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THE LINEARIZED MAXIMUM PRINCIPLE FOR QUASI-LINEAR NEUTRAL OPTIMAL PROBLEMS WITH DISCONTINUOUS INITIAL CONDITION AND VARIABLE DELAYS IN CONTROLS

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Let \( J = [a, b] \subset R \) be a finite interval, \( O \subset R^n, G \subset R^r \) be open sets; Let the function \( f : J \times O^s \times G^u \rightarrow R^n \) satisfy the following conditions: for almost all \( t \in J \) function \( f(t, x_1, \ldots, x_k, u_1, \ldots, u_m) \) is continuously differentiable with respect to \( x_1 \in O, i = 1, \ldots, s, u_m \in G, m = 1, \ldots, \nu \); for any fixed \( (x_1, \ldots, x_k, u_1, \ldots, u_m) \in O^s \times G^u \) the functions \( f(t, x_1, \ldots, x_k, u_1, \ldots, u_m), f_x(\cdot) i = 1, \ldots, s, f_u(\cdot), m = 1, \ldots, \nu \) are measurable on \( J \); for arbitrary compacts \( K \subset O, N \subset G \) there exists a function \( m_{KN}(\cdot) \in L(J, R_+), R_+ = [0, \infty), \) such that for any \( (x_1, \ldots, x_k, u_1, \ldots, u_m) \in K^s \times N^u \) and for almost all \( t \in J \) the following inequality is fulfilled

\[
| f(t, x_1, \ldots, x_k, u_1, \ldots, u_m) | + \sum_{i=1}^s | f_x(i) | + \sum_{m=1}^u | f_u(m) | \leq m_{KN}(t).
\]

Let the scalar functions \( \tau_i(t), i = 1, \ldots, s, \theta_m(t), m = 1, \ldots, \nu, t \in R \) and \( \eta_j(t), j = 1, \ldots, k, t \in R \), be absolutely continuous and continuously differentiable, respectively, and satisfy the conditions: \( \tau_i(t) \leq t, \tau_i(t) > 0, i = 1, \ldots, s; \theta_m(t) \leq t, \theta_m(t) > 0, m = 1, \ldots, \nu; \eta_j(t) < t, \eta_j(t) > 0, j = 1, \ldots, k. \) Let \( \Phi \) be the set of continuously differentiable functions \( \varphi : J_1 = [\tau, b] \rightarrow M, \tau = \min \{ \eta_1(a), \ldots, \eta_k(a), \tau_1(a), \ldots, \tau_s(a) \} \), where \( M \subset O \) is a convex set, \( \| \varphi \| = \sup \{ | \varphi(t) | : t \in J_1 \} \); \( \Omega \) be the set of measurable functions \( u : J_2 = [\theta, b] \rightarrow U \), such that \( \text{cl} \{ u(t) : t \in J_2 \} \) is a compact lying in \( G \); \( \theta = \min \{ \theta_1(a), \ldots, \theta_s(a) \} \), where \( U \subset G \) is a convex set, \( \| u \| = \sup \{ | u(t) | : t \in J_2 \} \); \( A_j(t), t \in J, i = 1, \ldots, k, \) be continuous \( n \times n \) matrix functions. The scalar functions \( q^j(t_0, t_1, x_0, x_1) \), \( i = 1, \ldots, l, \) are continuously differentiable on the set \( J_2 \times \Omega^2 \).

To every element \( \lambda = (t_0, t_1, x_0, \varphi, u) \in E = J_2 \times O \times \Phi \times \Omega \) let us correspond the differential equation

\[
\dot{x}(t) = \sum_{j=1}^k A_j(t) \dot{x}(\eta_j(t)) + f(t, x(\tau_1(t)), \ldots, x(\tau_s(t)), u(\theta_1(t)), \ldots, u(\theta_m(t)))
\]

with discontinuous initial condition

\[
x(t) = \varphi(t), \quad t \in [\tau, t_0), x(t_0) = x_0.
\]

**Definition 1.** Let \( \lambda = (t_0, t_1, x_0, \varphi, u) \in E, t_0 < b. \) The function \( x(t) = x(t; \lambda) \in O, t \in [\tau, t_1], t_1 \in (t_0, b) \) is said to be a solution corresponding to the element \( \lambda \), defined on the interval \( [\tau, t_1] \), if on the interval \( [\tau, t_0] \) the function \( x(t) \) satisfies the condition (2), while on the interval \([t_0, t_1]\) it is absolutely continuous and almost everywhere satisfies the equation (1).

