ON OSCILLATORY SOLUTIONS OF THIRD ORDER DIFFERENTIAL EQUATIONS

(Reported on June 9, 2003)

The proposed note deals with the asymptotic properties of solutions of the differential equation

\[(r(t)u''(t))' = p_0(t)u + p_1(t)u' + q(t),\]  \hspace{1cm} (1)

where \(r : [0, +\infty] \to [0, +\infty]\), \(p_i : [0, +\infty] \to \mathbb{R} (i = 1, 2)\) and \(q : [0, +\infty] \to \mathbb{R}\) are continuous functions.

The following theorem is valid.

Theorem. Let

\[\int_0^{+\infty} \frac{t}{r(t)} \, dt < +\infty,\]

\[\int_0^{+\infty} |p_i(t)| \, dt < +\infty \quad (i = 0, 1), \quad \int_0^{+\infty} |q(t)| \, dt < +\infty.\]  \hspace{1cm} (2)

Then an arbitrary oscillatory solution of the equation (1) satisfies the conditions

\[
\lim_{t \to +\infty} u(t) = \lim_{t \to +\infty} u'(t) = \lim_{t \to +\infty} r(t)u''(t) = 0.
\]  \hspace{1cm} (3)

Proof. Let the sequence \((t_k)_{k=1}^{+\infty}\) be such that

\[u(t_k) = 0, \quad 1 < t_k < t_{k+1} \quad (k = 1, 2, \ldots).\]

Then for each natural \(k\) there exists \(\tilde{t}_k \in [t_k, t_{k+2}]\) such that

\[u''(\tilde{t}_k) = 0.\]

Hence (1) implies

\[u''(t) = \frac{1}{r(t)} \int_{t_k}^{t} \left[ p_0(s)u(s) + p_1(s)u'(s) + q(s) \right] \, ds.\]

If now we set

\[\rho_{ik} = \max \left\{ |u^{(i)}(t)| : t_k \leq t \leq t_{k+2} \right\} \quad (i = 0, 1),\]

\[\rho_{2k} = \max \left\{ r(t)|u''(t)| : t_k \leq t \leq t_{k+2} \right\}\]

and

\[\varepsilon_k = \int_{t_k}^{t_{k+2}} \left( \frac{t}{r(t)} + |p_0(t)| + |p_1(t)| + |q(t)| \right) \, dt,\]

2000 Mathematics Subject Classification. 34C10, 34C11.

Key words and phrases. Oscillatory solution, asymptotic properties.
then from the latter identity we find that

$$|u''(t)| \leq \frac{1}{r(t)} \varepsilon_k (\rho_{0k} + \rho_{1k} + 1) \text{ for } t_k \leq t \leq t_{k+2}$$

(4)

and

$$\rho_{2k} \leq \varepsilon_k (\rho_{0k} + \rho_{1k} + 1) \text{ for } t_k \leq t \leq t_{k+2}.$$  

(5)

On the other hand, by virtue of conditions (2) we have

$$\lim_{k \to +\infty} \varepsilon_k = 0.$$  

(6)

Therefore without loss of generality it can be assumed that

$$\varepsilon_k < \frac{1}{2} \text{ for } k = 1, 2, \ldots.$$  

(7)

By the Green formula, for each natural \(k\) we have

$$u(t) = \int_{t_k}^{t_{k+2}} G_k(t, s) u''(s) \, ds, \quad u'(t) = \int_{t_k}^{t_{k+2}} \frac{\partial G_k(t, s)}{\partial s} u'(s) \, ds$$

for \(t_k \leq t \leq t_{k+2},\)

(8)

where

$$G_k(t, s) = \begin{cases} 
\frac{(t - t_k)(s - t_{k+2})}{t_{k+2} - t_k} & \text{for } t < s, \\
\frac{(t_{k+2} - t_k)(s - t_k)}{t_{k+2} - t_k} & \text{for } t > s.
\end{cases}$$

Moreover,

$$|G_k(t, s)| \leq s - t_k < s, \quad \left| \frac{\partial G_k(t, s)}{\partial t} \right| \leq 1 \text{ for } t_k \leq t, s \leq t_{k+2}.$$  

By virtue of these estimates and inequalities (4), from (8) we find that

$$\rho_{0k} \leq \varepsilon_k (\rho_{0k} + \rho_{1k} + 1) \int_{t_k}^{t_{k+2}} \frac{s}{r(s)} \, ds \leq \varepsilon_k^2 (\rho_{0k} + \rho_{1k} + 1),$$

$$\rho_{1k} \leq \varepsilon_k (\rho_{0k} + \rho_{1k} + 1) \int_{t_k}^{t_{k+2}} \frac{ds}{r(s)} \leq$$

$$\leq \varepsilon_k (\rho_{0k} + \rho_{1k} + 1) \int_{t_k}^{t_{k+2}} \frac{s \, ds}{r(s)} \leq \varepsilon_k^2 (\rho_{0k} + \rho_{1k} + 1).$$

Therefore

$$\rho_{0k} + \rho_{1k} \leq 2 \varepsilon_k^2 (\rho_{0k} + \rho_{1k}) + 2 \varepsilon_k^2.$$  

Hence by (5)–(7) it follows that

$$\rho_{0k} + \rho_{1k} \leq 4 \varepsilon_k^2, \quad \rho_{2k} \leq 2 \varepsilon_k \quad (k = 1, 2, \ldots)$$

and

$$\lim_{k \to +\infty} \rho_{ik} = 0 \quad (i = 0, 1, 2).$$

Therefore equalities (3) are fulfilled.

\[ \square \]

The proven theorem completes the previously known results on asymptotic behavior of solutions of linear differential equations of third order (see [1]–[9] and the references cited therein).
Acknowledgement

The author is thankful to Prof. I. Kiguradze for his valuable remarks and suggestions in preparing this manuscript.

References


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