ON THE FROBENIUS NUMBER OF A MODULAR DIOPHANTINE INEQUALITY

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Abstract. We present an algorithm for computing the greatest integer that is not a solution of the modular Diophantine inequality \( ax \mod b \leq x \), with complexity similar to the complexity of the Euclid algorithm for computing the greatest common divisor of two integers.

Keywords: numerical semigroup, Diophantine inequality, Frobenius number, multiplicity

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1. INTRODUCTION

Given two integers \( m \) and \( n \) with \( n \neq 0 \), we denote by \( m \mod n \) the remainder of the division of \( m \) by \( n \). Following the terminology used in [6], a proportionally modular Diophantine inequality is an expression of the form \( ax \mod b \leq cx \), where \( a, b \) and \( c \) are positive integers. The set \( S(a, b, c) \) of integer solutions of this inequality is a numerical semigroup, that is, it is a subset of \( \mathbb{N} \) (here \( \mathbb{N} \) denotes the set of nonnegative integers) that is closed under addition, contains the zero element and its complement in \( \mathbb{N} \) is finite. We say that a numerical semigroup is proportionally modular if it is the set of integer solutions of a proportionally modular Diophantine inequality.

The integers \( a, b \) and \( c \) in the inequality \( ax \mod b \leq cx \) are, respectively, the factor, the modulus and the proportion of the inequality. Following the terminology used in [7], proportionally modular Diophantine inequalities with proportion 1, that

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is, such that \( c = 1 \), are simply called modular Diophantine inequalities. A numerical semigroup is modular if it is the set of integer solutions of a modular Diophantine inequality.

If \( S \) is a numerical semigroup, then the greatest integer that does not belong to \( S \) is an important invariant of \( S \), called the Frobenius number of \( S \) (see [3]) and denoted here by \( g(S) \). Giving a formula for the Frobenius number of \( S(a, b, 1) \), as a function of \( a \) and \( b \), is still an open problem. Some progress was made in [7] and [4]. In [2] an algorithm to determine the Frobenius number of \( S(a, b, c) \) is described. The aim of the present paper is to give an algorithm that computes the Frobenius number of \( S(a, b, 1) \), with complexity similar to the complexity of the Euclid algorithm for computing the greatest common divisor of two integers. This algorithm has considerably smaller complexity than the one presented in [2] in most of the cases.

2. Preliminaries

Given a nonempty subset \( A \) of \( \mathbb{Q}_0^+ \) (here \( \mathbb{Q}_0^+ \) is the set of nonnegative rational numbers), we will denote by \( \langle A \rangle \) the submonoid of \( (\mathbb{Q}_0^+, +) \) generated by \( A \), that is, \( \langle A \rangle = \{ \lambda_1 a_1 + \ldots + \lambda_n a_n ; \ n \in \mathbb{N} \setminus \{0\}, \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in A \} \). Clearly, \( \langle A \rangle \cap \mathbb{N} \) is a submonoid of \( (\mathbb{N}, +) \), represented here by \( S(A) \). We will refer to \( S(A) \) as the submonoid of \( \mathbb{N} \) associated to \( A \).

Let \( p \) and \( q \) be two positive rational numbers with \( p < q \). We use the notation

\[
[p, q] = \{ x \in \mathbb{Q} ; p \leq x \leq q \} \quad \text{and} \quad ]p, q[ = \{ x \in \mathbb{Q} ; p < x < q \}.
\]

The following result is a reformulation of [6, Corollary 9].

**Proposition 1.**

1. Let \( a, b \) and \( c \) be positive integers such that \( c < a < b \). Then \( S(\lfloor \frac{b}{a_1}, \frac{b}{a_2} \rfloor) = S(a, b, c) \).
2. Conversely, if \( a_1, b_1, a_2 \) and \( b_2 \) are positive integers such that \( \frac{b_1}{a_1} < \frac{b_2}{a_2} \), then \( S(\lfloor \frac{b_1}{a_1}, \frac{b_2}{a_2} \rfloor) = S(a_1 b_2, b_1 a_2, a_1 b_2 - a_2 b_1) \).