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Definition 2. The element \( \lambda \in E \) is said to be admissible if the corresponding solution \( x(t) = x(t; \lambda) \) satisfies the conditions
\[
q^i(t_0, t_1, x_0, x(t_1)) = 0, \quad i = 1, \ldots, l. \tag{3}
\]
The set of admissible elements will be denoted by \( E_0 \).

Definition 3. The element \( \tilde{\lambda} = (\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{\varphi}, \tilde{u}) \in E_0 \) is said to be locally optimal, if there exists a number \( \delta > 0 \) such that for an arbitrary element \( \lambda \in E_0 \) satisfying
\[
| t_0 - \tilde{t}_0 | + | t_1 - \tilde{t}_1 | + | \tilde{x}_0 - x_0 | + \| \tilde{\varphi} - \varphi \| + \| \tilde{u} - u \| \leq \delta
\]
the inequality
\[
q^0(\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{x}(\tilde{t}_1)) \leq q^0(t_0, t_1, x_0, x(t_1)) \tag{4}
\]
holds, where \( \tilde{x}(t) = x(t; \tilde{\lambda}) \).

The problem (1)-(4) is said to be optimal problem with discontinuous initial condition and it consists in finding a locally optimal element.

In order to formulate the main results, we will introduce the following notation
\[
s_1 = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(0), \tilde{\varphi}(\tilde{t}_0), \tilde{\varphi}(\tilde{t}_0), \tilde{\varphi}(\tilde{t}_0), \tilde{\varphi}(\tilde{t}_0)), \quad i = 0, \ldots, p;
\]
\[
s_1 = (\gamma_0, \tilde{x}(\gamma_0)), \ldots, \tilde{x}(\gamma_n), x_0, \tilde{x}(\gamma_n), \ldots, \tilde{x}(\gamma_n));
\]
\[
s_1^i = (\gamma_i, \tilde{x}(\gamma_i)), \ldots, \tilde{x}(\gamma_i), x_0, \tilde{x}(\gamma_i), \ldots, \tilde{x}(\gamma_i));
\]
\[
\tilde{\rho}_i = \rho_i(\tilde{t}_0), \quad \gamma_i(t) = \tau^{-1}_i(t), \quad \rho_j(t) = \eta^{-1}_j(t); \quad \omega = (t, x_1, \ldots, x_s);
\]
\[
f_1(\omega) = f(\omega, \tilde{u}(\tilde{t}_1), \ldots, \tilde{u}(\tilde{t}_s)), \quad \tilde{f}_{\alpha_1}[t] = f_{\alpha_1}(t, \tilde{x}(\gamma_1), \ldots, \tilde{x}(\gamma_s); \ldots, \tilde{u}(\tilde{t}_1), \ldots, \tilde{u}(\tilde{t}_s));
\]

\[
Q = (q^0, \ldots, q^p), \quad \tilde{Q}_{\alpha_1} = Q_{\alpha_1}(\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{x}(\tilde{t}_1)).
\]

Theorem 1. Let the element \( \tilde{\lambda} = (\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{\varphi}, \tilde{u}) \in E_0, \tilde{t}_0 > a \) be locally optimal and the following conditions be fulfilled:

1) \( \gamma_i = \tilde{t}_0, \quad i = 1, \ldots, p; \gamma_{p+1} < \cdots < \gamma_s < \tilde{t}_1, \rho_j < \tilde{t}_1, \quad j = 1, \ldots, k; \)

2) there exists a number \( \delta > 0 \) such that
\[
\gamma_i(t) \leq \cdots \leq \gamma_p(t), \quad t \in (\tilde{t}_0 - \delta, \tilde{t}_0);
\]

3) there exist the finite limits: \( \tilde{\varphi}_i = \tilde{x}(\gamma_i, \tilde{t}_0), \quad i = 1, \ldots, s, \quad \tilde{x}(\gamma_j, \tilde{t}_1)); \quad j = 1, \ldots, k;
\]
\[
\lim_{\omega \to \gamma_i} f_1(\omega) = f_1^{-1}, \quad \omega \in (\tilde{t}_0 - \delta, \tilde{t}_0) \times O^s, \quad i = 0, \ldots, p,
\]
\[
\lim_{(\omega_1, \omega_2) \to (\gamma_i, \gamma_j)} \left[ f_1(\omega_1) - f_1(\omega_2) \right] = f_1^{-1}, \quad \omega_1, \omega_2 \in (\gamma_i - \delta, \gamma_i) \times O^s, \quad i = p + 1, \ldots, s,
\]
\[
\lim_{\omega \to \gamma_j} f_1(\omega) = f_1^{-1}, \quad \omega \in (\tilde{t}_1 - \delta, \tilde{t}_1) \times O^s.
\]

