COMPARISON THEOREMS FOR DIFFERENTIAL EQUATIONS WITH SEVERAL DEVIATIONS. THE CASE OF PROPERTY B.

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1. INTRODUCTION

In the present paper we consider the following differential equations

\[ u^{(n)}(t) - \sum_{i=1}^{m} p_i(t)u(t_\tau_i(t)), \]

\[ \psi^{(n)}(t) - \sum_{j=1}^{r} q_j(t)\psi(t_\sigma_j(t)), \]

where \( n, m, r \in \mathbb{N}, n \geq 3, p_i, q_j \in L_{\infty}([R_+; R_+], \tau_i, \sigma_j \in C([R_+; R_+]), \lim_{t \to +\infty} \tau_i(t) = \lim_{t \to +\infty} \sigma_j(t) = +\infty \) \((i = 1, \ldots, m; j = 1, \ldots, r)\).

Definition 1.1. We say that the equation (1.1) has Property B if any of its proper solutions either is oscillatory or satisfies

\[ |u^{(i)}(t)| \uparrow +\infty \quad \text{for} \quad t \uparrow +\infty \quad (i = 0, \ldots, n - 1), \]

when \( n \) is odd, and either is oscillatory or satisfies either (1.3) or

\[ |u^{(i)}(t)| \downarrow 0 \quad \text{for} \quad t \uparrow +\infty \quad (i = 0, \ldots, n - 1), \]

when \( n \) is even.

Below we give comparison theorems allowing to deduce Property B of the equation (1.1) from Property B of the equation (1.2). The results obtained here generalize those of [1]. The result obtained in [1] is a generalization of a theorem of T. Chanturia (see [2], Theorem 1.5) even in the case of ordinary differential equations \((\tau_i(t) \equiv \sigma_j(t) \equiv t, i = 1, \ldots, m; j = 1, \ldots, r)\). For analogous results concerning Property A see [3].

2. GENERAL COMPARISON THEOREMS

Let \( \varphi \in C([t_0, +\infty), (0, +\infty)) \). Below we use the following notation

\[ p_{\tau_i}(t) = \begin{cases} p_i(t), & \text{if } \varphi(t) \leq \tau_i(t), \\ 0 & \text{if } \varphi(t) > \tau_i(t), \quad t \in [t_0, +\infty), \quad (i = 1, \ldots, m). \end{cases} \]

\[ q_{\sigma_j}(t) = \begin{cases} q_j(t), & \text{if } \varphi(t) \leq \sigma_j(t), \\ 0 & \text{if } \varphi(t) > \sigma_j(t), \quad t \in [t_0, +\infty), \quad (i = 1, \ldots, r). \end{cases} \]

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Theorem 2.1. Let

\[ \tau_i(t) \leq t, \quad \forall t \in R, \quad (i = 1, \ldots, m), \]

\[ \int_{-\infty}^{+\infty} \sum_{i=1}^{m} p_i(t) \tau_i^{n-1}(t) dt = +\infty, \]

and there exist natural numbers \( k \in N, \ m_j, \ r_j \in N, \ (j = 1, \ldots, k) \) and nondecreasing functions \( \varphi_j \in C(R_+; (0, +\infty)) \) \((j = 0, \ldots, k - 1)\) such that

\[ 1 \leq m_1 < m_2 < \cdots < m_k = m, \quad 1 \leq r_1 < r_2 < \cdots < r_k = r, \]

\[ \lim_{t \to +\infty} \varphi_j(t) = +\infty \quad (j = 0, \ldots, k - 1), \]

the below inequality \((2.7_{n=2})\) holds when \( n \) is even,

\[ \int_{t}^{+\infty} s^{n-1} \sum_{i=m_j+1}^{r_j+1} \tau_i^{n-1}(s) \left( \frac{p_{r_i, \varphi_j}(s)}{\varphi_j(s)} \right) ds \geq \]

\[ \int_{t}^{+\infty} s^{n-1} \sum_{i=m_j+1}^{r_j+1} \tau_i^{n-1}(s) \left( \frac{p_{r_i, \varphi_j}(s)}{\varphi_j(s)} \right) ds \]  \( (2.7_1) \)

\[ \forall t \geq t_0 \quad (j = 0, \ldots, k - 1), \]

and \((2.7_1)\) and \((2.7_{n=2})\) hold when \( n \) is odd, where \( t_0 \) is sufficiently large, \( m_0 = r_0 = 0, \)
the functions \( p_{r_i, \varphi_j} \) and \( q_{r_i, \varphi_j} \) are defined by \((2.1)\) and \((2.2)\), respectively. Let, moreover, the equation \((1.2)\) have Property \( B \). Then the equation \((1.1)\) also has Property \( B \).

