T. Kiguradze

EXISTENCE AND UNIQUENESS THEOREMS ON PERIODIC SOLUTIONS TO MULTIDIMENSIONAL LINEAR HYPERBOLIC EQUATIONS

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In \( \mathbb{R}^n \) consider the linear hyperbolic equations

\[
    u^{(m)} = \sum_{\alpha \in \mathcal{E}^m} p_\alpha (x_\alpha) u^{(\alpha)} + \sum_{\alpha \in \mathcal{O}^m} p_\alpha (x_\alpha) u^{(\alpha)} + q(x),
\]

and

\[
    u^{(m)} = p_0(x) + q(x), \tag{2}
\]

where \( u = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_+ \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+ \) are multiindices, and

\[
    u^{(\alpha)} = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u}{\partial x_{1}^{\alpha_1} \cdots \partial x_{n}^{\alpha_n}}.
\]

We make use of following notations and definitions.

\( \mathbb{Z}_+ \) is the set of all nonnegative integers; \( \mathbb{Z}^n_+ \) is the set of all multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n); \|\alpha\| = \alpha_1 + \cdots + \alpha_n; 0 = (0, \ldots, 0) \in \mathbb{Z}^n_+ \).

The inequalities between the multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) are understood componentwise.

It will be assumed that \( m > 0 \).

If for some multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \) we have \( \alpha_1 = \cdots = \alpha_{i_k} = 0 \ (i_1 < \cdots < i_k) \), and \( \alpha_{j_1}, \ldots, \alpha_{j_{n-k}} > 0 \ (j_1 < \cdots < j_{n-k}) \), \( \{j_1, \ldots, j_{n-k}\} \subset \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\} \), then by \( x_\alpha \) (by \( x^\alpha \)) denote the vector \( (x_{i_1}, \ldots, x_{i_k}) \in \mathbb{R}^k \) (the vector \( (x_{j_1}, \ldots, x_{j_{n-k}}) \in \mathbb{R}^{n-k} \)).

By \( \mathcal{E}^m \) and \( \mathcal{O}^m \), respectively, denote the sets of all even and odd multiindices not exceeding \( m \) and different from \( m \), i.e.,

\[
    \mathcal{E}^m = \{ \alpha \in \mathbb{Z}^n_+ \setminus \{ m \} : \alpha \leq m, \ \alpha_1, \ldots, \alpha_n \ \text{are even} \},
\]

\[
    \mathcal{O}^m = \{ \alpha \in \mathbb{Z}^n_+ \setminus \{ m \} : \alpha \leq m, \ \alpha_1 + \cdots + \alpha_n \ \text{is odd} \}.
\]

By \( \mathcal{S}^m \) denote the set of nonzero multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n) \) whose components either equal to the corresponding components of \( m \), or equal to 0, i.e.,

\[
    \mathcal{S}^m = \{ \alpha \neq 0 : \alpha_i \in \{0, m_i\} \ (i = 0, \ldots, n) \}.
\]

Let \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n \) be a vector with positive components. Then by \( \Omega \) denote the rectangular box \( [0, \omega_1] \times \cdots \times [0, \omega_n] \) in \( \mathbb{R}^n \). Moreover, for an arbitrary multiindex \( \alpha \), similarly as we did above, introduce the vectors \( \omega_\alpha = (\omega_{\alpha_1}, \ldots, \omega_{\alpha_k}) \in \mathbb{R}^k \) and \( \omega^\alpha = (\omega_{j_1}, \ldots, \omega_{j_{n-k}}) \in \mathbb{R}^{n-k} \), and the rectangular boxes \( \Omega_\alpha = [0, \omega_{\alpha_1}] \times \cdots \times [0, \omega_{\alpha_k}] \) in \( \mathbb{R}^k \) and \( \Omega^\alpha = [0, \omega_{j_1}] \times \cdots \times [0, \omega_{j_{n-k}}] \) in \( \mathbb{R}^{n-k} \).

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We say that a function \( z : \mathbb{R}^n \to \mathbb{R} \) is \( \omega \)-periodic, if
\[
z(x_1, \ldots, x_j + \omega, \ldots, x_n) \equiv z(x_1, \ldots, x_n) \quad (j = 1, \ldots, n).
\]

It will be assumed that the functions \( p_\alpha (\alpha \in \mathcal{E}^m \cup \mathcal{O}^m) \) and \( q \), respectively, are \( \omega_\alpha \)-periodic and \( \omega \)-periodic continuous functions.

Let \( \ell = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n \). By \( \mathcal{C}^\ell \) denote the space of continuous functions \( u : \mathbb{R}^n \to \mathbb{R} \), having continuous partial derivatives \( u(\alpha) (\alpha \leq \ell) \).

