Consider a linear homogeneous system of generalized ordinary differential equations
\[ dx(t) = dA(t) \cdot x(t), \tag{1} \]
where \( A : [0, +\infty[ \rightarrow \mathbb{R}^{n \times n} \) is a real matrix-function with locally bounded variation components.

In this paper we give some sufficient conditions imposed on the components of matrix-function \( A \), which guarantee the stability of the system (1) in the Liapunov sense with respect to small perturbations. This conditions are differed from those given in [1]. Analogous conditions for ordinary differential equations are given in [2].

The following notations and definitions will be used in the paper:
\( \mathbb{R} = ]-\infty, +\infty[ \), \( \mathbb{R}_+ = [0, +\infty[ \), \( [a, b] \) and \( ]a, b[ \) (\( a, b \in \mathbb{R} \)) are, respectively, a closed and open intervals; \( \mathbb{R}^{n \times m} \) is the space of all real \( n \times m \) matrices \( X = (x_{ij})_{i,j=1}^{n,m} \) with the norm \( \|X\| = \max_{j=1,\ldots,m} \sum_{i=1}^{n} |x_{ij}| \); \( O_{n \times m} \) (or \( O \)) is zero \( n \times m \)-matrix; \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \) is the space of all real column \( n \)-vectors \( x = (x_i)_{i=1}^{n} \); \( I_n \) is the identity \( n \times n \)-matrix; \( V_0^b(X) = \sup_{0 \leq \tau \leq b} V_0^\tau(X) \), where \( V_0^\tau(X) \) is the sum of total variations on \([0, \tau] \) of the components \( x_{ij} \) (\( i = 1, \ldots, n; j = 1, \ldots, m \)) of the matrix-function \( X : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m} \); \( V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m} \), where \( v(x_{ij})(0) = 0 \) and \( v(x_{ij})(t) = V_0^t(x_{ij}) \) for \( 0 < t < +\infty \) (\( i = 1, \ldots, n; j = 1, \ldots, m \)).

\( X(-) \) and \( X(+) \) are the left and the right limits of the matrix-function \( X : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m} \) at the point \( t \); \( d_1 X(t) = X(t) - X(-) \), \( d_2 X(t) = X(+) - X(t) \);

\( BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times m}) \) is the set of all matrix-functions of bounded variations on every closed interval from \( \mathbb{R}_+ \).

\( s_0 : BV_{loc}(\mathbb{R}_+, \mathbb{R}) \rightarrow BV_{loc}(\mathbb{R}_+, \mathbb{R}) \) is an operator defined by
\[ s_0(x)(t) \equiv x(t) - \sum_{0 < \tau \leq t} d_1 x(\tau) - \sum_{0 < \tau < t} d_2 x(\tau). \]

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If \( g : \mathbb{R}_+ \to \mathbb{R} \) is a nondecreasing function, \( x : \mathbb{R}_+ \to \mathbb{R} \) and \( 0 \leq s < t < +\infty \), then
\[
\int_s^t x(\tau) d\tau = \int_s^t x(\tau) d\tau_1(\tau) - \int_s^t x(\tau) d\tau_2(\tau) + \sum_{s \leq \tau < t} x(\tau) d\tau_1(\tau) - \sum_{s \leq \tau < t} x(\tau) d\tau_2(\tau),
\]
where \( \tau_1 : \mathbb{R}_+ \to \mathbb{R} \) and \( \tau_2 : \mathbb{R}_+ \to \mathbb{R} \) are continuous nondecreasing functions, such that \( \tau_1(t) = \tau_2(t) = \tau(\tau) \) is Lebesgue-Stieltjes integral over the open interval \( ]s,t[ \) with respect to the measure corresponding to the function \( g \).