Theorem 2.2. Let

\[ \tau_i(t) \geq t, \quad \forall t \in R, \quad (i = 1, \ldots, m), \]

and there exist natural numbers \( k, \ m_j, \ r_j \in N, \ (j = 1, \ldots, k) \) and nondecreasing functions \( \varphi_j \in C(R_+; (0, +\infty)) \) \((j = 0, \ldots, k - 1)\) satisfying \((2.5)\) and \((2.6)\) such that the below inequality \((2.7_2)\) and \((2.9)\) hold,

\[ \int_{t}^{+\infty} s^{n-1} \sum_{i=1}^{m} p_i(t) dt = +\infty \]

when \( n \) is even and \((2.4)\) and \((2.7_1)\) hold when \( n \) is odd, where the functions \( p_{r_i, \varphi_j} \) and \( q_{r_i, \varphi_j} \) are defined by \((2.1)\) and \((2.2)\), respectively. Let, moreover, the equation \((1.2)\) have Property \( B \). Then the equation \((1.1)\) also has Property \( B \).

Theorem 2.3. Let the conditions \((2.3)\) and \((2.4)\) be fulfilled, and for sufficiently large \( t_0 \) there exist \( t_1 = t_1(t_0) \geq t_0, \) natural numbers \( k, \ m_j, \ r_j \in N, \ (j = 1, \ldots, k) \) and nondecreasing functions \( \varphi_j \in C(R_+; (0, +\infty)) \) \((j = 0, \ldots, k - 1)\) satisfying \((2.5)\) and \((2.6)\) such that the below inequality \((2.10_{n=2})\) holds when \( n \) is even,

\[ \int_{t_0}^{t} s^{n-1} \sum_{i=m_j+1}^{r_j+1} \tau_i^{n-1}(s) \left( \frac{p_{r_i, \varphi_j}(s)}{\varphi_j(s)} \right) ds \geq \]


\[
\geq \int_{t_0}^{\tau} s^{n-1-1} \sum_{i=t_j+1}^{r_j+1} \frac{\varphi_i(s)}{\sigma_i(s)} \left( q_{\tau_i,\tau_i}(s) + \frac{\varphi_i(s)}{\sigma_i(s)} (q_i(s) - q_{\tau_i,\tau_i}(s)) \right) ds \tag{2.10_1}
\]

\[ \forall \ t \geq t_0 \ (j = 0, \ldots, k - 1), \]

and (2.10_1) and (2.10_2) hold when \( n \) is odd, where the functions \( p_{\tau_i,\tau_i} \) and \( q_{\tau_i,\tau_i} \) are defined by (2.1) and (2.2) respectively. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 2.4.** Let (2.8) be fulfilled, and for sufficiently large \( \tau \) there exist \( t_i - t_i(t_0) \geq t_0 \), natural numbers \( k, m, t_j \in N (j = 1, \ldots, k) \) and nondecreasing functions \( \varphi_j(t) (j = 0, \ldots, k - 1) \) satisfying (2.5) and (2.6) respectively, such that the inequalities (2.9) and (2.10a) hold when \( n \) is even and (2.4) and (2.10b) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

3. **Effective Comparison Theorems**

Everywhere below we assume that \( \sigma_i(t) (i = 1, \ldots, r) \) are nondecreasing functions.