Since the inequality \( ax \mod b \leq cx \) has the same solutions as the inequality \( (a \mod b)x \mod b \leq cx \), we can assume that \( a < b \). Moreover, if \( c \geq a \), then \( S(a, b, c) = \mathbb{N} \). Therefore, we can suppose that \( a, b \) and \( c \) are positive integers such that \( c < a < b \). Consequently, the condition imposed in (1) of the above proposition is not restrictive.

The next proposition is [8, Proposition 5].
Proposition 2. If $I$ is an interval of positive rational numbers (not necessarily closed), then $S(I)$ is a proportionally modular numerical semigroup.

As an immediate consequence of Propositions 1 and 2 we have the following result.

Proposition 3. Let $I$ be an interval of rational numbers greater than one. Then $S(I)$ is a proportionally modular numerical semigroup. Moreover, every proportionally modular numerical semigroup not equal to $\mathbb{N}$ is of this form.

The following lemma can be easily deduced from [8, Lemma 2] and will be used several times in this paper.

Lemma 4. Let $I$ be an interval of positive rational numbers and let $x$ be a positive integer. Then $x \in S(I)$ if and only if there exists a positive integer $y$ such that $x/y \in I$.

If $S$ is a numerical semigroup, then the smallest positive integer that belongs to $S$ is the multiplicity of $S$ (see [1]) and it is denoted by $m(S)$. If $a_1, b_1, a_2$ and $b_2$ are positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, then [9, Algorithm 12] allows us to compute the multiplicity of $S([\frac{b_1}{a_1}, \frac{b_2}{a_2}])$. In essence, this algorithm follows the steps of the Euclid algorithm for computing the greatest common divisor of two integers.

Note that, by Proposition 1, we have $S(a, b, 1) = S([\frac{b}{a}, \frac{b}{a-1}])$. In Theorem 18 we will see that

$$g(S\left([\frac{b}{a}, \frac{b}{a-1}]\right)) = b - m\left(S\left([\frac{b}{a}, \frac{b}{a-1}]\right)\right)$$

and in Theorem 9 that

$$S\left([\frac{b}{a}, \frac{b}{a-1}]\right) = S\left([\frac{2b^2 + 1}{2ab}, \frac{2b^2 - 1}{2b(a-1)}]\right).$$

Therefore

$$g\left(S\left([\frac{b}{a}, \frac{b}{a-1}]\right)\right) = b - m\left(S\left([\frac{2b^2 + 1}{2ab}, \frac{2b^2 - 1}{2b(a-1)}]\right)\right)$$

and $m(S([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}]))$ can be computed by applying [9, Algorithm 12].
3. A proportionally modular representation for an open modular numerical semigroup

If \( x_1 < x_2 < \ldots < x_k \) are integers, then we use \( \{x_1, x_2, \ldots, x_k, \rightarrow\} \) to denote the set \( \{x_1, x_2, \ldots, x_k\} \cup \{z \in \mathbb{Z}; z > x_k\} \). Following the terminology used in [8], a numerical semigroup \( S \) is a half-line if \( S = \{0\} \cup \{m(S), \rightarrow\} \), and it is open modular if either \( S \) is a half-line or there exist integers \( a \) and \( b \) such that \( 2 \leq a < b \) and \( S = S([\frac{b}{a}, \frac{b}{a-1}]) \).

If \( S = S([\frac{b}{a}, \frac{b}{a-1}]) \), then by Proposition 2 we know that \( S \) is a proportionally modular numerical semigroup and therefore it admits a proportionally modular representation, that is, there exist positive integers \( x, y, z \) such that \( S = S(x, y, z) \).

Observe that, in view of Proposition 1, it suffices to find positive integers \( a_1, b_1, a_2 \) and \( b_2 \) such that \( S([\frac{b}{a_1}, \frac{b}{a_2}]) = S([\frac{b}{a_1}, \frac{b}{a_2}]) \). Finding these positive integers is the fundamental aim of this section. To this end, we need some preliminary results and concepts.

