ON BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

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Let \(-\infty < a < b < +\infty, I = [a, b], p : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}^n)\) and \(\ell : C(I; \mathbb{R}^n) \to \mathbb{R}^n\) be linear bounded operators, \(q \in L(I; \mathbb{R}^n)\) and \(c_0 \in \mathbb{R}^n\). On the basis of the results from [1], in the present paper we establish new sufficient conditions for unique solvability of the boundary value problem

\[
\frac{dx(t)}{dt} = \varepsilon p(x)(t) + q(t), \quad (1)
\]

\[
\ell(x) = c_0, \quad (2)
\]

where \(\varepsilon > 0\) is a small parameter.

Throughout the paper, the following notation will be used.

\(\mathbb{R} = ]-\infty, \infty[, \mathbb{R}^+ = [0, \infty];\)

\(x_t\) is the characteristic function of the interval \(I\), i.e.,

\[
x_t(t) = \begin{cases} 
1 & \text{for } t \in I, \\
0 & \text{for } t \not\in I
\end{cases}
\]

\(\mathbb{R}^n\) is the space of \(n\)-dimensional column vectors \(x = (x_i)_{i=1}^n\) with the elements \(x_i \in \mathbb{R}\) \((i = 1, \ldots, n)\) and the norm

\[
\|x\| = \sqrt{\sum_{i=1}^n |x_i|^2};
\]

\(\mathbb{R}^{n \times n}\) is the space of \(n \times n\) matrices \(X = (x_{ik})_{i,k=1}^n\) with the elements \(x_{ik} \in \mathbb{R}\) \((i, k = 1, \ldots, n)\) and the norm

\[
\|X\| = \sqrt{\sum_{i,k=1}^n |x_{ik}|^2};
\]

if \(x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathbb{R}^n\) and \(X = (x_{ik})_{i,k=1}^n, Y = (y_{ik})_{i,k=1}^n \in \mathbb{R}^{n \times n}\), then \(x \leq y \iff x_i \leq y_i \quad (i = 1, \ldots, n)\) and \(X \leq Y \iff x_{ik} \leq y_{ik} \quad (i, k = 1, \ldots, n)\),

\[
|x| = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}, \quad |X| = \left(\sum_{i,k=1}^n |x_{ik}|^2\right)^{1/2};
\]

\(\det(X)\) is the determinant of the matrix \(X\);

\(X^{-1}\) is the inverse matrix to \(X\);

\(r(X)\) is the spectral radius of the matrix \(X\);

\(E\) is the unit matrix;

\(\Theta\) is the zero matrix;

\(C(I; \mathbb{R}^n)\) is the space of continuous vector functions \(x : I \to \mathbb{R}^n\) with the norm

\[
\|x\|_C = \max\left\{\|x(t)\| : t \in I\right\};
\]

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if \( z = (x_i)_{i=1}^n \in C(I; \mathbb{R}^n) \), then
\[
|x|_C = \left( \sum_{i=1}^n |x_i(t)|^2 \right)^{1/2};
\]
\( L(I; \mathbb{R}^n) \) is the space of integrable vector functions \( x : I \to \mathbb{R}^n \) with the norm
\[
|x|_L = \int_a^b \|x(t)\| \, dt;
\]
\( L(I; \mathbb{R}^{n \times n}) \) is the space of integrable matrix functions \( X : I \to \mathbb{R}^{n \times n} \);
if \( Z \in C(I; \mathbb{R}^{n \times n}) \) is a matrix function with the columns \( z_1, \ldots, z_n \) and \( g : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}^n) \) is a linear operator, then \( g(Z) \) stands for the matrix function with columns \( g(z_1), \ldots, g(z_n) \).

Below we will assume that \( p : C(I; \mathbb{R}^n) \to L(I; \mathbb{R}^n) \) is a strongly bounded operator, i.e., there exists \( \eta \in L(I; \mathbb{R}_+^n) \) such that
\[
\|p(x)(t)\| \leq \eta(t) \|x\|_C \quad \text{for } t \in I, \quad x \in C(I; \mathbb{R}^n).
\]

**Theorem 1.** Let
\[
\mathcal{P}_0(t) = E, \quad \mathcal{P}_k(t) = \int_a^t p(\mathcal{P}_{k-1})(s) \, ds \quad \text{for } t \in I \quad (k = 1, 2, \ldots)
\]
and there exist a nonnegative integer \( k_0 \) such that
\[
\det \left( \mathcal{P}_{k_0} \right) \neq 0
\]
and, in case \( k_0 \geq 1 \),
\[
\mathcal{P}_{k_0}(t) = \Theta \quad (k = 0, \ldots, k_0 - 1).
\]
Then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in [0, \varepsilon_0] \) the problem \((1),(2)\) has a unique solution.