**Theorem 3.1.** Let \( m = r \), the conditions (2.3) and (2.4) be fulfilled, the below inequality (3.1_{\text{m} = 2}) hold, when \( n \) is even
\[
\int_{t_0}^{\tau} s^{n-1-1} \varphi^{-1}(s) \left( p_{\tau,\tau}(s) + \frac{\tau(s)}{\sigma(s)} (p_i(s) - p_{\tau,\tau}(s)) \right) ds \geq
\]

\[ \geq \int_{t_0}^{\tau} s^{n-1-1} \varphi^{-1}(s) q_i(s) ds \ \forall \ t \geq t_0 \ (i = 1, \ldots, m) \tag{3.1_1}
\]

and the inequalities (3.1_1) and (3.1_{\text{m} = 2}) hold when \( n \) is odd, where \( t_0 \) is sufficiently large and the functions \( p_{\tau,\tau}, \sigma_i \) are defined by (2.1).

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.1.** Let the conditions (2.3), (2.4) be fulfilled, \( m - r, \sigma_i(t) \leq \tau_i(t) (i = 1, \ldots, m) \) the below inequality (3.2_{\text{m} = 2}) hold, when \( n \) is even
\[
\int_{t_0}^{\tau} s^{n-1-1} \varphi^{-1}(s) p_i(s) ds \geq \int_{t_0}^{\tau} s^{n-1-1} \varphi^{-1}(s) q_i(s) ds \ \forall \ t \geq t_0 \ (i = 1, \ldots, m), \tag{3.2_1}
\]

and the inequalities (3.2_1) and (3.2_{\text{m} = 2}) hold when \( n \) is odd, where \( t_0 \) is sufficiently large. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.2.** Let the conditions (2.3), (2.4) be fulfilled, \( m - r, \sigma_i(t) \geq \tau_i(t) (i = 1, \ldots, m) \) the below inequality (3.3_{\text{m} = 2}) hold, when \( n \) is even
\[
\int_{t_0}^{\tau} s^{n-1-1} \varphi^{-1}(s) p_i(s) ds \geq \int_{t_0}^{\tau} s^{n-1-1} \varphi^{-1}(s) q_i(s) ds \ \forall \ t \geq t_0 \ (i = 1, \ldots, m), \tag{3.3_1}
\]
and the inequalities (3.3.1) and (3.3.2) hold when \( n \) is odd, where \( t_0 \) is sufficiently large. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 3.2.** Let \( m = r \), the inequality (2.8) be fulfilled, the conditions (2.9) and (3.1) hold when \( n \) is even and the conditions (2.4) and (3.1) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.3.** Let \( m = r \), the inequality (2.8) be fulfilled \( \sigma_i(t) \leq \tau_i(t) \) \((i = 1, \ldots, m)\), the conditions (2.9) and (3.3.2) hold when \( n \) is even and the conditions (2.4) and (3.3.2) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.4.** Let \( m = r \), the inequality (3.8) be fulfilled \( \sigma_i(t) \geq \tau_i(t) \) \((i = 1, \ldots, m)\), the conditions (2.9) and (3.3.2) hold when \( n \) is odd and the conditions (2.4) and (3.3.2) hold when \( n \) is even. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 3.3.** Let \( m = r \), the conditions (2.3) and (2.4) be fulfilled and for any sufficiently large to there exist \( t_1 - t_1(t_0) \geq t_0 \) such that the below inequality (3.3.2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-i} \sigma_i(s) \left( \frac{\dot{\sigma}_i(s) \ddot{\sigma}_i(s)}{\tau_i(s)} - \frac{\sigma_i(s)}{\tau_i(s)} \ddot{\sigma}_i(s) \right) ds \geq 0, \quad (3.4)
\]

and the inequalities (3.4) and (3.3.2) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.5.** Let \( m = r \), the conditions (2.3), (2.4) be fulfilled, \( \sigma_i(t) \leq \tau_i(t) \) \((i = 1, \ldots, m)\), and for any sufficiently large to there exist \( t_1 - t_1(t_0) \geq t_0 \) such that the below inequality (3.3.2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-i-1} \sigma_i(s) \ddot{\sigma}_i(s) ds \geq \int_{t_0}^{t} s^{n-i} \sigma_i(s) \ddot{\sigma}_i(s) ds, \quad \text{if } t \geq t_1(i = 1, \ldots, m), \quad (3.5)
\]