If \( G = (g_{ik})_{i,k=1}^{n,n} : \mathbb{R}_+ \to \mathbb{R}^{n \times n} \) is a nondecreasing matrix-function, \( X = (x_{ik})_{i,k=1}^{n,m} : \mathbb{R}_+ \to \mathbb{R}^{n \times m} \), then
\[
\int_s^t dG(\tau) \cdot X(\tau) = \left( \sum_{k=1}^n \int_s^t x_{kj}(\tau) d\tau(\tau) \right)_{i,j=1}^{n,m} \quad \text{for} \quad 0 \leq s \leq t < +\infty.
\]

If \( G_j : \mathbb{R}_+ \to \mathbb{R}^{n \times n} \) (\( j = 1,2 \)) are nondecreasing matrix-functions, \( G(t) \equiv G_1(t) - G_2(t) \) and \( X : \mathbb{R}_+ \to \mathbb{R}^{n \times m} \), then
\[
\int_s^t dG(\tau) \cdot X(\tau) = \int_s^t dG_1(\tau) \cdot X(\tau) - \int_s^t dG_2(\tau) \cdot X(\tau) \quad \text{for} \quad 0 \leq s \leq t < +\infty.
\]

\( r(H) \) is the spectral radius of the matrix \( H \in \mathbb{R}^{n \times n} \).

Under a solution of the system (1) we understand a vector function \( x \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^n) \) such that
\[
x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) \quad (0 \leq s \leq t < +\infty).
\]

We will assume that \( A = (a_{ik})_{i,k=1}^{n,n} \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times n}) \), \( A(0) = O_{n \times n} \) and
\[
det(I_n + (-1)^j d_j A(t)) \neq 0 \quad \text{for} \quad t \in \mathbb{R}_+ \ (j = 1,2).
\]

Let \( x_0 \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^n) \) be a solution of the system (1).

**Definition 1.** Let \( \xi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a nondecreasing function such that
\[
\lim_{t \to +\infty} \xi(t) = +\infty.
\]

The solution \( x_0 \) of the system (1) is called \( \xi \)-exponentially asymptotically stable, if there exists a positive number \( \eta \) such that for every \( \epsilon > 0 \) there exists a positive number \( \delta = \delta(\epsilon) > 0 \) such that an arbitrary solution \( x \) of the system (1), satisfying the inequality
\[
\|x(t_0) - x_0(t_0)\| < \delta
\]
for some \( t_0 \in \mathbb{R}_+ \), admits the estimate
\[
\|x(t) - x_0(t)\| < \epsilon \exp(-\eta(\xi(t) - \xi(t_0))) \quad \text{for} \quad t \geq t_0.
\]

Stability, uniformly stability and asymptotically stability of the solution \( x_0 \) are defined analogously as for systems of ordinary differential equations (see [2]), i.e. in case when
A(t) is the diagonal matrix-function with diagonal elements equal to t. Note that exponentially asymptotically stability ([2]) is particular case of ξ-exponentially asymptotically stability if we assume ξ(t) ≡ t.

**Definition 2.** The system (1) is called stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable) if every solution of this system is stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable).

**Definition 3.** The matrix-function A is called stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable) if the system (1) is stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable).

If $X \in BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$, then $A(X,·) : BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m}) \rightarrow BV_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ is an operator defined by

$$A(X,Y)(t) = Y(t)+ \sum_{0<\tau\leq t} d_1X(\tau) \cdot (I_n - d_1X(\tau))^{-1} \cdot d_1Y(\tau) - \sum_{0<\tau\leq t} d_2X(\tau) \cdot (I_n + d_2X(\tau))^{-1} \cdot d_2Y(\tau) \text{ for } t \in \mathbb{R}_+;$$

If $a \in BV_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ and $1 + (-1)^j d_ja(t) \neq 0$ for $t \in \mathbb{R}_+$ $(j = 1, 2)$, then $J : BV_{loc}(\mathbb{R}_+,\mathbb{R}_+) \rightarrow BV_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ is an operator defined by

$$J(a)(t) = \sum_{0<s \leq t} (d_1a(s) + \ln|1 - d_1a(s)|) + \sum_{0<s < t} (d_2a(s) - \ln|1 + d_2a(s)|) \text{ for } t \in \mathbb{R}_+. $$