Then there exist a non-zero vector \( \pi = (\pi_0, \ldots, \pi_1), \pi_0 \leq 0, \) and a solution \( \chi(t) = (\chi_1(t), \ldots, \chi_n(t)), \psi(t) = (\psi_1(t), \ldots, \psi_n(t)) \) of the system
\[
\begin{align*}
\dot{\chi}(t) &= -\sum_{i=1}^n \psi(\gamma_i(t)) f_1(\gamma_i(t)) \gamma_i(t), \\
\dot{\psi}(t) &= \chi(t) + \sum_{j=1}^k \psi(\rho_j(t)) A_j(\rho_j(t)) \rho_j(t), \quad t \in [\tilde{t}_0, \tilde{t}_1], \\
\psi(t) &= 0, t > \tilde{t}_1,
\end{align*}
\]
such that the following conditions are fulfilled:
Then there exists a non-zero vector 

Here

c)

the condition

\[
\int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t) \sum_{m=1}^\nu \tilde{f}_m(t) \omega(t_m(t)) dt \geq \int_{\tilde{t}_0}^{\tilde{t}_1} \psi(t) \sum_{m=1}^\nu \tilde{f}_m(t) u(t_m(t)) dt, \quad \forall u \in \Omega,
\]

\[
\sum_{i=p+1}^s \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma_i(t)) \tilde{f}_{\gamma_i(\tilde{t}_0)} \gamma_i(t) \tilde{v}(t) dt + \sum_{j=1}^k \int_{\eta_j(\tilde{t}_0)}^{\tilde{t}_0} \psi(\rho_j(t)) A_j[\rho_j(t)] \rho_j(t) \tilde{v}(t) dt \geq
\]

\[
\geq \sum_{i=p+1}^s \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} \psi(\gamma_i(t)) \tilde{f}_{\gamma_i(\tilde{t}_0)} \gamma_i(t) \varphi(t) dt +
\]

\[
+ \sum_{j=1}^k \int_{\eta_j(\tilde{t}_0)}^{\tilde{t}_0} \psi(\rho_j(t)) A_j[\rho_j(t)] \rho_j(t) \tilde{v}(t) dt, \quad \forall \varphi \in \Phi;
\]

b) the conditions for the moments \(\tilde{t}_0, \tilde{t}_1:\)

\[
\pi \tilde{Q}_{i_0} \geq -\psi(\tilde{t}_0 - |\tilde{v}(\tilde{t}_0)|) - \sum_{i=1}^k A_j(\tilde{t}_0) \tilde{\varphi}(\eta_j(\tilde{t}_0)) + \sum_{i=0}^p (\tilde{\gamma}_i - \tilde{\gamma}_i^-) f_i^- +
\]

\[
+ \sum_{i=p+1}^s \psi(\gamma_i - \gamma_i^-) f_i^- + \tilde{\chi}(\tilde{t}_0) \tilde{v}(\tilde{t}_0),
\]

\[
\pi \tilde{Q}_{i_1} \geq -\psi(\tilde{t}_1) \left[ \sum_{j=1}^k A_j(\tilde{t}_1) \tilde{\varphi}(\eta_j(\tilde{t}_1)) + f_{i+1}^+ \right];
\]

c) the condition for the solution \(\chi(t), \psi(t)\):

\[
\pi \tilde{Q}_{x_0} = -\chi(\tilde{t}_0), \quad \pi \tilde{Q}_{x_1} = \psi(\tilde{t}_1) = \chi(\tilde{t}_1).
\]

Here

\[
\tilde{\gamma}_0 = 1, \quad \tilde{\gamma}_i = \hat{\gamma}_i^- + \tilde{\gamma}_i, \quad i = 1, \ldots, p, \quad \tilde{\gamma}_{p+1} = 0;
\]