and the inequalities (3.5) and (3.3.1) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.6.** Let \( m = r \), the conditions (2.3), (2.4) be fulfilled, \( \sigma_i(t) \geq \tau_i(t) \) \((i = 1, \ldots, m)\), and for any sufficiently large to there exist \( t_1 - t_1(t_0) \geq t_0 \) to such that the below inequality (3.3.2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-i} \sigma_i(s) ds \geq \int_{t_0}^{t} s^{n-i} \dot{\sigma}_i(s) ds, \quad \text{if } t \geq t_1 \quad (i = 1, \ldots, m), \quad (3.6)
\]

and the inequalities (3.6) and (3.3.2) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.
Theorem 3.4. Let \( m = t \), the inequality (2.8) be fulfilled, and for any sufficiently large \( t_0 \) there exist \( t_1 = t_2 = \cdots = t_0 \) such that the conditions (2.9) and (3.5a) hold when \( n \) is even and the conditions (2.4) and (3.4) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Corollary 3.7. Let \( m = t \), the inequality (2.8) be fulfilled, \( \sigma_i(t) \leq \tau_i(t) (i = 1, \ldots, m) \), the conditions (2.9) and (3.5a) hold when \( n \) is even and the conditions (2.4) and (3.4) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Corollary 3.8. Let \( m = t \), the inequality (2.8) be fulfilled, \( \sigma_i(t) \geq \tau_i(t) (i = 1, \ldots, m) \), the conditions (2.9) and (3.5a) hold when \( n \) is even and the conditions (2.4) and (3.4) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Below we use the following notation:

\[
\tau_i(t) = \min\{\tau_i(t) : i = 1, \ldots, m\}, \quad \tau^*(t) = \max\{\tau_i(t) : i = 1, \ldots, m\}, \\
\sigma_i(t) = \min\{\sigma_i(t) : i = 1, \ldots, r\}, \quad \sigma^*(t) = \max\{\sigma_i(t) : i = 1, \ldots, r\}.
\]

Theorem 3.5. Let \( \tau^*(t) \leq t \) for \( t \in R_+ \), the condition (2.4) be fulfilled, the below inequality (3.76) hold. when \( n \) is even

\[
\int_{t}^{t_0} s^{n-1} \sum_{i=1}^{m} \tau_i^{-1}(s) \left( p_{\tau_i,\sigma^*}(s) + \frac{\tau_i(s)}{\sigma^*(s)} (p_{i}(s) - p_{\tau_i,\sigma^*}(s)) \right) ds \geq \int_{t}^{t_0} s^{n-1} \sum_{i=1}^{r} \sigma_i^{-1}(s) \eta_i(s) ds \quad \text{if } t \geq t_0,
\]

and the inequalities (3.71) and (3.76) hold when \( n \) is odd, where \( t_0 \) is sufficiently large and the functions \( p_{\tau_i,\sigma^*} \) are defined by (2.1). Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Corollary 3.9. Let \( \tau^*(t) \leq t \) for \( t \in R_+ \), and the condition (2.4) be fulfilled along with one of the following four conditions (to is sufficiently large):

1) \( \tau_i(t) \geq \sigma^*(t) \) for \( t \in R_+ \), the below inequality (3.76) holds, when \( n \) is even

\[
\int_{t}^{t_0} s^{n-1} \sum_{i=1}^{m} \tau_i^{-1}(s) p_i(s) ds \geq \int_{t}^{t_0} s^{n-1} \sum_{i=1}^{r} \sigma_i^{-1}(s) \eta_i(s) ds \quad \text{if } t \geq t_0,
\]

and the inequalities (3.8) and (3.76) hold when \( n \) is odd;

2) \( \tau_i(t) \geq \sigma^*(t) \) for \( t \in R_+ \), the below inequality (3.76) holds, when \( n \) is even

\[
\int_{t}^{t_0} s^{n-1} \frac{\tau_i(s)}{\sigma^*(s)} \sum_{i=1}^{m} \tau_i^{-1}(s) p_i(s) ds \geq \int_{t}^{t_0} s^{n-1} \sum_{i=1}^{r} \sigma_i^{-1}(s) \eta_i(s) ds \quad \text{if } t \geq t_0,
\]

and the inequalities (3.9) and (3.76) hold when \( n \) is odd;

