NECESSARY CONDITIONS OF OPTIMALITY FOR OPTIMAL PROBLEMS WITH DELAYS AND WITH A DISCONTINUOUS INITIAL CONDITION

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Let $J = [a, b]$ be a finite interval; $O \subset \mathbb{R}^n, G \subset \mathbb{R}^r$ be open sets and let the function $f : J \times O^s \times G^r \to \mathbb{R}^n$ satisfy the following conditions:

1) for a fixed $t \in J$ the function $f(t, x_1, \ldots, x_s, u_1, \ldots, u_r)$ is continuous with respect to $(x_1, \ldots, x_s, u_1, \ldots, u_r) \in O^s \times G^r$ and continuously differentiable with respect to $(x_1, \ldots, x_s) \in O^s$;

2) for a fixed $(x_1, \ldots, x_s, u_1, \ldots, u_r) \in O^s \times G^r$ the functions $f, f_{x_i}, i = 1, \ldots, s$, are measurable with respect to $t$. For arbitrary compacts $K \subset O, V \subset G$ there exists a function $m_{K, V}(\cdot) \in L(J, \mathbb{R}^n)$ such that

\[ |f(t, x_1, \ldots, x_s, u_1, \ldots, u_r)| + \sum_{i=1}^s |f_{x_i}(\cdot)| \leq m_{K, V}(t), \]

\[ \forall (t, x_1, \ldots, x_s, u_1, \ldots, u_r) \in J \times K^s \times V^r. \]

Let now $\tau_i(t), i = 1, \ldots, s, t \in J$, be absolutely continuous functions, satisfying the conditions: $\tau_i(t) \leq t$, $\dot{\tau}_i(t) > 0$; $\Delta$ be a space of piecewise continuous functions $\varphi : J_1 = [\tau, b] \to N$, $\tau = \min(\tau_1(a), \ldots, \tau_s(a))$, with a finite number of discontinuity points of the first kind; The functions $\theta_i(t), i = 1, \ldots, r, t \in R$, satisfy cone-measurability condition i.e. there exists absolutely continuous function $\theta(t) < t$, $\theta(t) > 0$ such that $\theta_i(t) = \theta^{k_i}(t)$, where $k_i > \cdots k_1 \geq 0$ are natural numbers, $\theta^0(t) = \theta^{(0^{-1})}(t)$, $\theta^0(t) = t$; $\Omega$ is the set of measurable functions $u : J_2 = [\theta, b] \to U$, $\theta = \min\{\theta_1(a), \ldots, \theta_r(a)\}$, satisfying the conditions $c_i(u(t) : t \in J_2)$ is compact lying in $G$, $U \subset G$ is an arbitrary set, $J_2 = [\theta, b], \theta = \theta_v(a)$; $q^i : J^2 \times O^2 \to \mathbb{R}, i = 0, \ldots, l$, are continuously differentiable functions.

We consider the differential equation in $\mathbb{R}^n$

\[ \dot{x}(t) = f(t, x(t_0), \ldots, x(\tau_i(t)), u(\theta_1(t)), \ldots, u(\theta_r(t))), \]

\[ t \in [t_0, t_1] \subset J, \]

with the discontinuity condition

\[ x(t) = \varphi(t), \quad t \in [\tau, t_0), \quad x(t_0) = x_0. \]
Definition 2. The element $\sigma \in A$ is said to be admissible if the corresponding solution $x(t) = x(t; \sigma)$ satisfies the conditions

$$q'(t_0, t_1, x(t_0), x(t_1)) = 0, \quad i = 0, \ldots, l.$$ 

The set of admissible elements will be denoted by $A_0$.

Definition 3. The element $\hat{\sigma} = (\hat{t}_0, \hat{t}_1, \hat{x}_0, \hat{\nu}, \hat{u}) \in A_0$ is said to be optimal if for an arbitrary element $\sigma \in A_0$ the inequality

$$q^0(\hat{t}_0, \hat{t}_1, x(\hat{t}_0), x(\hat{t}_1)) \leq q^0(t_0, t_1, x(t_0), x(t_1)); \quad x(t) = x(t; \hat{\sigma})$$ 

holds.