If \( S \) is a numerical semigroup, then \( \mathbb{N} \setminus S \) is finite. The elements of \( \mathbb{N} \setminus S \) are the so-called gaps of \( S \). The cardinality of \( \mathbb{N} \setminus S \) is known as the singularity degree of \( S \) (see [1]).

The Frobenius number and the singularity degree of an open modular numerical semigroup can be easily computed by using the following result.

**Lemma 5** [8, Theorem 11]. Let \( 2 \leq a < b \) be integers, \( \alpha = \gcd\{a, b\} \) and \( \beta = \gcd\{a-1, b\} \). Then \( S([\frac{b}{a}, \frac{b}{a-1}]) \) is a proportionally modular numerical semigroup with Frobenius number \( b \) and singularity degree \( \frac{1}{2}(b-1 + \alpha + \beta) \).

The next lemma is straightforward to prove.

**Lemma 6.** Let \( 2 \leq a < b \) be integers. Then \( b-1 \in S([\frac{b}{a}, \frac{b}{a-1}]) \).

**Proof.** A simple check shows that \( \frac{b}{a} < \frac{b-1}{a-1} < \frac{b}{a-1} \). By applying Lemma 4 we have that \( b-1 \in S([\frac{b}{a}, \frac{b}{a-1}]) \). \( \square \)

It is well-known (see for instance [5]) that every numerical semigroup \( S \) is finitely generated and therefore there exists a finite subset \( A \) of \( \mathbb{N} \) such that \( S = \langle A \rangle \). We say that \( A \) is a minimal system of generators of \( S \) if no proper subset of \( A \) generates \( S \). It is also well-known (see [5]) that \( S^* \setminus (S^* + S^*) \) is the unique minimal system of generators of \( S \), with \( S^* = S \setminus \{0\} \). The cardinality of the minimal system of generators of \( S \) is also an important invariant of \( S \) called the embedding dimension of \( S \) (see [1]).

From Lemmas 5 and 6 we deduce the following result which gives an upper bound to the minimal generators of \( S([\frac{b}{a}, \frac{b}{a-1}]) \).
Lemma 7. Let $2 \leq a < b$ be integers. Then every minimal generator of $S(\frac{b}{a}, \frac{b}{a-1})$ is smaller than $2b$.

Proof. From Lemmas 5 and 6 we know that $\{b-1, b+1, \rightarrow\} \subseteq S(\frac{b}{a}, \frac{b}{a-1})$. We conclude the proof by pointing out that every positive integer greater than or equal to $2b$ belongs to $\{b-1, b+1, \rightarrow\} + \{b-1, b+1, \rightarrow\}$. □

A simple check proves the next result.

Lemma 8. Let $2 \leq a < b$ be integers. Then $\frac{b}{a} < \frac{2b^2+1}{2ab} < \frac{2b^2-1}{2b(a-1)} < \frac{b}{a-1}$.

We are now ready to state the principal result of this section.

Theorem 9. Let $2 \leq a < b$ be integers. Then $S(\frac{b}{a}, \frac{b}{a-1}) = S(\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)})$.

Proof. From Lemma 8 we have that $[\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}] \subseteq [\frac{b}{a}, \frac{b}{a-1}]$ and so $S([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}]) \subseteq S([\frac{b}{a}, \frac{b}{a-1}])$. To prove the other inclusion we only need to show that every minimal generator of $S([\frac{b}{a}, \frac{b}{a-1}])$ belongs to $S([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}])$.

Let $x$ be a minimal generator of $S([\frac{b}{a}, \frac{b}{a-1}])$. Then by Lemma 4 there exists a positive integer $y$ such that $\frac{b}{a} < \frac{y}{x} < \frac{b}{a-1}$. Moreover, by applying Lemma 7 we have that $x \leq 2b - 1$ and, since $1 < \frac{b}{a} < \frac{y}{x}$, we deduce that $y < 2b - 1$. Let us show that $\frac{2b^2+1}{2ab} < \frac{y}{x} < \frac{2b^2-1}{2b(a-1)}$. As $\frac{b}{a} < \frac{y}{x}$, we have $by < ax$ and so $ax - by \geq 1$. Hence $2abx - 2b^2y \geq 2b$. Since $y < 2b - 1$, we infer that $2abx - 2b^2y \geq y$, and consequently $\frac{2b^2+1}{2ab} \leq \frac{x}{y}$. Arguing in a similar way with $\frac{x}{y} < \frac{b}{a-1}$, we get $2b^2y - 2b(a-1)x \geq y$, which is equivalent to $\frac{x}{y} \leq \frac{2b^2-1}{2b(a-1)}$. Finally, by applying Lemma 4 we obtain that $x \in S([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}])$. □

As an immediate consequence of the previous theorem we have the following result.