3) \( \tau^*(t) \geq \sigma^*(t) \) for \( t \in R_+ \), the below inequality (3.76) holds, when \( n \) is even

\[
\int_{t}^{t_0} \frac{s^{n-1}}{\tau^*(s)} \sum_{i=1}^{m} \tau_i(s) p_i(s) ds \geq \int_{t}^{t_0} s^{n-1} \sum_{i=1}^{r} \sigma_i^{-1}(s) \eta_i(s) ds \quad \text{if } t \geq t_0,
\]

and the inequalities (3.10) and (3.76) hold when \( n \) is odd;
4) \( \tau^*(t) \leq \sigma^*(t) \) for \( t \in \mathbb{R}_+ \), the below inequality (3.11_{n-2}) holds, when \( n \) is even

\[
\int_{t}^{+\infty} s^{n-1} \sum_{i=1}^{m} \eta_i^*(s)p_i(s)ds \geq \int_{t}^{+\infty} s^{n-1} \sum_{i=1}^{r} \sigma_i^*(s)q_i(s)ds \quad \text{if} \quad t \geq t_0, \quad (3.11_i)
\]

and the inequalities (3.11_i) and (3.11) hold when \( n \) is odd.

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Theorem 3.6. Let \( \tau^*(t) \leq t \) for \( t \in \mathbb{R}_+ \), the condition (2.4) be fulfilled, the below inequality (3.12_{n-2}) hold, when \( n \) is even

\[
\int_{t}^{+\infty} s^{n-1} \sum_{i=1}^{m} \eta_i^*(s)\left(p_{r_i,s}(s) + \frac{\eta(s)}{\sigma(s)}(p_i(0) - p_{r_i,s}(s))\right)ds \geq \int_{t}^{+\infty} s^{n-1} \sum_{i=1}^{r} \sigma_i^*(s)q_i(s)ds \quad \text{if} \quad t \geq t_0, \quad (3.12_i)
\]

and the inequalities (3.12_i) and (3.12_{n-2}) holds when \( n \) is odd, where to is sufficiently large and the functions \( p_{r_i,s} \) are defined by (2.1). Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Corollary 3.10. Let \( \tau^*(t) \geq t \) for \( t \in \mathbb{R}_+ \), the condition (2.4) be fulfilled along with one of the following four conditions (to is sufficiently large):

1) \( \tau^*(t) \geq \sigma(t) \) for \( t \in \mathbb{R}_+ \), the below inequality (3.13_{n-2}) holds, when \( n \) is even

\[
\int_{t}^{+\infty} s^{n-1} \sum_{i=1}^{m} \eta_i^*(s)p_i(s)ds \geq \int_{t}^{+\infty} s^{n-1} \sum_{i=1}^{r} \sigma_i^*(s)q_i(s)ds \quad \text{if} \quad t \geq t_0, \quad (3.13_i)
\]

and the inequalities (3.13_i) and (3.13_{n-2}) hold when \( n \) is odd;

2) \( \tau^*(t) \leq \sigma(t) \) for \( t \in \mathbb{R}_+ \), the below inequality (3.14_{n-2}) holds, when \( n \) is even

\[
\int_{t}^{+\infty} s^{n-1} \sigma_*(s) \eta^*(s)p_i(s)ds \geq \int_{t}^{+\infty} s^{n-1} \sigma_*(s) \eta^*(s)p_i(s)ds \quad \text{if} \quad t \geq t_0, \quad (3.14_i)
\]

and the inequalities (3.14_i) and (3.14_{n-2}) hold when \( n \) is odd;

3) \( \tau^*(t) \geq \sigma(t) \) for \( t \in \mathbb{R}_+ \), the below inequality (3.15_{n-2}) holds, when \( n \) is even

\[
\int_{t}^{+\infty} s^{n-1} \sigma_*(s) \eta^*(s)p_i(s)ds \geq \int_{t}^{+\infty} s^{n-1} \sigma_*(s) \eta^*(s)p_i(s)ds \quad \text{if} \quad t \geq t_0, \quad (3.15_i)
\]

and the inequalities (3.15_i) and (3.15_{n-2}) hold when \( n \) is odd;

