ON LYAPUNOV STABILITY OF A CLASS OF LINEAR SYSTEMS OF DIFFERENCE EQUATIONS

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In the present note we consider the linear system of difference equations

$$\Delta y(k - 1) = G_1(k - 1)y(k - 1) + G_2(k)y(k) + G_3(k)y(k + 1) + g(k) \quad (k = 1, 2, \ldots), \quad (1)$$

where $G_j(k) \in \mathbb{R}^{n \times n}$ and $g(k) \in \mathbb{R}^n$ ($j = 1, 2, 3; k = 0, 1, \ldots$).

We give effective necessary and sufficient conditions guaranteeing the stability of the system (1) in Lyapunov sense with respect to small perturbations. They are the analogues of the well-known conditions for the stability of linear ordinary differential systems with constant coefficients (see, e.g., [1], [2]).

The following notation and definitions will be used in the paper.

$\mathbb{N} = \{1, 2, \ldots\}$ is the set of all natural numbers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$; $\mathbb{R} = ] - \infty, +\infty[$. $\mathbb{R}_+ = [0, +\infty[$. $\lceil t \rceil$ is the integer part of $t \in \mathbb{R}$.

$\mathbb{R}^{n \times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i=1}^{n} \sum_{j=1}^{m} |x_{ij}|;$$

$O_{n \times m}$ (or $O$) is the zero $n \times m$-matrix.

If $X \in \mathbb{R}^{n \times n}$, then $X^{-1}$, $\det X$ and $r(X)$ are, respectively, the matrix inverse to $X$, the determinant and the spectral radius of $X$; $I_n$ is the identity $n \times n$-matrix; $\delta_{ij}$ is the Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ ($i, j = 1, 2, \ldots$).

$\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column $n$-vectors $x = (x_i)_{i=1}^{n}$.

If $J \subseteq \mathbb{N}_0$ and $Q \subseteq \mathbb{R}^{n \times m}$, then $E(J; Q)$ is the set of all matrix-functions $Y : J \rightarrow Q$.

$\Delta$ is the first order difference operator, i.e.,

$$\Delta y(k - 1) = y(k) - y(k - 1) \quad (k = 1, 2, \ldots) \quad \text{for} \quad y \in E(\mathbb{N}_0; \mathbb{R}^n).$$

Let $y_0 \in E(\mathbb{N}_0; \mathbb{R}^n)$ be a solution of the difference system (1) and let $G \in E(\mathbb{N}_0; \mathbb{R}^{n \times n})$ be an arbitrary matrix-function.

**Definition 1.** A solution $y_0 \in E(\mathbb{N}_0; \mathbb{R}^n)$ of the system (1) is called $G$-stable if for every $\varepsilon > 0$ and $k_0 \in \mathbb{N}_0$ there exists $\delta(\varepsilon, k_0) > 0$ such that for every solution $y$ of the system (1) for which

$$\|(I_n + G(k_0))(y(k_0) - y_0(k_0))\| + \|y(k_0 + 1) - y_0(k_0 + 1)\| < \delta$$

the estimate

$$\|(I_n + G(k))(y(k) - y_0(k))\| + \|y(k + 1) - y_0(k + 1)\| < \varepsilon \quad \text{for} \quad k \geq k_0$$

holds.

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Definition 2. A solution $y_0 \in E(\mathbb{N}_0; \mathbb{R}^n)$ of the system (1) is called $G$-asymptotically stable if it is $G$-stable and for every $k_0 \in \mathbb{N}_0$ there exists $\Delta = \Delta(k_0) > 0$ such that for every solution $y$ of the system (1) for which
\[ ||(I_n + G(k_0))(y(k_0) - y_0(k_0))|| + ||y(k_0 + 1) - y_0(k_0 + 1)|| < \Delta \]
the condition
\[ \lim_{k \to \infty} ((I_n + G(k))(y(k) - y_0(k))|| + ||y(k + 1) - y_0(k + 1)||) = 0 \]
holds.

We say that $y_0$ is stable (asymptotically stable) if it is $O_{n \times n}$-stable ($O_{n \times n}$-asymptotically stable).

Definition 3. The system (1) is called $G$-stable ($G$-asymptotically stable) if every its solution is $G$-stable ($G$-asymptotically stable).