**Theorem 1.** Let the components $a_{ik} (i, k = 1, \ldots, n)$ of the matrix-function $A$ satisfy the conditions

$$1 + (-1)^j d_ja_{ii}(t) \neq 0 \text{ for } t \geq t^* \quad (j = 1, 2; i = 1, \ldots, n),$$

$$\int_{t^*}^{t} \exp(a_{ii}(t) - J(a_{ii})(t) - a_{ii}(\tau) + J(a_{ii})(\tau))d\tau(b_{ik})(\tau) \leq h_{ik}$$

for $t \geq t^*$ $(i \neq k; i, k = 1, \ldots, n)$

and

$$\sup\{a_{ii}(t) - J(a_{ii})(t) : t \in \mathbb{R}_+\} < +\infty \quad (i = 1, \ldots, n),$$

where $b_{ik}(t) \equiv A(a_{ii}, a_{ik})(t) (i, k = 1, \ldots, n)$, $t^*$ and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \ldots, n)$. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0 (i = 1, \ldots, n)$, be such that

$$r(H) < 1.$$  

(5)

Then the matrix-function $A$ is stable.

**Theorem 2.** Let the components $a_{ik} (i, k = 1, \ldots, n)$ of the matrix-function $A$ satisfy the conditions (3), (4) and

$$\sup\{a_{ii}(t) - J(a_{ii})(t) - a_{ii}(\tau) + J(a_{ii})(\tau) : t \geq \tau \geq 0\} < +\infty,$$

where $t^* \in \mathbb{R}_+$ and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \ldots, n)$ are such that the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0 (i = 1, \ldots, n)$, satisfies the condition (5). Then the matrix-function $A$ is uniformly stable.
Let, moreover, the matrix $b_{ik}(t) - b_{ii}(t)$ for $t \geq \tau \geq t^*$ $(i \neq k; i, k = 1, \ldots, n)$,

$$V_i^* b_{ik} \leq -h_{ik}(t) (b_{ii}(t) - b_{ii}(\tau)) \quad \text{for} \quad t \geq \tau \geq t^* \quad (i \neq k; i, k = 1, \ldots, n),$$

where $t_+ \in \mathbb{R}_+$, $b_{ik}(t) \equiv A(a_{ii}, a_{ik})(t)$ $(i, k = 1, \ldots, n)$, $b_{ii} (i = 1, \ldots, n)$ are non-increasing functions, and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \ldots, n)$ are such that the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0 (i = 1, \ldots, n)$, satisfies the condition (5). Then the matrix-function $A$ is uniformly stable.

**Theorem 3.** Let the components $a_{ik} (i, k = 1, \ldots, n)$ of the matrix-function $A$ satisfy the conditions (3),

$$a_{ii}(t) - J(a_{ii})(t) - a_{ii}(t^*) + J(a_{ii})(t^*) \leq -\xi(t) + \xi(t^*) \quad \text{for} \quad t \geq t^* \quad (i = 1, \ldots, n)$$

and

$$\int_{t^*}^t \exp(\xi(t)) - \xi(\tau) + a_{ii}(t) - J(a_{ii})(t) - a_{ii}(\tau) + J(a_{ii})(\tau) \, dv_h(t) \leq h_{ii} \quad \text{for} \quad t \geq t^* \quad (i \neq k; i, k = 1, \ldots, n),$$

where $t^*$ and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \ldots, n)$, $b_{ik}(t) \equiv A(a_{ii}, a_{ik})(t)$ $(i, k = 1, \ldots, n)$. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0 (i = 1, \ldots, n)$, satisfies the condition (5), and the function $\xi \in BV_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the condition (2). Then the matrix-function $A$ is asymptotically stable.