4) \( \tau^*(t) \leq \sigma(t) \) for \( t \in \mathbb{R}_+ \), the below inequality (3.16_{n-2}) holds, when \( n \) is even

\[
\int_{t}^{+\infty} s^{n-1} \sigma_*(s) \eta^*(s)p_i(s)ds \geq \int_{t}^{+\infty} s^{n-1} \sigma_*(s) \eta^*(s)p_i(s)ds \quad \text{if} \quad t \geq t_0, \quad (3.16_i)
\]

and the inequalities (3.16_i) and (3.16_{n-2}) hold when \( n \) is odd.
Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 3.7.** Let \( \tau(t) \geq t \) for \( n \), the conditions (2.9) and (3.7a) be fulfilled when \( n \) is even and the conditions (2.4) and (3.7a) holds when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.11.** Let \( \tau(t) \geq t \) for \( t \in R \), and one of the following four conditions be fulfilled:
1) \( \tau(t) \leq \sigma^*(t) \), the conditions (2.9) and (3.9a) hold when \( n \) is even and the conditions (2.4) and (3.9a) hold when \( n \) is odd;
2) \( \tau(t) \geq \sigma^*(t) \), the conditions (2.9) and (3.8a) hold when \( n \) is even and the conditions (2.4) and (3.8a) hold when \( n \) is odd;
3) \( \tau(t) \leq \sigma^*(t) \), the conditions (2.9) and (3.11a) hold when \( n \) is even and the conditions (2.4) and (3.11a) hold when \( n \) is odd;
4) \( \tau(t) \geq \sigma^*(t) \), the conditions (2.9) and (3.10a) hold when \( n \) is even and the conditions (2.4) and (3.10a) hold when \( n \) is odd.

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 3.8.** Let \( \tau(t) \geq t \), the conditions (2.9) and (3.12a) be fulfilled when \( n \) is even and the conditions (2.4) and (3.12a) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.12.** Let \( \tau(t) \geq t \) for \( t \in R \), and one of the following four conditions be fulfilled:
1) \( \tau(t) \leq \sigma(t) \), the conditions (2.9) and (3.14a) hold when \( n \) is even and the conditions (2.4) and (3.14a) hold when \( n \) is odd;
2) \( \tau(t) \geq \sigma(t) \), the conditions (2.9) and (3.15a) hold when \( n \) is even and the conditions (2.4) and (3.15a) hold when \( n \) is odd;
3) \( \tau(t) \geq \sigma(t) \), the conditions (2.9) and (3.16a) hold when \( n \) is even and the conditions (2.4) and (3.16a) hold when \( n \) is odd;
4) \( \tau(t) \leq \sigma(t) \), the conditions (2.9) and (3.17a) hold when \( n \) is even and the conditions (2.4) and (3.17a) hold when \( n \) is odd.

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 3.9.** Let \( \tau'(t) \leq t \), the condition (2.4) be fulfilled and for any sufficiently large to there exist \( t_1 - t_1(t_0) \geq t \), such that the below inequality (3.17a) holds, when \( n \) is even

\[
\int_{t_0}^t s^{n-1} \sum_{i=1}^m r_i(s)(p_i(s) - p_i(s)) + \frac{\sigma^*(s)}{n(s)} p_i(s) ds \geq 0 \quad t \geq t_1, \tag{3.17a}
\]

and the inequalities (3.17b) and (3.17c) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.
Corollary 3.13. Let \( r^*(t) \leq t \), the condition (2.4) be fulfilled and for any sufficiently large to there exist \( t_1 - t_1(t_0) \geq t_0 \) such that one of the following four conditions is fulfilled:

1) \( r_1(t) \leq \sigma^*(t) \) for \( t \in R_+ \), the below inequality (3.18n-2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-1} \tau_r(s) \sum_{i=1}^{m} \tau_i^{-1}(s)p_i(s)ds \geq \int_{t_0}^{t} s^{n-1} \sigma^*(s) \sum_{i=1}^{r} \sigma_i^{-1}(s)q_i(s)ds \quad \text{if} \quad t \geq t_1, \quad (3.18_1)
\]

and the inequalities (3.18_1) and (3.18n-2) hold when \( n \) is odd;