It is evident that the system (1) is $G$-stable ($G$-asymptotically stable ) if and only if its corresponding homogeneous system
\[ \Delta y(k - 1) = G_1(k - 1)y(k - 1) + G_2(k)y(k) + G_3(k)y(k + 1) \quad (k = 1, 2, \ldots) \quad (1_0) \]
is $G$-stable ($G$-asymptotically stable). On the other hand, the system (10) is $G$-stable ($G$-asymptotically stable) if and only if its zero solution is $G$-stable ($G$-asymptotically stable). Thus the $G$-stability ($G$-asymptotic stability) of the system (1) is the common property of all solutions of this system and the vector-function $g_0$ does not affect this property. Therefore, it is the property of the triple $(G_1, G_2, G_3)$. Hence the following definition is natural.

Definition 4. The triple $(G_1, G_2, G_3)$ is $G$-stable ($G$-asymptotically stable) if the system (10) is $G$-stable ($G$-asymptotically stable).

Remark 1. It is evident that the triple $(G_1, G_2, G_3)$ is $G$-stable if and only if every solution of the system (10) is $G$-bounded, i.e., there exists $M > 0$ such that
\[ ||(I_n + G(k))y(k)|| + ||y(k + 1)|| \leq M \quad (k = 0, 1, \ldots). \]

Analogously, the triple $(G_1, G_2, G_3)$ is $G$-asymptotically stable if and only if every solution $y$ of the system (10) is $G$-convergent to zero, i.e.,
\[ \lim_{k \to \infty} ((I_n + G(k))y(k)|| + ||y(k + 1)||) = 0. \]

Remark 2. If the matrix-function $G$ is such that
\[ \det(I_n + G(k)) \neq 0 \quad (k = 0, 1, \ldots) \]
and
\[ ||G(k)|| + ||(I_n + G(k))^{-1}|| < M \quad (k = 0, 1, \ldots) \]
for some $M > 0$, then the triple $(G_1, G_2, G_3)$ is $G$-stable ($G$-asymptotically stable) if and only if it is stable (asymptotically stable).

Theorem 1. Let the matrix-functions $G_1, G_2, G_3 \in E(\mathbb{N}_0; \mathbb{R}^{n \times n})$ be such that
\[ \det(I_n + G_1(k)) \neq 0 \quad (k = 1, 2, \ldots) \]
and
\[ G(k) = (G_{ij}(k))_{i,j=1}^n, \]
where $G(k) = (G_{ij}(k))_{i,j=1}^2$, $G_{11}(k) \equiv (G_1(k) + G_2(k))(I_n + G_1(k))^{-1}$, $G_{12}(k) \equiv G_3(k)$, $G_{21}(k) \equiv -(I_n + G_1(k))^{-1}$, $G_{22}(k) \equiv O_{n \times n}$. 

\[ G_{11}(k) \equiv (G_1(k) + G_2(k))(I_n + G_1(k))^{-1}, \quad G_{12}(k) \equiv G_3(k), \]
\[ G_{21}(k) \equiv -(I_n + G_1(k))^{-1}, \quad G_{22}(k) \equiv O_{n \times n}. \]
$B_l \in \mathbb{R}^{2n \times 2n}$ ($l = 1, \ldots, m$) are pairwise permutable constant matrices, and $\beta_l \in E(\mathbb{N}; \mathbb{R}_+)$ ($l = 1, \ldots, m$) are such that

$$\lim_{k \to +\infty} \beta_l(k) = +\infty \quad (l = 1, \ldots, m).$$

Then:

a) the triple $(G_1, G_2, G_3)$ is $G_1$-stable if and only if every eigenvalue of the matrices $B_l$ ($l = 1, \ldots, m$) has the nonpositive real part and, in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;

b) the triple $(G_1, G_2, G_3)$ is $G_1$-asymptotically stable if and only if every eigenvalue of the matrices $B_l$ ($l = 1, \ldots, m$) has the negative real part.

**Corollary 1.** Let $G_j(k) \equiv G_{0j}$ ($j = 1, 2, 3$) be constant matrix-functions and

$$\det(I_n + G_{0j}) \neq 0, \quad \det G_{03} \neq 0,$$

where $G_{0j} \in \mathbb{R}^{n \times n}$ ($j = 1, 2, 3$) are constant matrices. Let, moreover, $\lambda_i$ ($i = 1, \ldots, m$) be pairwise different eigenvalues of the $2n \times 2n$-matrix

$$G_{01} = (G_{01} + G_{02})(I_n + G_{01})^{-1}, \quad G_{02} = (I_n + G_{01})^{-1}, \quad G_{03} = G_{03},$$

where $A_j = (\alpha_{jil})_{i,l=1}^n$ ($j = 1, 2$), are constant $n \times n$-matrices such that

$$\det(I_n - A_j) \neq 0, \quad \det((I_n - A_j)) \neq 0.$$

Then:

a) the triple $(G_{01}, G_{02}, G_{03})$ is stable if and only if $|1 - \lambda_i| \geq 1$ ($i = 1, \ldots, m$) and, in addition, if $|1 - \lambda_i| = 1$ for some $i \in \{1, \ldots, m\}$, then the elementary divisor corresponding to $\lambda_i$ is simple;

b) the triple $(G_{01}, G_{02}, G_{03})$ is asymptotically stable if and only if $|1 - \lambda_i| > 1$ ($i = 1, \ldots, m$).

