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ON ADVANCED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PROPERTIES A AND B

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Dedicated to the blessed memory of Professor T. Chanturia

In the present paper we give new results on oscillatory properties of the functional differential equation

$$u^{(n)}(t) = (-1)^k \int_{\tau_0(t)}^{\tau(t)} f(u(s))p(s, t) ds.$$  \hspace{1cm} (1_k)$$

Throughout the paper it will be assumed that $n \geq 2$, $k \in \{1, 2\}$ and the following conditions are fulfilled:

(i) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous nondecreasing function such that

$$-f(x) = f(-x) > 0, \quad \frac{ds}{f(s)} = +\infty \quad \text{for} \quad x > 0, \quad \lim_{x \to +\infty} f(x) = +\infty;$$

(ii) the functions $\tau_0$ and $\tau : [0, +\infty[ \rightarrow [0, +\infty[\ (j = 1, \ldots, m)$ are continuous and

$$\tau(t) > \tau_0(t) \geq t \quad \text{for} \quad t \geq 0;$$

(iii) the function $p : [0, +\infty[ \times [0, +\infty[ \rightarrow \mathbb{R}$ is nondecreasing in the first argument, and Lebesgue integrable on each finite interval of $[0, +\infty[$ in the second argument.

Particular cases of $(1_k)$ are the following differential equations frequently occurring in the oscillation theory (see [1–17] and the references therein):

$$u^{(n)}(t) = (-1)^k \sum_{j=1}^{m} p_j(t)u(\tau_j(t))^{\lambda} \text{sgn}(u(\tau_j(t)))$$  \hspace{1cm} (2_k)$$

and

$$u^{(n)}(t) = (-1)^k \sum_{j=1}^{m} p_j(t)u(\tau_j(t)),$$  \hspace{1cm} (3_k)$$

where $\lambda \in [0, 1[, \quad \text{the functions} \quad p_j : [0, +\infty[ \rightarrow [0, +\infty[\ (j = 1, \ldots, m)$ are Lebesgue integrable on each finite interval of $[0, +\infty[,$ and $\tau_j : [0, +\infty[ \rightarrow [0, +\infty[\ (j = 1, \ldots, m)$ are continuous functions satisfying the inequalities

$$\tau_j(t) \geq t \quad (j = 1, \ldots, m).$$

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By a solution of equation (1k) on an interval \([a, +\infty) \subset [0, +\infty]\) we understand a function \(u : [a, +\infty) \to \mathbb{R}\) which is absolutely continuous together with its first \(n-1\) derivatives on each finite interval of \([0, +\infty]\) and satisfies (1k) almost everywhere on \([a, +\infty]\).

A solution \(u\) of equation (1k) is said to be proper if it is defined on an interval \([a, +\infty) \subset [0, +\infty]\) and
\[
\sup\{|u(s)| : s \geq t\} > 0 \quad \text{for} \quad t \geq a.
\]

A proper solution of equation (1k) is said to be oscillatory if it has a sequence of zeros converging to \(+\infty\).

We use the following definitions from [9] and [3].

**Definition 1.** Equation (1k) has property \(A\) if every proper solution of this equation for \(n\) even is oscillatory and for \(n\) odd either is oscillatory or satisfies the condition
\[
|u^{(i)}(t)| \downarrow 0 \quad \text{as} \quad t \to +\infty \quad (i = 0, 1, \ldots, n-1).
\]

**Definition 2.** Equation (1k) has property \(B\) if every proper solution of this equation for \(n\) even either is oscillatory or satisfies (4) or satisfies the condition
\[
|u^{(i)}(t)| \uparrow +\infty \quad \text{as} \quad t \to +\infty \quad (i = 0, 1, \ldots, n-1),
\]
and for \(n\) odd either is oscillatory or satisfies (5).

We introduce the following notation.
\[q(t) = p(\tau(t), t) - p(\tau_0(t), t), \quad q_l(t) = t^{n-l} \sum_{j=1}^{m} \left[ \tau_j(t) \right]^{l-1} p_j(t) \quad (l = 1, \ldots, n).\]

\(N_{n,k}\) is the set of \(l \in \{1, \ldots, n-1\}\) for which \(l + n + k\) is even.