Corollary 10. Let $2 \leq a < b$ be integers and let $\alpha$ and $\beta$ be rational numbers such that $\frac{b}{a} < \alpha \leq \frac{2b^2+1}{2ab} < \frac{2b^2-1}{2b(a-1)} \leq \beta < \frac{b}{a-1}$. Then $S([\alpha, \beta]) = S([\frac{b}{a}, \frac{b}{a-1}])$.

From this we deduce the following.

Corollary 11. Let $2 \leq a < b$ be integers. If $k$ is an integer greater than or equal to $2b^2$, then $S([\frac{b}{a}, \frac{b}{a-1}]) = \{x \in \mathbb{N} \mid (ka-1)x \text{ mod } kb \leq (k-2)x\}$.

Proof. A simple check shows that

$$\frac{b}{a} < \frac{kb}{ka-1} < \frac{2b^2+1}{2ab} < \frac{2b^2-1}{2b(a-1)} \leq \frac{kb}{k(a-1)+1} < \frac{b}{a-1}.$$ 

By applying Corollary 10, we have that $S([\frac{b}{a}, \frac{b}{a-1}]) = S([\frac{kb}{ka-1}, \frac{kb}{k(a-1)+1}])$. We conclude the proof by using Proposition 1. □
The next result is an immediate consequence of Lemma 5 and Corollary 11.

**Corollary 12.** Let \(2 \leq a < b\) be integers. Set \(\alpha = \gcd\{a, b\}\), \(\beta = \gcd\{a - 1, b\}\), and let \(k\) be an integer greater than or equal to \(2b^2\). Then the numerical semigroup 
\[S(ka - 1, kb, k - 2)\]
has Frobenius number \(b\) and singularity degree \(\frac{1}{2}(b - 1 + \alpha + \beta)\).

4. **An algorithm for computing the Frobenius number of a modular numerical semigroup**

In this section, our first goal will be to prove Theorem 18, which establishes a relationship between the Frobenius number of 
\[S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\]
and the multiplicity of 
\[S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\]
for \(a\) and \(b\) integers such that \(2 \leq a < b\). Before that, we need to recall and establish some results.

The next result is deduced from Proposition 1 and [7, Corollary 6].

**Lemma 13.** Let \(2 \leq a < b\) be integers. If \(x \in \mathbb{N} \setminus S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\), then 
\[b - x \in S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\].

The following lemma follows from [7, Lemma 11] and describes the integers \(x\) for which both \(x\) and \(b - x\) belong to 
\[S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\].

**Lemma 14.** Let \(2 \leq a < b\) be integers. Then \(\{x, b - x\} \subseteq S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\) if and only if
\[x \in \{0, \frac{b}{\alpha}, 2\frac{b}{\alpha}, \ldots, (\alpha - 1)\frac{b}{\alpha}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta - 1)\frac{b}{\beta}, b\}\],
where \(\alpha = \gcd\{a, b\}\) and \(\beta = \gcd\{a - 1, b\}\).

The next result gives an upper bound for the Frobenius number of 
\[S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\].

**Lemma 15.** Let \(1 \leq c < a < b\) be integers. Then the Frobenius number of 
\[S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-c} \rfloor)\]
is smaller than \(b - 1\).