2) \( r_1(t) \geq \sigma^*(t) \) for \( t \in R_+ \), the below inequality (3.19n-2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-2} \tau_1(s) \sum_{i=1}^{m} \tau_i^{-1}(s)p_i(s)ds \geq \int_{t_0}^{t} s^{n-1} \sigma^*(s) \sum_{i=1}^{r} \sigma_i^{-1}(s)q_i(s)ds \quad \text{if} \quad t \geq t_1, \quad (3.19_1)
\]

and the inequalities (3.19_1) and (3.19n-2) hold when \( n \) is odd;

3) \( r^*(t) \leq \sigma^*(t) \) for \( t \in R_+ \), the below inequality (3.20n-2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-1} \tau_1(s) \sum_{i=1}^{m} \tau_i^{-1}(s)p_i(s)ds \geq \int_{t_0}^{t} s^{n-1} \sigma^*(s) \sum_{i=1}^{r} \sigma_i^{-1}(s)q_i(s)ds \quad \text{if} \quad t \geq t_1, \quad (3.20_1)
\]

and the inequalities (3.20_1) and (3.20n-2) hold when \( n \) is odd;

4) \( r^*(t) \geq \sigma^*(t) \) for \( t \in R_+ \), the below inequality (3.21n-2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-1} \tau_1(s) \sum_{i=1}^{m} \tau_i^{-1}(s)p_i(s)ds \geq \int_{t_0}^{t} s^{n-1} \sigma^*(s) \sum_{i=1}^{r} \sigma_i^{-1}(s)q_i(s)ds \quad \text{if} \quad t \geq t_1, \quad (3.21_1)
\]

and the inequalities (3.21_1) and (3.21n-2) hold when \( n \) is odd.

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Theorem 3.10. Let \( r^*(t) \leq t \), for \( t \in R_+ \) the condition (2.4) be fulfilled and for any sufficiently large to there exist \( t - t_1(t_0) \geq t_0 \) such that the below inequality (3.22n-2) holds, when \( n \) is even

\[
\int_{t_0}^{t} s^{n-1} \sum_{i=1}^{m} \tau_i^{-1}(s)\left((p_i(s) - P_{r_1,\tau_1}(s)) + \frac{\sigma^*(s)}{\tau_1(s)} P_{r_1,\tau_1}(s)\right)ds \geq \int_{t_0}^{t} s^{n-1} \sum_{i=1}^{r} \sigma_i^{-1}(s)q_i(s)ds \quad \text{if} \quad t \geq t_1, \quad (3.22_1)
\]

and the inequalities (3.22_1) and (3.22n-2) hold when \( n \) is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

Corollary 3.14. Let \( r^*(t) \leq t \), for \( t \in R_+ \) the condition (2.4) be fulfilled and for any sufficiently large to there exist \( t - t_1(t_0) \geq t_0 \) such that one of the following four conditions is fulfilled:
1) $\tau_t(t) \leq \sigma_t(t)$ for $t \in R_+$, the below inequality (3.33n-2) holds, when $n$ is even

$$\int_{t_0}^{t} s^{n-1} \tau_t(s) \sum_{i=1}^{m} \sigma_i(s) \eta_i(s) ds \geq \int_{t_0}^{t} s^{n-1} \sum_{i=1}^{r} \sigma_i(s) \eta_i(s) ds \quad \forall \ t \geq t_1, \ (3.23_i)$$

and the inequalities (3.23_i) and (3.33n-2) hold when $n$ is odd;

2) $\tau_t(t) \geq \sigma_t(t)$ for $t \in R_+$, the below inequality (3.34n-2) holds, when $n$ is even

$$\int_{t_0}^{t} s^{n-1} \sigma_t(s) \sum_{i=1}^{m} \tau_i^{-1}(s) \eta_i(s) ds \geq \int_{t_0}^{t} s^{n-1} \sum_{i=1}^{r} \sigma_i(s) \eta_i(s) ds \quad \forall \ t \geq t_1, \ (3.24_i)$$

and the inequalities (3.24_i) and (3.34n-2) hold when $n$ is odd;