**Theorem 2.** Let the element \(\lambda = (\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{\varphi}, \tilde{u}) \in E_0, \tilde{t}_1 < b\) be locally optimal and the condition 1) of Theorem 1 and the following conditions be fulfilled:

4) there exists a number \(\delta > 0\) such that

\[
\gamma(t) \leq \gamma(\tilde{t}_0), \quad t \in [\tilde{t}_0, \tilde{t}_0 + \delta);
\]

5) there exist the finite limits:

\[
\lim_{\omega \to \sigma^t} (\tilde{\omega}) = f_{i+1}^+, \quad \omega \in [\tilde{t}_0, \tilde{t}_0 + \delta] \times O^*, \quad i = 0, \ldots, p.
\]

Then there exists a non-zero vector \(\pi = (\pi_0, \ldots, \pi_s), \pi_0 \leq 0\) and a solution \(\chi(t), \psi(t)\) of the system (5) such that the conditions a) and c) are fulfilled. Moreover,

\[
\pi \tilde{Q}_{i_0} \leq -\psi(\tilde{t}_0 - |\tilde{v}(\tilde{t}_0)|) - \sum_{j=1}^k A_j(\tilde{t}_0) \tilde{\varphi}(\eta_j(\tilde{t}_0)) + \sum_{i=0}^p (\tilde{\gamma}_i - \tilde{\gamma}_i^-) f_i^- +
\]

\[
+ \sum_{i=p+1}^s \psi(\gamma_i) f_i^+ + \tilde{\chi}(\tilde{t}_0) \tilde{v}(\tilde{t}_0),
\]

\[
\pi \tilde{Q}_{i_1} \leq -\psi(\tilde{t}_1) \left[ \sum_{j=1}^k A_j(\tilde{t}_1) \tilde{\varphi}(\eta_j(\tilde{t}_1)) + f_{i+1}^+ \right].
\]
Here
\[ \hat{\gamma}_0^+ = 1, \quad \hat{\gamma}_i^+ = \gamma_i^+, \quad i = 1, \ldots, p, \quad \hat{\gamma}_{p+1}^+ = 0. \]

**Theorem 3.** Let the element \( \bar{\lambda} = (\bar{t}_0, \bar{t}_1, \bar{x}_0, \bar{u}) \in E_0 \), \( \bar{t}_0, \bar{t}_1 \in (a, b) \) be locally optimal and the conditions of Theorems 1, 2 and the following conditions be fulfilled: the functions \( \dot{\bar{x}}(\eta_j(\bar{t}_1)), j = 1, \ldots, k \), are continuous;
\[ \gamma_i, \bar{t}_0 \notin \{ \eta_k \} (\eta_k (\bar{t}_1)), \quad \bar{t}_1 \in (a, b) \]
\( k_m = 1, \ldots, k \), \( i = p + 1, \ldots, s \);
\[ \sum_{i=0}^{p}(\hat{\gamma}_{i+1}^- - \hat{\gamma}_i^-) f_i^- = \sum_{i=0}^{p}(\hat{\gamma}_{i+1}^+ - \hat{\gamma}_i^+) f_i^+ = f_0, \]
\[ f_i^+ \hat{\gamma}_i^- = f_i^+ \hat{\gamma}_i^+, \quad i = p + 1, \ldots, s, \quad f_{s+1}^+ = f_{s+1}. \]
Then there exists a non-zero vector \( \pi = (\pi_0, \ldots, \pi_l) \), \( \pi_0 \leq 0 \) and a solution \( \chi(t), \psi(t) \) of the system (5) such that the condition a) and c) are fulfilled. Moreover,
\[ \pi \hat{Q}_0 = -\psi(\bar{t}_0) \tilde{\varphi}(\bar{t}_0) + \sum_{j=1}^{k} A_j(\bar{t}_0) \tilde{\varphi}(\eta_j(\bar{t}_0)) + f_0 + \sum_{i=p+1}^{s} \psi(\gamma_i) f_i + \chi(\bar{t}_0) \tilde{\varphi}(\bar{t}_0), \]
\[ \pi \hat{Q}_0 = -\psi(\bar{t}_0) \sum_{j=1}^{k} A_j(\bar{t}_0) \tilde{\varphi}(\eta_j(\bar{t}_1)) + f_{s+1}. \]

Finally we note that the optimal control problems for various classes of delay and neutral differential equations with discontinuous initial condition are considered in [1]–[4].

**References**


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