**Proof.** By Proposition 1 we know that 
\[S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-c} \rfloor) = \{x \in \mathbb{N}; ax \mod b \leq cx\}\].
Note that, if \(x \geq b - 1\), then \(ax \mod b \leq b - 1 \leq c(b - 1) \leq cx\) and therefore 
\(x \in S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-c} \rfloor)\). \(\Box\)

Next we discard some values for the multiplicity of 
\[S(\lfloor \frac{b}{a} \rfloor, \lfloor \frac{b}{a-1} \rfloor)\].
Lemma 16. Let $2 \leq a < b$ be integers, $\alpha = \gcd\{a, b\}$, $\beta = \gcd\{a - 1, b\}$ and $S' = S(\frac{b}{a}, \frac{b}{a-1})$. Then $m(S') \notin \{\frac{b}{a}, \frac{2b}{a}, \ldots, (\alpha - 1)\frac{b}{a}, \frac{b}{\beta}, \frac{2b}{\beta}, \ldots, (\beta - 1)\frac{b}{\beta}, b\}$.

Proof. By Lemma 4 there exists a positive integer $y$ such that $\frac{b}{a} < \frac{m(S')}{y} < \frac{b}{a-1}$. Let us assume that $m(S') = k\frac{b}{\alpha}$ with $k \in \{1, \ldots, \alpha\}$. Then $\frac{b}{a} = k\frac{b}{\alpha} = \frac{m(S')}{k\alpha} < \frac{m(S')}{y} < \frac{b}{a-1}$. Hence $S(\frac{m(S')}{k\alpha}, \frac{m(S')}{k\alpha-1}) \subseteq S'$. In view of Lemma 5 we have that $g(S(\frac{m(S')}{k\alpha}, \frac{m(S')}{k\alpha-1}))) = m(S')$ and also $g(S') = b$. So we deduce that $b \leq m(S')$. From Lemma 6 we know that $m(S') \leq b - 1$, which is not possible. Similarly we can prove that $m(S') \neq k\frac{b}{\beta}$ for $k \in \{1, \ldots, \beta\}$.

Now we study which elements of $S(\frac{b}{a}, \frac{b}{a-1})$ belong to $S(\frac{b}{a}, \frac{b}{a-1})$.

Lemma 17. Let $2 \leq a < b$ be integers, $\alpha = \gcd\{a, b\}$ and $\beta = \gcd\{a - 1, b\}$. If $x \in S(\frac{b}{a}, \frac{b}{a-1}) \setminus \{0, \frac{b}{a}, 2\frac{b}{a}, \ldots, (\alpha - 1)\frac{b}{a}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta - 1)\frac{b}{\beta}, b\}$, then $x \in S(\frac{b}{a}, \frac{b}{a-1})$.

Proof. Since $x \in S(\frac{b}{a}, \frac{b}{a-1})$, by Lemma 4 there exists a positive integer $y$ such that $\frac{b}{a} \leq \frac{x}{y} \leq \frac{b}{a-1}$. If $\frac{x}{y} = \frac{b}{a}$, then $x = k\frac{b}{a}$ for some positive integer $k$. Suppose that $k \geq \alpha + 1$. Let us prove that $k\frac{b}{\alpha} \in S(\frac{b}{a}, \frac{b}{a-1})$. To this end, in view of Lemma 4, it suffices to see that $\frac{b}{a} < \frac{kb}{\alpha(a-1)} < \frac{b}{a-1}$. But a simple check shows that these inequalities hold. The case $\frac{x}{y} = \frac{b}{a-1}$ is analogous to the previous one.

We are now ready to state the theorem announced at the beginning of this section.

Theorem 18. Let $2 \leq a < b$ be integers. Define $S = S(\frac{b}{a}, \frac{b}{a-1})$ and $S' = S(\frac{b}{a}, \frac{b}{a-1})$. Then $g(S) = b - m(S')$.

Proof. From Lemmas 14 and 16 we deduce that $b - m(S') \notin S$. By Lemma 17 we obtain that, if $x \in \{1, \ldots, m(S') - 1\}$, then either $x \notin S$ or $x \in \{0, \frac{b}{a}, 2\frac{b}{a}, \ldots, (\alpha - 1)\frac{b}{a}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta - 1)\frac{b}{\beta}, b\}$. Hence, by Lemmas 13 and 14 we have that $\{b - 1, b - 2, \ldots, b - m(S') - 1\} \subseteq S$. Moreover, Lemma 15 asserts that $\{b - 1, \rightarrow\} \subseteq S$. Therefore $g(S) = b - m(S')$.