**Corollary 2.** Let the components $a_{ik} (i, k = 1, \ldots, n)$ of the matrix-function $A$ satisfy the conditions (3) and (6), where $t_+ \in \mathbb{R}_+$, $b_{ik}(t) \equiv A(a_{ii}, a_{ik})(t)$ $(i, k = 1, \ldots, n)$, $b_{ii} (i = 1, \ldots, n)$ are non-increasing functions, and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \ldots, n)$ are such that the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0 (i = 1, \ldots, n)$, satisfies the condition (5). Let, moreover,

$$\lim_{t \to +\infty} a_0(t) = +\infty,$$

where

$$a_0(t) = \min \{|a_{ii}(t) - J(a_{ii})(t) - a_{ii}(t^*) + J(a_{ii})(t^*)| : i = 1, \ldots, n\} \quad (t \geq t^*).$$

Then the matrix-function $A$ is uniformly and asymptotically stable.

**Corollary 3.** Let the components $a_{ik} (i, k = 1, \ldots, n)$ of the matrix-function $A$ satisfy the conditions (3),

$$a_{ii}(t) - J(a_{ii})(t) - a_{ii}(t^*) + J(a_{ii})(t^*) \leq -\gamma(t - t^*) \quad \text{for} \quad t \geq t^* \quad (i = 1, \ldots, n)$$

and

$$\int_{t^*}^t \exp(\gamma(t - \tau)) + a_{ii}(t) - J(a_{ii})(t) - a_{ii}(\tau) + J(a_{ii})(\tau) \, dv_h(t) \leq h_{ik}$$

for $t \geq t^* \quad (i \neq k; i, k = 1, \ldots, n)$, where $\gamma > 0$, $t^*$ and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \ldots, n)$, $b_{ik}(t) \equiv A(a_{ii}, a_{ik})(t)$ $(i, k = 1, \ldots, n)$. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0 (i = 1, \ldots, n)$, satisfy the condition (5). Then $A$ is exponentially asymptotically stable.
Corollary 4. Let the components $a_{ik}$ $(i, k = 1, \ldots, n)$ of the matrix-function $A$ satisfy the conditions (3), (6) and (7), where $\gamma > 0$, $t^*$ and $h_{ik} \in \mathbb{R}_+$ $(i \neq k; i, k = 1, \ldots, n)$, $b_{ik}(t) \equiv A(a_{ii}, a_{ik})(t)$ $(i, k = 1, \ldots, n)$. Let, moreover, the matrix $H = \{h_{ik}\}_{i,k=1}^n$, where $h_{ii} = 0$ $(i = 1, \ldots, n)$, satisfy the condition (5). Then $A$ is exponentially asymptotically stable.

Theorem 4. Let $\overline{A} = (\overline{a}_{ik}) \in BV_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times n})$ be a matrix-function such that

$$
\|d_j\overline{A}(t)\| < 1 \quad \text{for} \quad t \geq 0,
$$

$$
so(a_{ii})(t) - so(a_{ii})(s) \leq so(\overline{a}_{ii})(t) - so(\overline{a}_{ii})(s)
$$

for $t > s \geq 0$; $(i = 1, \ldots, n)$,

$$
|s_0(a_{ik})(t) - s_0(a_{ik})(s)| \leq s_0(\overline{a}_{ik})(t) - s_0(\overline{a}_{ik})(s)
$$

for $t > s \geq 0$; $(i \neq k; i = 1, \ldots, n)$

and

$$
|d_j a_{ik}(t)| \leq d_j \overline{a}_{ik}(t) \quad \text{for} \quad t \geq 0 \quad (j = 1, 2; i, k = 1, \ldots, n).
$$

Let, moreover, $\overline{a}_{ik}$ $(i \neq k; i, k = 1, \ldots, n)$ are nondecreasing functions, $\overline{A}$ be stable (uniformly stable, asymptotically stable or $\xi$-exponentially asymptotically stable). Then $A$ will be stable (uniformly stable, asymptotically stable or $\xi$-exponentially asymptotically stable), too.

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References


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