \leq \mu_q\) if \(q_n < q\), \(\lambda_q \leq D^0_{T_1}(\delta_n)\) if \(q < q_n\), \(D_{T_1}(\gamma_{q_n}) \leq D^0_{T_1}(\delta_n)\) if \(q_i < q_n\), \(D_{T_1}(\delta_n) \leq D^0_{T_1}(\gamma_{q_j})\) if \(q_n < q_j\), when 1 \(\leq i, j \leq n - 1\). Now setting \(\gamma_{q_n} = \delta_n\) we obtain \(T_1\)-fuzzy sets \(\gamma_{q_1}, \gamma_{q_2}, \ldots, \gamma_{q_n}\) that satisfy \((S_{n+1})\). Therefore the collection \(\{\gamma_{q_i} : i = 1, 2, \ldots\}\) has the required property. This completes the proof.

**Definition 7.** Let \((X, T_1, T_2, \leq)\) and \((Y, S_1, S_2, \leq)\) be ordered fuzzy bitopological spaces. A mapping \(f : (X, T_1, T_2, \leq) \rightarrow (Y, S_1, S_2, \leq)\) is called increasing (decreasing) \(T_1\)-fuzzy continuous if \(f^{-1}(\lambda)\) is a \(T_1\)-fuzzy open increasing (decreasing) subset of \((X, T_1, T_2, \leq)\) for every \(S_1\)- or \(S_2\)-fuzzy open subset \(\lambda\) of \((Y, S_1, S_2, \leq)\). If \(f\) is both increasing and decreasing \(T_1\)-fuzzy continuous, then it is called ordered \(T_1\)-fuzzy continuous.

**Definition 8.** Let \((X, T_1, T_2, \leq)\) be an ordered fuzzy bitopological space. A function \(f : X \rightarrow \mathbb{R}(I)\) is called lower (upper) \(T_1\)-fuzzy continuous if \(f^{-1}(R_t)(f^{-1}(L_t))\) is increasing or decreasing \(T_1\)-fuzzy open for each \(t \in \mathbb{R}\). Similarly we can define a lower and upper \(T_1\)-fuzzy continuous function.

**Lemma 1.** Let \((X, T_1, T_2, \leq)\) be an ordered fuzzy bitopological space, let \(\lambda \in I^X\), and let

\[
f : X \rightarrow \mathbb{R}(I) \text{ be such that } f(x)(t) = \begin{cases} 
1 & \text{if } t < 0, \\
\lambda(x) & 0 \leq t \leq 1, \\
0 & \text{if } t > 1,
\end{cases}
\]
for all \( x \in X \). Then \( f \) is lower (upper) \( T_1 \)-fuzzy continuous iff \( \lambda \) is a \( T_1 \)-fuzzy open (closed) increasing or decreasing set.

**Definition 9.** The characteristic function of \( \lambda \in I^X \) is the map \( \chi_\lambda : X \to [0,1] \) defined by \( \chi_\lambda(x) = (\lambda(x)) \), \( x \in X \).

**Proposition 5.** Let \((X,T_1,T_2,\leq)\) be an ordered fuzzy bitopological space, and let \( \lambda \in I^X \). Then \( \chi_\lambda \) is lower (upper) \( T_1 \)-fuzzy continuous iff \( \lambda \) is a \( T_1 \)-fuzzy open (closed) increasing or decreasing set.

**Proof.** The proof follows from Lemma 1. \(\square\)

**Proposition 6.** Let \((X,T_1,T_2,\leq)\) be an ordered fuzzy bitopological space. Then the following statements are equivalent.

(a) \((X,T_1,T_2,\leq)\) is pairwise ordered fuzzy extremally disconnected.

(b) If \( g,h : X \to \mathbb{R}(I) \), \( g \) is lower \( T_1 \)-fuzzy continuous, \( h \) is upper \( T_2 \)-fuzzy continuous and \( g \leq h \), then there exists an increasing \( T_1 \)- and \( T_2 \)-fuzzy continuous function \( f : (X,T_1,T_2,\leq) \to \mathbb{R}(I) \) such that \( g \leq f \leq h \).

(c) If \( 1 - \lambda \) is \( T_2 \)-fuzzy open increasing and \( \mu \) is \( T_1 \)-fuzzy open decreasing such that \( \mu \leq \lambda \), then there exists an increasing \( T_1 \)- and \( T_2 \)-fuzzy continuous function \( f : (X,T_1,T_2,\leq) \to [0,1] \) such that \( \mu \leq (1 - L_1)f \leq R_0f \leq \lambda \).