Now, we present an algorithm that allows us to compute the Frobenius number of $S(\frac{b}{a}, \frac{b}{a-1})$ for $a$ and $b$ integers such that $2 \leq a < b$. In view of Proposition 1, we have $S(\frac{b}{a}, \frac{b}{a-1}) = S(a, b, 1)$. Therefore, this algorithm computes the Frobenius number of a modular numerical semigroup.

In [9] we gave an algorithm for computing the multiplicity of a proportionally modular numerical semigroup defined by a closed interval. Thus the idea is to combine this algorithm with Theorems 9 and 18.

Algorithm 19. Input: $a$ and $b$ integers such that $2 \leq a < b$.
Output: The Frobenius number of $S(\frac{b}{a}, \frac{b}{a-1})$. 373
(1) Compute the multiplicity $m$ of $S(\frac{2b^2+1}{2a}, \frac{2b^2-1}{2a(a-1)})$ by using [9, Algorithm 12].
(2) Return $b - m$.

Next we briefly recall [9, Algorithm 12]. In order to do this, we need to introduce some concepts.

Let $a_1, b_1, a_2$ and $b_2$ be positive integers. Define

$$R\left(\frac{b_1}{a_1}, \frac{b_2}{a_2}\right) = \left[\frac{a_2}{b_2 \mod a_2}, \frac{a_1}{b_1 \mod a_1}\right].$$

Given a closed interval $I$ of positive rational numbers we can construct recursively the following sequence of closed intervals:

$I_1 = I$,
$I_{n+1} = R(I_n)$, if $I_n$ contains no integers, and $I_{n+1} = I_n$, otherwise.

We will refer to $\{I_n\}_{n \in \mathbb{N} \setminus \{0\}}$ as the sequence of intervals associated with $I$.

Given a rational number $q$ we denote by $\lfloor q \rfloor$ the integer $\max\{z \in \mathbb{Z} \setminus \{0\} \mid z \leq q\}$ and by $\lceil q \rceil$ the integer $\min\{z \in \mathbb{Z} \setminus \{0\} \mid q \leq z\}$. Let $I$ be a closed interval. If $I$ does not contain an integer, then $\lfloor x \rfloor = \lfloor y \rfloor$ for every $x, y \in I$. This integer is denoted by $\lfloor I \rfloor$.

We are now ready to recall [9, Algorithm 12].

Algorithm 20. Input: $I$ a closed interval of positive rational numbers such that $S(I) \neq \mathbb{N}$.

Output: The multiplicity of the semigroup $S(I)$.

(1) Compute the sequence of intervals associated to $I$ until we find the first interval of the sequence that contains an integer. Let us denote such intervals by $I_1, I_2, \ldots, I_l$.

(2) If $I_l = [\alpha, \beta]$, then $P(I_l) = \lceil \alpha \rceil / 1$.

(3) Calculate $P(I_1)$ by applying successively $P(I_{n-1}) = P(I_n)^{-1} + \lfloor I_{n-1} \rfloor$.

(4) The multiplicity of $S(I)$ is the numerator of $P(I_1)$.

We end this section with an example that illustrates Algorithm 19.

Example 21. Let us compute the Frobenius number of the modular numerical semigroup $S(17, 108, 1)$. By Proposition 1, we have $S(17, 108, 1) = S([\frac{108}{17}, \frac{108}{16}])$.

(1) (a) $I_1 = \left[\frac{23329}{3672}, \frac{23327}{3456}\right], \quad I_2 = \left[\frac{3456}{2591}, \frac{3672}{1297}\right]$.

Note that $2 \in I_2$.

(b) $P(I_2) = \frac{2}{1}$.

(c) $P(I_1) = \frac{1}{2} + 6 = \frac{13}{2}$.

(d) The multiplicity of $S([\frac{23329}{3672}, \frac{23327}{3456}])$ is 13.

(2) The Frobenius number of $S([\frac{108}{17}, \frac{108}{16}])$ is $108 - 13 = 95$. 

374
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