The problem of optimal control consists in finding optimal element. In order to formulate the main results, we will need the following notations:

- Let $\gamma^i = \gamma_i(t_0^-)$, $i = 1, \ldots, s$, $\gamma_i(t)$ is the function inverse to $\tau_i(t)$; $\gamma_i = \gamma_i(t_0)$;
- $\omega^- = (t_0, x_0, \hat{x}_0, \hat{x}_0, \hat{x}_0, \hat{x}_0, \cdots, \hat{x}_0, \hat{x}_0, \hat{x}_0, \hat{x}_0, \hat{x}_0, \cdots, \hat{x}_0)$, $i = 0, \ldots, p$;
- $\omega^- = (t_0, \hat{x}(\gamma_1(t_0)), \cdots, \hat{x}(\gamma_{s+1}(t_0)), \hat{x}_0, \hat{x}_0, \hat{x}_0, \cdots, \hat{x}_0, \hat{x}_0, \hat{x}_0, \cdots, \hat{x}_0, \hat{x}_0, \cdots, \hat{x}_0)$, $i = p + 1, \ldots, s$.

Analogously is defined $\hat{\gamma}^+$, $\hat{\omega}^+$, $\hat{\omega}^+$. Theorem 1. Let $\hat{\sigma} \in A_0$ be optimal element, $t_0 \in (a, b)$, $\hat{t}_1 \in (a, b]$ and the following conditions are hold:

- $\tau_i(t_0) = \hat{t}_0$, $i = 1, \ldots, p$; $\tau_i(t_0) < \hat{t}_0$, $\tau_i(t_1) > \hat{t}_0$, $i = p + 1, \ldots, s$; there exists the left semi-neighborhood $V^-_{\hat{t}_0}$ of the point $t_0$ such that

$$t < \gamma_i(t) < \cdots < \gamma_s(t), \quad \forall t \in V^-_{\hat{t}_0}.$$ 

This next, $\gamma_{p+1} < \cdots < \gamma_s$.

1. There exist the finite limits:

$$\hat{\gamma}_i^- = \gamma_i(t_0^-), \quad i = 1, \ldots, s;$$

$$\lim_{\omega \to \omega^-} \hat{f}(\omega) = f^-_{\hat{\omega}^-}, \quad \omega = (t, x_1, \ldots, x_s) \in R^-_{t_0} \times O^s, \quad i = 0, \ldots, p,$$

where $\hat{f}(\omega) = f(\omega, \hat{u}(\hat{\theta}_1(t))), \ldots, \hat{u}(\hat{\theta}_s(t)))$,

$$\lim_{(\omega_1, \omega_2) \to (\omega^-_{\hat{\omega}^-})} \hat{f}(\omega) = f(\omega_1, \omega_2) = f^-_{\omega^-}, \quad \omega_1, \omega_2 \in R^-_{\gamma} \times O^s, \quad i = p + 1, \ldots, s.$$

Then there exists non-zero vector $\pi = (\pi_0, \ldots, \pi_s)$, $\pi_0 \leq 0$, and a solution $\psi(t)$, $t \in [\hat{t}_0, \gamma]$, $\gamma = \max(\gamma_1, \ldots, \gamma_s)$ of the equation

$$\dot{\psi}(t) = \sum_{i=1}^{s} \psi(\gamma_i(t)) f^i_{\gamma_i(t)} \gamma_i(t), \quad t \in [\hat{t}_0, \hat{t}_1],$$

$$\psi(t) = 0, \quad t \in (\hat{t}_1, \gamma),$$

$$t \in [\hat{t}_0, \hat{t}_1].$$
such that the following conditions are fulfilled:

\[
\sum_{i=p+1}^{r_1(t_0)} \int_{t_0}^{t_1} \psi(\gamma_i(t)) \bar{f}_{\gamma_i} [\gamma_i(t)] \dot{\gamma}_i(t) \dot{\varphi}(t) dt \geq \\
\geq \sum_{i=p+1}^{r_1(t_0)} \int_{t_0}^{t_1} \psi(\gamma_i(t)) \bar{f}_{\gamma_i} [\gamma_i(t)] \varphi(t) dt, \forall \varphi(\cdot) \in \Delta, \tag{5}
\]

\[
\int_{t_0}^{t_1} \bar{f}(t) dt \geq \int_{t_0}^{t_1} \psi(t) f(t, \bar{x}(\tau_1(t)), \ldots, \bar{x}(\tau_s(t)), u(\theta_1(t)), \ldots, u(\theta_s(t))) dt, \forall \psi(\cdot) \in \Omega, \tag{6}
\]

\[
\pi \bar{Q}_{x_0} = -\psi(t_0), \pi \bar{Q}_{x_1} = \psi(t_1), \tag{7}
\]

\[
\pi \bar{Q}_{x_0} \geq -\psi(t_0) \sum_{i=0}^{p} (\tilde{\gamma}_{i+1}^+ - \tilde{\gamma}_i^-) f_i^- + \sum_{i=p+1}^{r_1(t_0)} \psi(\gamma_i) f_i^- \tilde{\gamma}_i^-, \tag{8}
\]

\[
\pi \bar{Q}_{t_1} \geq -\psi(t_1) f_{p+1}^-. \tag{9}
\]