**Theorem 2.** Let $G_j(k) \equiv G_{0j}$ ($j = 1, 2, 3$) be constant matrix-functions such that

$$G_{01} = (A_1 - A_3)(I_n - A_1 + A_3)^{-1},$$

$$G_{02} = I_n + (A_1 + A_2 - 2I_n)(I_n + G_{01}), \quad G_{03} = (I_n - A_2),$$

where $A_j = (\alpha_{jil})_{i,l=1}^n$ ($j = 1, 2$), are constant $n \times n$-matrices such that

$$\det(I_n - A_1 + A_3) \neq 0, \quad \det((I_n - A_2)) \neq 0.$$

Let, moreover,

$$\alpha_{jii} < 0 \quad (j = 1, 2; i = 1, \ldots, n) \quad \text{and} \quad r(H) < 1,$$

where $H = (H_{mj})_{m,j=1}^2$.

$$H_{jj} = ((1 - \delta_i)\alpha_{jii}\alpha_{jii}^{-1})_{i,l=1}^n \quad (j = 1, 2),$$

$$H_{21} = ((\alpha_{2il})\alpha_{2il}^{-1})_{i,l=1}^n, \quad H_{12} = (\alpha_{2il}\alpha_{2il}^{-1})_{i,l=1}^n,$$

Then the triple $(G_{01}, G_{02}, G_{03})$ is asymptotically stable. Conversely, if this triple is asymptotically stable,

$$\alpha_{jil} \geq 0, \quad \alpha_{2il} \geq 1 \quad (j = 1, 2, 3; \quad i \neq l; \quad i, l = 1, \ldots, n)$$

and

$$\alpha_{j+iil} - \delta_{il} + \sum_{l \neq i} \alpha_{jil} \alpha_{j+iil} \leq \min\{1 - \alpha_{jii}, 1 + \alpha_{jii}\} \quad (j = 1, 2; \quad i = 1, \ldots, n),$$

then the condition (2) holds as well.
To prove these results we use the following concept. Consider the system of the so-called generalized linear ordinary differential equations
\[ dx(t) = dA(t) \cdot x(t) + df(t) \quad \text{for} \quad t \in \mathbb{R}_+, \quad (3) \]
where \( A : \mathbb{R}_+ \to \mathbb{R}^{n \times n} \) and \( f : \mathbb{R}_+ \to \mathbb{R}^n \) are, respectively, the matrix and vector-functions with the components having bounded variation on every closed interval from \( \mathbb{R}_+ \) (see, i.e. \([3]\)).

Under a solution of the system (2) we understand a vector-function \( x : \mathbb{R}_+ \to \mathbb{R}^n \) with the components having bounded variations on every closed interval from \( \mathbb{R}_+ \) and such that
\[ x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for} \quad 0 \leq t \leq s, \]
where the integral is understood in Lebesgue–Stiltjes sense.

The difference system (1) is a particular case of the system (2). Namely, \( y \in E(\mathbb{N}_0; \mathbb{R}^n) \) is a solution of the system (1) if and only if the vector-function \( x(t) = (z_1([t]) + G_1([t])) y(t) \) is a solution of the \( 2n \times 2n \)-system (2), where
\[ A(t) = O_{2n \times 2n} \quad \text{for} \quad 0 \leq t \leq 1, \quad A(t) = \sum_{k=1}^{[t]} G(k) \quad \text{for} \quad t \geq 1, \]
\[ f(t) = O_{2n} \quad \text{for} \quad 0 \leq t \leq 1, \quad f(t) = \sum_{k=1}^{[t]} G(k) \quad \text{for} \quad t \geq 1. \]

Thus Theorem 1 and its corollaries immediately follow from the corresponding results contained in [4] for the system (1).

As to the proof of Theorem 2, we use a system of the form (2) different from the one constructed above, in order to apply the analogous result from [4].

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**References**


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