ON THE SOLVABILITY OF NONLINEAR OPERATOR EQUATIONS IN A BANACH SPACE

(Reported on July 7, 2003)

Let $B$ be a Banach space with a norm $\| \cdot \|_B$ and $h : B \to B$ be a completely continuous nonlinear operator. In this paper, we give theorems on the existence of a solution of the operator equation

$$x = h(x),$$

(1)


The use will be made of the following notation.

$\Theta$ is the zero element of the space $B$.

$D$ is the closure of the set $D \subset B$.

$B \times B = \{ (x, y) : x \in B, \ y \in B \}$ is the Banach space with the norm

$$\|(x, y)\|_{B \times B} = \|x\|_B + \|y\|_B.$$

$\Lambda(B \times B)$ is the set of completely continuous operators $g : B \times B \to B$ such that:

(i) $g(x, \cdot) : B \to B$ is a linear operator for every $x \in B$;

(ii) for any $x$ and $y \in B$ the equation

$$z = g(x, z) + y$$

has a unique solution $z$ and

$$\|z\|_B \leq \gamma \|y\|,$$

where $\gamma$ is a positive constant, independent of $x$ and $y$.

$\Lambda_0(B \times B)$ is the set of completely continuous operators $g : B \times B \to B$ such that:

(i) $g(x, \cdot) : B \to B$ is a linear operator for any $x \in B$;

(ii) the set

$$\{ g(x, y) : x \in B, \ \|y\|_B \leq 1 \}$$

is relatively compact;

(iii) $y \notin \{ g(x, y) : x \in B \}$ for $y \in B$ and $y \neq \Theta$.

Let $g_0 \in \Lambda_0(B \times B)$. We say that a linear bounded operator $\varphi : B \to B$ belongs to the set $\mathcal{L}_y$ if there exists a sequence $x_k \in B$ ($k = 1, 2, \ldots$) such that

$$\lim_{k \to \infty} g(x_k, y) = \varphi(y) \text{ for } y \in B.$$

Along with $B$, we consider a partially ordered Banach space $B_0$ in which the partial order is generated by a cone $K$, i.e., for any $u$ and $v \in B_0$, it is said that $u$ does not exceed $v$, and is written $u \leq v$ if $v - u \in K$.

A linear operator $\eta : B_0 \to B_0$ is said to be positive if it transforms the cone $K$ into itself.

An operator $\nu : B \to B_0$ is said to be positively homogeneous if $\nu(\lambda x) = \lambda \nu(x)$ for $\lambda \geq 0, \ x \in B$.

By $r(\eta)$ we denote the spectral radius of the operator $\eta$.

2000 Mathematics Subject Classification. 47H10, 34K13.

Key words and phrases. Nonlinear operator equation in a Banach space, a priori boundedness principle, functional differential equation, periodic solution.
Lemma 1. $\Lambda_0(\mathcal{B} \times \mathcal{B}) \subset \Lambda(\mathcal{B} \times \mathcal{B})$.

Theorem 1 (A priori boundedness principle). Let there exist an operator $g \in \Lambda(\mathcal{B} \times \mathcal{B})$ and a positive constant $\rho_0$ such that for any $\lambda \in [0,1]$ an arbitrary solution of the equation
\[ x = (1 - \lambda)g(x, x) + \lambda h(x) \]
admits the estimate
\[ \|x\|_B \leq \rho_0. \]  
Then the equation (1) is solvable.

Corollary 1. Let there exist a linear completely continuous operator $g : \mathcal{B} \to \mathcal{B}$ and a positive constant $\rho_0$ such that the equation
\[ y = g(y) \]
has only a trivial solution, and for any $\lambda \in [0,1]$ an arbitrary solution of the equation
\[ x = (1 - \lambda)g(x, x) + \lambda h(x) \]
admits the estimate (2). Then the equation (1) is solvable.

On the basis of Lemma 1 and Theorem 1 we prove the following theorem.

Theorem 2. Let there exist an operator $g \in \Lambda_0(\mathcal{B} \times \mathcal{B})$, a partially ordered Banach space $\mathcal{B}_0$ with a cone $\mathcal{K}$ and positively homogeneous continuous operators $\mu$ and $\nu : \mathcal{B} \to \mathcal{K}$ such that
\[ \mu(y) - \nu(y - z) \not\in \mathcal{K} \text{ for } y \not\in \Theta, \quad z \in \{g(x, y) : x \in B\} \]
and
\[ \nu(h(x) - g(x, x) - h_0(x)) \leq \mu(x) + \mu_0(x) \text{ for } x \in \mathcal{B}, \]
where $h_0 : \mathcal{B} \to \mathcal{B}$ and $\mu_0 : \mathcal{B} \to \mathcal{K}$ satisfy the conditions
\[ \lim_{\|x\|_B \to \infty} \frac{\|h_0(x)\|_{\mathcal{B}_0}}{\|x\|_B} = 0, \quad \lim_{\|x\|_B \to \infty} \frac{\|\mu_0(x)\|_{\mathcal{B}_0}}{\|x\|_B} = 0. \]
Then the equation (1) is solvable.