Here \( f(t) = f(t, \bar{x}(\tau_1(t)), \ldots, \bar{x}(\tau_s(t)), \bar{f}_{\gamma_i}[t] = \bar{f}_{\gamma_i}(t, \bar{x}(\tau_1(t)), \ldots, \bar{x}(\tau_s(t)); \gamma_0^+ = 1, \gamma_i^- = \gamma_i^+, i = 1, \ldots, p, \gamma_{p+1} = 0; \)

The tilde over \( Q = (q^0, \ldots, q^s)^T \) means that the corresponding gradient is calculated at the point \((t_0, t_1, x(t_0), x(t_1))\)

**Remark 1.** If

\[
\text{rank}(\bar{Q}_{x_0}, \bar{Q}_{x_1}) = 1 + l,
\]

then in theorem 1 \( \psi(t) \neq 0 \). If \( \hat{x}(t_0) = \bar{x}_0 \), then \( f_0^- = \cdots = f_p^- = 0, i = p+1, \ldots, s \), the condition (8) has the form

\[
\pi \bar{Q}_{t_0} \geq \psi(t_0) f_0^-.
\]

If \( \gamma_0^- < \cdots < \gamma_s^- < 1 \), then the condition (3) is held.

**Theorem 2.** Let \( \tilde{\sigma} \in A_0 \) be optimal element, \( \tilde{t}_0 \in (a, b), \tilde{t}_1 \in (a, b) \) and the following conditions hold:

1. \( \tau_i(t_0) = \tilde{t}_0, i = 1, \ldots, p; \tau_i(t_1) = \tilde{t}_1, i = p + 1, \ldots, s; \) there exists the right semi-neighborhood \( V^+(t_0) \) of the point \( t_0 \) such that

\[
t < \gamma_i(t) < \gamma_i(t), \forall t \in V^+_t_0; \tag{10}
\]

next, \( \gamma_{p+1}^- < \cdots < \gamma_s^-; \)

2. There exist the finite limits:

\[
\lim_{\omega \to \omega_i^+} \bar{f}^+ = f_i^+, \omega = (t, x_1, \ldots, x_s) \in R^+_t_0 \times O^*, i = 0, \ldots, p,
\]

\[
\lim_{(\omega_1, \omega_2) \to (\omega_i^+, \omega_i^+)} [f(\omega_1) - f(\omega_2)] = f_i^+, \omega_1, \omega_2 \in R^+_t_0 \times O^*, i = p + 1, \ldots, s,
\]

\[
\lim_{\omega \to \omega_i^+} \bar{f}(\omega) = f_i^+, \omega \in R^+_t_1 \times O^*.
\]
then there exists a non-zero vector \( \pi = (\pi_0, \ldots, \pi_1) \), \( \pi_0 \leq 0 \), and a solution \( \psi(t) \) of the equation (4) such that the conditions (5)–(7) are fulfilled. Moreover,

\[
\pi Q_{t_0} \leq -\psi(t_0) \left( \sum_{i=0}^{p} (\tilde{\gamma}_{i+1}^+ - \tilde{\gamma}_{i-}^-) f_i^- + \sum_{i=p+1}^{s} \psi(\gamma_i) f_i^+ \right), 
\]

where \( \tilde{\gamma}_i^+ = 1, \tilde{\gamma}_i^- = 4_i^+ \), \( i = 1, \ldots, p, \tilde{\gamma}_{p+1} = 0 \).

Remark. If \( \tilde{\varphi}(\tilde{t}_0^+) = \tilde{x}_0 \), then \( f_i^+ = \cdots = f_p^+ \), \( f_i^- = 0 \), \( i = p+1, \ldots, s \), the condition (11) has the form

\[
\pi Q_{t_0} \leq -\psi(t_0) f_0^+.
\]

If \( 1 < \tilde{\gamma}_1^+ < \cdots < \tilde{\gamma}_p^+ \), then the condition (10) holds.

**Theorem 3.** Let \( \hat{\pi} \in A_0 \) be an optimal element, \( \hat{t}_0, \hat{t}_1 \in (a,b) \) and the assumptions of theorems 1, 2 are hold. Let, besides

\[
\begin{align*}
\sum_{i=0}^{p} (\tilde{\gamma}_{i+1}^+ - \tilde{\gamma}_{i-}^-) f_i^- &= \sum_{i=0}^{p} (\tilde{\gamma}_{i+1}^+ - \tilde{\gamma}_{i-}^-) f_i^+ = f_0, \\
f_i^- \tilde{\gamma}_i^- = f_i^+ \tilde{\gamma}_i^+ = f_i, & \text{ } i = p+1, \ldots, s, \ f_{s+1}^+ = f_{s+1},
\end{align*}
\]

then there exists non-zero vector \( \pi = (\pi_0, \ldots, \pi_1) \), \( \pi_0 \leq 0 \) and a solution \( \psi(t) \) of the equation (4) such that the condition (5)–(7) are fulfilled. Moreover,

\[
\begin{align*}
\pi Q_{t_0} &= \psi(t_0) f_0 + \sum_{i=p+1}^{s} \psi(\gamma_i) f_i, & \pi Q_{t_1} &= -\psi(t_1) f_{s+1},
\end{align*}
\]