3) $\tau_t(t) \leq \sigma_t(t)$ for $t \in R_+$, the below inequality (3.35n-2) holds

$$\int_{t_0}^{t} s^{n-1} \sigma_t(s) \sum_{i=1}^{m} \tau_i^{-1}(s) \eta_i(s) ds \geq \int_{t_0}^{t} s^{n-1} \sum_{i=1}^{r} \sigma_i(s) \eta_i(s) ds \quad \forall \ t \geq t_1, \ (3.25_i)$$

when $n$ is even and the inequalities (3.25_i) and (3.35n-2) hold when $n$ is odd;

4) $\tau_t(t) \geq \sigma_t(t)$ for $t \in R_+$, the below inequality (3.36n-2) holds, when $n$ is even

$$\int_{t_0}^{t} s^{n-1} \sigma_t(s) \sum_{i=1}^{m} \tau_i^{-1}(s) \eta_i(s) ds \geq \int_{t_0}^{t} s^{n-1} \sum_{i=1}^{r} \sigma_i(s) \eta_i(s) ds \quad \forall \ t \geq t_1, \ (3.26_i)$$

and the inequalities (3.26_i) and (3.36n-2) hold when $n$ is odd.

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 3.11.** Let $\tau_t(t) \geq t$ and for any sufficiently large $t_0$ there exist $t_1 - t_1(t_0) \geq t_0$ such that the conditions (2.9) and (3.17i) are fulfilled when $n$ is even and the conditions (2.4) and (3.17i) hold when $n$ is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Corollary 3.15.** Let $\tau_t(t) \geq t$ and for any sufficiently large $t_0$ there exist $t_1 - t_1(t_0) \geq t_0$ such that one of the following four conditions is fulfilled:

1) $\tau_t(t) \leq \sigma_t(t)$, the conditions (2.9) and (3.15x2) hold when $n$ is even and the conditions (2.4) and (3.18i) hold when $n$ is odd;

2) $\tau_t(t) \geq \sigma_t(t)$, the conditions (2.9) and (3.19x2) hold when $n$ is even and the conditions (2.4) and (3.19i) hold when $n$ is odd;

3) $\tau_t(t) \leq \sigma_t(t)$, the conditions (2.9) and (3.20x2) hold when $n$ is even and the conditions (2.4) and (3.20i) hold when $n$ is odd;

4) $\tau_t(t) \geq \sigma_t(t)$, the conditions (2.9) and (3.21x2) hold when $n$ is even and the conditions (2.4) and (3.21i) hold when $n$ is odd.

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

**Theorem 3.12.** Let $\tau_t(t) \geq t$ and for any sufficiently large $t_0$ there exist $t_1 - t_1(t_0) \geq t_0$ such that the conditions (2.9) and (3.22x2) are fulfilled when $n$ is even and the conditions (2.4) and (3.22i) hold when $n$ is odd. Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.
Corollary 3.16. Let $\tau_2(t) \geq t$ and for any sufficiently large $t_0$ there exist $t_i - t_i(t_0) \geq t_0$ such that one of the following four conditions is fulfilled:

1) $\tau_2(t) \leq \sigma_2(t)$, the conditions (2.3) and (3.25) hold when $n$ is even and the conditions (2.4) and (3.26) hold when $n$ is odd;

2) $\tau_2(t) \geq \sigma_2(t)$, the conditions (2.3) and (3.25) hold when $n$ is even and the inequalities (2.4) and (3.26) hold when $n$ is odd;

3) $\tau_2(t) \leq \sigma_2(t)$, the conditions (2.3) and (3.25) hold when $n$ is even and the inequalities (2.4) and (3.26) hold when $n$ is odd;

4) $\tau_2(t) \geq \sigma_2(t)$, the conditions (2.3) and (3.25) hold when $n$ is even and the conditions (2.4) and (3.26) hold when $n$ is odd.

Let, moreover, the equation (1.2) have Property B. Then the equation (1.1) also has Property B.

References


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