**Proof.** Note in the first place that since the operators \( p \) and \( I \) are bounded, there exists a matrix \( B \in \mathbb{R}^{n \times n} \) such that
\[
\int_a^b \|p(x)(t)\| \, dt \leq B \|x\|_C, \quad \|I(x)\| \leq B \|x\|_C \quad \text{for } x \in C(I; \mathbb{R}^n).
\]

For any \( \varepsilon > 0 \) and \( x \in C(I; \mathbb{R}^n) \) set
\[
p_c(x)(t) = \varepsilon p(x)(t), \quad p_c^k(x)(t) = p_c^{k-1}(x)(t) - \int_a^t p_c(p_c^{k-2}(x))(s) \, ds \quad (k = 1, 2, \ldots),
\]
\[
\Lambda_{k;\varepsilon} = (p_c^{k}(E) + \cdots + p_c^{k-1}(E)) \quad (k = 1, 2, \ldots).
\]
Then
\[
p_c^k(x)(t) = \varepsilon^k p^k(x) \quad (k = 0, 1, \ldots),
\]
where \( p^k : C(I; \mathbb{R}^n) \to C(I; \mathbb{R}^n) \quad (k = 0, 1, \ldots) \) is a sequence of operators such that
\[
p_c^0(x)(t) = x(t), \quad p_c^k(x)(t) = \int_a^t p_c(p_c^{k-1}(x))(s) \, ds \quad (k = 1, 2, \ldots).
\]

On the other hand, by \((3)-(5)\) we have
\[
\Lambda_{k+1,\varepsilon} = -\varepsilon \mathcal{P}_{k}(\Lambda_{k,\varepsilon}), \quad \det(\Lambda_{k+1,\varepsilon}) \neq 0.
\]
Assuming that
\[
p_c^{k+1,1}(x)(t) = p_1^1(x)(t) - \Lambda_{k+1,\varepsilon}^{-1}(p_c^{k+1}(x)),
\]
by virtue of conditions \((6)-(8)\) we find
\[
\|p_c^{k+1,1}(x)\|_C \leq A_c \|x\|_C \quad \text{for } x \in C(I; \mathbb{R}^n),
\]
where
\[
A_c = \varepsilon A_c, \quad A_c = B + \|\Lambda_{k+1,\varepsilon}^{-1}\|, \quad \varepsilon_0 = 1/r(A).
\]

Clearly, if
then
\[ r(A_{\varepsilon}) < 1 \quad \text{for} \quad \varepsilon \in [0, \varepsilon_0]. \]  
(11)

However, by Theorem 1.2 from [1], the conditions (9)–(11) guarantee the unique solvability of the problem (1), (2) for arbitrary \( \varepsilon \in [0, \varepsilon_0]. \)

A particular case of (2) is the boundary condition
\[ \sum_{j=1}^{\nu} A_j x(t_j) = c_0, \]
(12)

where \( t_j \in I \) and \( A_j \in \mathbb{R}^{n \times n} \) (\( j = 1, \ldots, \nu \)).

The proven theorem immediately implies

**Corollary 1.** Let either
\[ \det \left( \sum_{j=1}^{\nu} A_j \right) \neq 0, \]
or
\[ \sum_{j=1}^{\nu} A_j = \Theta \quad \text{and} \quad \det \left( \sum_{j=1}^{\nu} A_j \int_{t_j}^{t_{j+1}} p(E(s)) \, ds \right) \neq 0. \]

Then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in [0, \varepsilon_0] \) problem (1), (12) has a unique solution.

Now we consider the differential system
\[ \frac{dx(t)}{dt} - \varepsilon P(t)x(\tau(t)) + q_0(t) \]
(13)
with the boundary conditions
\[ x(t) - u(t) \quad \text{for} \quad t \notin I, \quad \ell(x) = c_0 \]
(14)
or
\[ x(t) - u(t) \quad \text{for} \quad t \notin I, \quad \sum_{j=1}^{\nu} A_j x(t_j) = c_0, \]
(15)
where \( P \in L(I; \mathbb{R}^{n \times n}), q_0 \in L(I; \mathbb{R}^n), \tau : I \to \mathbb{R} \) is a measurable function and \( u : \mathbb{R} \to \mathbb{R}^n \)
is a continuous bounded vector function.

If we assume that
\[ \tau_0(t) = \begin{cases} a & \text{for} \quad t < a \\ \tau(t) & \text{for} \quad a \leq \tau(t) \leq b \\ b & \text{for} \quad \tau(t) > b \end{cases} \]

\[ p(x)(t) - \chi_2(\tau(t)) P(t)x(\tau_0(t)), \]
\[ q(t) - \varepsilon \left( 1 - \chi_2(\tau(t)) \right) P(t) u(\tau(t)) + q_0(t), \]
then the problem (13), (14) reduces to the problem (1), (2). Hence for the problem (13), (14) Theorem 1 can be formulated in the form of

**Theorem 2.** Let
\[ P_0(t) - E, \quad P_k(t) - \int_{t_k}^{t} \chi_2(\tau(s)) P(s) P_{k-1}(\tau_0(s)) \, ds \quad (k = 1, 2, \ldots) \]
and there exist a nonnegative integer \( k_0 \) such that
\[ \det \left( I(P_{k_0}) \right) \neq 0 \]

and, in case \( k_0 \geq 1, \)
\[ \ell(P_0) = \Theta \quad (k = 0, \ldots, k_0 - 1). \]

Then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in [0, \varepsilon_0[ \) the problem (13), (14) has a unique solution.
Corollary 2. Let either
\[ \det \left( \sum_{j=1}^{\nu} A_j \right) \neq 0 \]
or
\[ \sum_{j=1}^{\nu} A_j = \Theta \quad \text{and} \quad \det \left( \sum_{j=1}^{\nu} A_j \int_{-\infty}^{t} x_j(\tau(s)) P(s) \, ds \right) \neq 0. \]

Then there exists \( c_0 > 0 \) such that for any \( \varepsilon \in [0, c_0] \) the problem (13), (15) has a unique solution.

In the case \( r(t) \equiv 0 \), Theorem 2 implies Corollary 3.1 from [2].

References


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