If

\[
\text{rank}(\tilde{Q}_{t_0}, \tilde{Q}_{t_1}, \tilde{Q} x_0, \tilde{Q} x_1) = 1 + l,
\]

then in theorem 3 \( \psi(t) \neq 0 \). If \( \tilde{\varphi}(\tilde{t}_0^-) = \tilde{\varphi}(\tilde{t}_0^+) = \tilde{x}_0 \), then \( f_i = 0, i = p+1, \ldots, s \). For the case \( s = \nu = 2, \tau_1(t) = \theta_2(t) = t \) the analogous theorems are given in [1].

Now we consider the case, when the functions \( \theta_i(t), i = 1, \ldots, \nu \), are absolutely continuous and \( \theta_i(t) \leq t, \theta_i(t) > 0 \). Next, \( U \subset G \) is a convex set and the function \( f(t, x_1, \ldots, x_s, u_1, \ldots, u_\nu) \) satisfies the following conditions: for a fixed \( t \in J \) it is continuously differentiable with respect to \( (x_1, \ldots, x_s, u_1, \ldots, u_\nu) \in O^u \times G^u \); for a fixed \( (x_1, \ldots, x_s, u_1, \ldots, u_\nu) \in O^u \times G^u \) the functions \( f_i, i = 1, \ldots, \nu, \ j = 1, \ldots, \nu \) are measurable with respect to \( t \); for arbitrary compacts \( K \subset O, V \subset G \) there exists a function \( m_{K,V}(\cdot) \in L(J, E_0^\nu) \) such that

\[
|f(t, x_1, \ldots, x_s, u_1, \ldots, u_\nu)| + \sum_{i=1}^{s} |f_{x_i}(\cdot)| + \sum_{i=1}^{\nu} |f_{u_i}(\cdot)| \leq m_{K,V}(t),
\]

\[
\forall (t, x_1, \ldots, x_s, u_1, \ldots, u_\nu) \in J \times K^u \times V^u.
\]

**Theorem 4.** Let \( \hat{\pi} \in A_0 \) be an optimal element, \( \hat{t}_0 \in (a,b), \hat{t}_1 \in (a,b) \) and the assumptions of Theorem 1 be fulfilled. Then there exist a non-zero vector \( \pi = (\pi_0, \ldots, \pi_1) \),
\[ \pi_0 \leq 0 \text{ and a solution } \psi(t) \text{ of the equation } (4) \text{ such that the conditions } (5), (7)-(9) \text{ are fulfilled. Moreover,} \]
\[ \sum_{j=1}^{l} \int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)f_{u_j}[t]u_j(t)dt \geq \sum_{j=1}^{l} \int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t)f_{u_j}[t]u_j(t)dt, \tag{14} \]

where
\[ f_{u_j}[t] = f_{u_j}(t, \tilde{x}(\tau_1(t)), \ldots, \tilde{x}(\tau_s(t)), \tilde{u}(\theta_1(t)), \ldots, \tilde{u}(\theta_{\nu}(t))). \]

**Theorem 5.** Let \( \tilde{\sigma} \in A_0 \) be an optimal element, \( \tilde{t}_0 \in [a, b] \), \( \tilde{t}_1 \in (a, b) \) and the assumptions of Theorem 2 be fulfilled. Then there exist a non-zero vector \( \pi = (\pi_0, \ldots, \pi_l) \), \( \pi_0 \leq 0 \) and a solution \( \psi(t) \) of the equation (4) such that the conditions (5), (7), (11), (12) are fulfilled.

**Theorem 6.** Let \( \tilde{\sigma} \in A_0 \) be an optimal element, \( \tilde{t}_0, \tilde{t}_1 \in (a, b) \) and the assumptions of Theorem 3 be fulfilled. Then there exist a non-zero vector \( \pi = (\pi_0, \ldots, \pi_l) \), \( \pi_0 \leq 0 \), and a solution \( \psi(t) \) of the equation (4) such that the conditions (5), (7), (13), (14) hold.

The case, when \( t_0 \) is fixed is considered in [2].

**References**


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