**Proof.** (a) \(\Rightarrow\) (b). Define \( H_r = L_rh \) and \( G_r = (1 - R_r)g \), \( r \in Q \). Thus we have two monotone increasing families of, respectively, \( T_1 \)-fuzzy open decreasing and \( T_2 \)-fuzzy closed decreasing subsets of \((X,T_1,T_2,\leq)\). Moreover, \( H_r \leq G_s \) if \( r < s \). By Proposition 4, there exists a monotone increasing family \( \{F_r\}_{r \in Q} \) of \( T_1 \)- and \( T_2 \)-fuzzy clopen decreasing sets of \((X,T_1,T_2,\leq)\) such that \( D_{T_1}(H_r) \leq F_s \) and \( F_r \leq D_{T_2}(G_s) \) whenever \( r < s \). Letting \( V_t = \bigwedge_{r \leq t} (1 - F_r) \) for all \( t \in \mathbb{R} \), we define a monotone decreasing family \( \{V_t : t \in \mathbb{R}\} \subseteq I^X \). Moreover, we have \( I_{T_1}(V_t) \subseteq I_{T_1}^0(V_s) \) whenever \( s < t \).

One can easily prove that \( f \) is well defined and it is the required extension by making use of Theorem 3.7 of Kubiak (1987) [10], of the concepts of ordered fuzzy topology and fuzzy bitopology.

(b) \(\Rightarrow\) (c). Suppose \( 1 - \lambda \) is a \( T_2 \)-fuzzy open increasing set and \( \mu \) is a \( T_1 \)-fuzzy open decreasing set, \( \mu \leq \lambda \). Then \( \chi_\mu \leq \chi_\lambda \) and \( \chi_\mu, \chi_\lambda \) are lower \( T_1 \)- and upper \( T_2 \)-fuzzy continuous functions, respectively. Hence by (b), there exists an increasing \( T_1 \)- and \( T_2 \)-fuzzy continuous function \( f : (X,T,\leq) \to \mathbb{R}(I) \) such that \( \chi_\mu \leq f \leq \chi_\lambda \). Clearly, \( f(x) \in [0,1] \) for all \( x \in X \) and \( \mu = (1 - L_1)\chi_\mu \leq (1 - L_1)f \leq R_0f \leq R_0\chi_\lambda = \lambda \).

(c) \(\Rightarrow\) (a). This follows from Proposition 3 and the fact that \((1 - L_1)f \) and \( R_0f \) are \( T_1 \)-fuzzy closed decreasing and \( T_2 \) fuzzy open decreasing sets, respectively.
3. Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces

In this section, Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces is studied.

**Proposition 7** (Tietze Extension Theorem). Let \((X, T_1, T_2, \leq)\) be a pairwise ordered fuzzy extremally disconnected space. Let \(A \in X\) such that \(\chi_A\) is \(T_1\)-fuzzy open increasing and \(T_2\)-fuzzy open increasing set and let \(f: (A, T_1/A, T_2/A, \leq) \to I\) be an increasing \(T_1\)– and \(T_2\)-fuzzy continuous and isotone function. Then \(f\) admits an extension \(F: (X, T_1, T_2, \leq) \to I\) with all its properties preserved if \(f\) satisfies the following \# property:

\[
[\lambda] < [\mu] \Rightarrow f^{-1}\{\chi_{[\lambda],[\lambda]}\} < f^{-1}\{\chi_{[\mu],[1]}\},
\]

where \(\delta < \theta \Leftrightarrow D_{T_2}f(\delta) \wedge I_{T_2}L(\sigma)(\theta) = 0\) and

\[
[[\lambda_1], [\lambda_2]] = \{[\mu] \in I(L): \lambda_1 \leq [\mu] \leq [\lambda_2]\}.
\]

**Proof.** Define \(g, h: X \to [0,1]\) by

\[
g(x) = \begin{cases} f(x) & \text{if } x \in A, \\ [\lambda_0] & \text{if } x \notin A; \end{cases}
\]

\[
h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ [\lambda_1] & \text{if } x \notin A; \end{cases}
\]

where \([\lambda_0]\) is the equivalence class [10] determined by \(\lambda_0: R \to I\) such that

\[
\lambda_0(t) = \begin{cases} 1 & \text{if } t < 0, \\ 0 & \text{if } t > 0, \end{cases}
\]

\([\lambda_1]\) is the equivalence class [10] determined by \(\lambda_1: R \to I\) such that

\[
\lambda_1(t) = \begin{cases} 1 & \text{if } t < 1, \\ 0 & \text{if } t > 1, \end{cases}
\]

\(g\) is lower \(T_1\)-fuzzy continuous, \(h\) is upper \(T_2\)-fuzzy continuous and \(g \leq h\). Hence by Proposition 6 there exists an increasing \(T_1\)– and \(T_2\)-fuzzy continuous function \(F: (X, T_1, T_2, \leq) \to [0,1]\) such that \(g(x) \leq F(x)\) \(\leq h(x)\) for all \(x \in X\). Hence for all \(x \in A\) we have \(g(x) \leq f(x) \leq h(x)\), so that \(F\) is the required extension of \(f\) over \(X\). Moreover, \(F\) is isotone as \(f\) satisfies the \# property. Hence the theorem.
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