For any \(l \in \{1, \ldots, n-1\}\) and \(a > 0\) the function \(v_{a,l} : [a, +\infty) \to [0, +\infty]\) is the lower solution of the Cauchy problem
\[v'(t) = \frac{1}{(n-l)!} \int_{\tau_0(t)}^{\tau(t)} f(s^{l-1} p(s)) d_s, \quad v(a) = 1.\]

**Theorem 1.** The condition
\[
\int_{0}^{+\infty} t^{n-l-1} q(t) dt = +\infty
\]

is necessary for equation (11) (equation (12)) to have property \(A\) (property \(B\)). If along with (6) the condition
\[
\int_{a}^{+\infty} t^{n-l-1} \left[ \int_{\tau_0(t)}^{\tau(t)} f(s^{l-1} v_{a,l}(s)) d_s p(s, t) \right] dt = +\infty
\]
holds for any \(a > 0\) and \(l \in N_{n,1}\) (for any \(a > 0\) and \(l \in N_{n,2}\), then equation (11) (equation (12)) has property \(A\) (property \(B\)).
Corollary 1. Let condition (6) be fulfilled. Then there exists a continuous function 
\( \tau_* : [0, +\infty[ \to [0, +\infty[ \) such that if 
\[
\tau_0(t) \geq \tau_*(t) \quad \text{for} \ t \geq 0,
\]
then equation (1) (equation (12)) has property A (property B).

Theorem 2. Let \( n \) be odd (even) and
\[
\liminf_{t \to +\infty} \frac{f(\tau_0^{-1}(t))}{t} > 0
\]
for any \( l \in \mathcal{N}_{n,1} \) (for any \( l \in \mathcal{N}_{n,2} \)). Then condition (6) is necessary and sufficient for equation (1) (equation (12)) to have property A (property B).

Theorems 1, 2 and Corollary 1 generalize respectively Theorems 1.1, 1.2 and Corollary 1.1 from [7]. For equations (21) and (31) from these results we have the following statements.

Corollary 2. The condition
\[
\int_0^{+\infty} t^{n-1} q_1(t) dt = +\infty
\]
is necessary for equation (21) (equation (22)) to have property A (property B). If along with (7) the condition
\[
\int_0^{+\infty} t^{n-1} \left[ \sum_{j=1}^{m} \tau_j(t)^{\lambda(l-1)} p_j(t) \left( \int_0^{\tau_j(t)} q(s) ds \right)^{\frac{1}{\lambda}} \right] dt = +\infty
\]
holds for any \( l \in \mathcal{N}_{n,1} \) (for any \( l \in \mathcal{N}_{n,2} \)), then equation (21) (equation (22)) has property A (property B).

Corollary 3. Let condition (7) be fulfilled. Then there exists a continuous function 
\( \tau_* : [0, +\infty[ \to [0, +\infty[ \) such that if
\[
\tau_j(t) \geq \tau_*(t) \quad \text{for} \ t \geq 0 \quad (j = 1, \ldots, m),
\]
then equation (21) (equation (22)) has property A (property B).

Corollary 4. Let \( n \) be odd (even) and
\[
\liminf_{t \to +\infty} \left[ t^{-\lambda} \tau_j(t) \right] > 0 \quad (j = 1, \ldots, m).
\]
Then condition (7) is necessary and sufficient for equation (21) (equation (22)) to have property A (property B).

Corollary 5. The condition (7) is necessary for equation (31) (equation (32)) to have property A (property B). If along with (7) the condition
\[
\int_0^{+\infty} t^{n-1} \left[ \sum_{j=1}^{m} \tau_j(t)^{\lambda(l-1)} \exp \left( \frac{1}{(n-l)! t^l} \int_0^{\tau_j(t)} q(s) ds \right) p_j(t) \right] dt = +\infty
\]
holds for any \( l \in \mathcal{N}_{n,1} \) (for any \( l \in \mathcal{N}_{n,2} \)), then equation (31) (equation (32)) has property A (property B).
Corollary 6. Let condition (7) be fulfilled. Then there exists a continuous function \( \tau : [0, +\infty) \to [0, +\infty] \) such that if inequalities (8) hold, then equation (3_1) (equation (3_2)) has property A (property B).

Corollary 7. Let \( n \) be odd (even) and
\[
\liminf_{t \to +\infty} [t^{-2}\tau_j(t)] > 0 \quad (j = 1, \ldots, m).
\]
Then condition (7) is necessary and sufficient for equation (3_1) (equation (3_2)) to have property A (property B).

Note that Corollaries 2–7 take into account the effect of advanced arguments since, as it is well-known (see [4]), in the case
\[
\tau_j(t) \equiv t \quad (j = 1, \ldots, m)
\]
condition (7) does not guarantee that equations (2_1) and (3_1) (equations (2_2) and (3_2)) have property A (property B).

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References


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