Corollary 2. Let there exist an operator $g \in \Lambda_0(\mathcal{B} \times \mathcal{B})$, a partially ordered Banach space $\mathcal{B}_0$ with a cone $\mathcal{K}$, a positively homogeneous operator $\nu : \mathcal{B} \to \mathcal{K}$ and a linear bounded positive operator $\eta : \mathcal{B}_0 \to \mathcal{K}$ such that
\[ \nu(\eta) < 1, \]
\[ \|\nu(x)\|_{\mathcal{B}_0} > \|x\|_B \text{ for } x \not\in \Theta \text{ and } \]
\[ \nu(h(x) - g(x, x) - h_0(x)) \leq \eta(\nu(x)) + \mu_0(x) \text{ for } x \in \mathcal{B}, \]
where $h_0 : \mathcal{B} \to \mathcal{B}$ and $\mu_0 : \mathcal{B} \to \mathcal{K}$ are operators satisfying (4). Then the equation (1) is solvable.

Corollary 3. Let there exist an operator $g \in \Lambda_0(\mathcal{B} \times \mathcal{B})$ such that
\[ \lim_{\|x\|_B \to \infty} \frac{\|h(x) - g(x, x)\|_{\mathcal{B}_0}}{\|x\|_B} = 0. \]
Then the equation (1) is solvable.

Theorem 3. Let the space $\mathcal{B}$ be separable. Let, moreover, there exist an operator $g \in \Lambda_0(\mathcal{B} \times \mathcal{B})$, a partially ordered Banach space $\mathcal{B}_0$ with a cone $\mathcal{K}$, and positively homogeneous continuous operators $\mu$ and $\nu : \mathcal{B} \to \mathcal{K}$ such that for every $\mathcal{F} \in \mathcal{L}_0$ the inequality
\[ \nu(y - \mathcal{F}(y)) \leq \mu(y) \]
has only a trivial solution and the condition (3) is fulfilled, where $h_0 : \mathcal{B} \to \mathcal{B}$ and $\mu_0 : \mathcal{B} \to \mathcal{K}$ are operators satisfying (4). Then the equation (1) is solvable.
Corollary 4. Let the space $\mathcal{B}$ be separable, let there exist an operator $g \in \Lambda_0(\mathcal{B} \times \mathcal{B})$ such that the condition (5) hold, and let for every $g \in \mathcal{L}_g$ the equation

$$y = g(y)$$

have only a trivial solution. Then the equation (1) is solvable.

Theorem 1 implies a priori boundedness principles proved in [1] and [4], while Theorems 2 and 3 imply the Conti–Opial type theorems proved in [2] and [3].

We give one more application of Theorem 1 concerning the existence of an $\omega$-periodic solution of the functional differential equation

$$u^{(n)}(t) = f(u)(t) + f_0(t). \quad (6)$$

Here $n \geq 1$, $\omega > 0$, $f_0 \in L_\omega$, $f : C_\omega \to L_\omega$ is a continuous operator, $C_\omega$ is the space of continuous $\omega$-periodic functions $u : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|u\|_{C_\omega} = \max \{|u(t)| : 0 \leq t \leq \omega\}$$

and $L_\omega$ is the space of integrable on $[0, \omega]$ $\omega$-periodic functions $v : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|v\|_{L_\omega} = \frac{\omega}{\int_{0}^{\omega} |v(t)| \, dt}.$$ 

By an $\omega$-periodic solution of the equation (6) we understand an $\omega$-periodic function $u : \mathbb{R} \to \mathbb{R}$ which is absolutely continuous together with $u^{(i)}$ ($i = 1, \ldots, n - 1$) and almost everywhere on $\mathbb{R}$ satisfies the equation (6).

On the basis of Corollary 1 we prove the following theorem.

Theorem 4. Let there exist $q \in L_\omega$, $\sigma \in \{-1, 1\}$ and a positive constant $\rho$ such that

$$0 \leq \sigma f(x)(t) \, \text{sgn} x(t) \leq q(t) \quad \text{for} \quad x \in C_\omega, \ t \in \mathbb{R},$$

and for any $x \in C_\omega$, satisfying the inequality

$$|x(t)| > \rho \quad \text{for} \quad t \in \mathbb{R},$$

the condition

$$\int_{0}^{\omega} f(x)(t) \, dt \neq 0$$

is fulfilled. Let, moreover,

$$\int_{0}^{\omega} f_0(t) \, dt = 0. \quad (7)$$

Then the equation (6) has at least one solution.

As an example, consider the differential equation

$$u^{(n)}(t) = \sum_{k=1}^{m} f_k(t) \frac{|u(\tau_k(t))|^{\mu_k} \, \text{sgn} u(\tau_k(t))}{1 + |u(\tau_k(t))|^{\mu_k}} + f_0(t), \quad (8)$$

where $f_k \in L_\omega$ ($k = 0, \ldots, n$), $\mu_k \geq \lambda_k > 0$ ($k = 1, \ldots, n$), and $\tau_k : \mathbb{R} \to \mathbb{R}$ ($k = 1, \ldots, n$) are measurable functions such that the fraction

$$\frac{\tau_k(t + \omega) - \tau_k(t)}{\omega}$$

is an integral number for any $t \in \mathbb{R}$ and $k \in \{1, \ldots, n\}$. 
Corollary 5. Let there exist a number $\sigma \in \{-1,1\}$ such that
$$\sigma f_k(t) \geq 0 \text{ for } t \in \mathbb{R} \text{ (} k = 1, \ldots, n \text{)}$$
and
$$\sigma \sum_{k=1}^{n} \int_{0}^{\infty} f_k(t) \, dt > 0.$$ 

Let, moreover, the condition (7) hold. Then the equation (8) has at least one $\omega$-periodic solution.

Acknowledgment

This work was supported by GRDF (Grant No. 3318).

References


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