By a solution of equation (1) (equation (2)) we will understand a classical solution, i.e., a function \( u \in \mathcal{C}^m \) satisfying equation (1) (equation (2)) everywhere in \( \mathbb{R}^n \).

In the case, where \( n = 2, m_1 = m_2 = 1 (n = 2, m_1 = m_2 = 2) \) sufficient conditions for existence and uniqueness of \( (\omega_1, \omega_2) \)-periodic solutions of equation (1) are given in [1–3, 6–8] (in [9, 10]). In the general case the problem on \( \omega \)-periodic solutions to equations (1) and (2) are little investigated. In the present paper optimal sufficient conditions of existence and uniqueness of \( \omega \)-periodic solutions to equation (1) (equation (2)) are given. Similar results for higher order nonlinear ordinary differential equations were obtained by I. Kiguradze and T. Kusano [5].

We consider equations (1) and (2) in two cases, where \( m \) is either even, or odd. Also note that equations (1) and (2) do contain partial derivatives with even or odd (according to the above definitions) multiindices only (e.g., neither of \( m \) and \( \alpha \) can equal to (1, 1, 1, 1)).

**Theorem 1.** Let \( m \) be even, and let
\[
(-1)^{\frac{|m|+1}{2}} p_\alpha(x_\alpha) \leq 0 \quad \text{for } x \in \mathbb{R}^n, \quad \alpha \in \mathcal{E}^m,
\]

\[
\mathbb{R}^n \setminus I_{p_0} = \mathbb{R}^n,
\]

where \( I_{p_0} = \{x \in \mathbb{R}^n : p_0(x) = 0\} \). Then equation (1) has at most one \( \omega \)-periodic solution.

**Theorem 2.** Let \( m \) be odd, and let there exist \( j \in \{1, 2\} \) such that along with (4) the inequality
\[
(-1)^{j+1} |\alpha| p_\alpha(x_\alpha) \leq 0 \quad \text{for } x \in \mathbb{R}^n, \quad \alpha \in \mathcal{E}^m
\]

holds. Then equation (1) has at most one \( \omega \)-periodic solution.

Theorems 1 and 2 almost immediately follow from the following lemma.

**Lemma 1.** Let \( u \in \mathcal{C}^m \) be an \( \omega \)-periodic function. Then
\[
\int_{\Omega^n} u(\alpha)(x) u(x) dx^\alpha = (-1)^{\frac{|m|}{2}} \int_{\Omega^n} |u(\frac{\partial}{\partial x})(x)|^2 dx^\alpha \quad \text{for } \alpha \in \mathcal{E}^m,
\]

\[
\int_{\Omega^n} u(\alpha)(x) u(x) dx^\alpha = 0 \quad \text{for } \alpha \in \mathcal{O}^m.
\]

One can easily prove the lemma using integration by parts and taking into consideration \( \omega \)-periodicity of \( u \).

**Proof of Theorem 1.** All we need to prove is that if \( q(x) \equiv 0 \), then equation (1) has only a trivial \( \omega \)-periodic solution. Indeed, let \( q(x) \equiv 0 \), and let \( u \) be an arbitrary \( \omega \)-periodic solution of equation (1). After multiplying equation (1) by \( u \) and integrating over the rectangular box \( \Omega \), by Lemma 1 and condition (3), we get
\[
\int_{\Omega} \left( |u(\frac{\partial}{\partial x})(x)|^2 + \sum_{\alpha \in \mathcal{E}^m} |p_\alpha(x_\alpha)| |u(\frac{\partial}{\partial x})(x)|^2 \right) dx = 0.
\]

(4) and (6) immediately imply that \( u(x) \equiv 0 \). \( \square \)

We omit the proof of Theorem 2, since it is similar to the proof of Theorem 1.
Theorem 3. Let \( m \) be even, and let
\[
0 \leq (-1)^{m-1} \frac{\omega_1 \cdots \omega_n}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^n} f(x) dx = \mathbb{R}^n.
\] (7)
Then equation (2) has at most one \( \omega \)-periodic solution.

To prove the theorem along with Lemma 1 we need the following

Lemma 2. Let \( m \) be even, and let \( u \in C^m \) be an \( \omega \)-periodic function. Then
\[
\int_{\Omega} \left| u^{(m)}(x) \right|^2 dx \leq \frac{\omega_1 \cdots \omega_n}{(2\pi)^{\frac{m}{2}}} \int_{\Omega} \left| u^{(m)}(x) \right|^2 dx.
\] (8)

This lemma immediately follows from Wirtinger's inequality ([4], Theorem 258).

Proof of Theorem 3. Assume the contrary: let \( q(x) \equiv 0 \) and equation (2) have a nontrivial \( \omega \)-periodic solution \( u \). Then we have
\[
u^{(m)}(x) = p_0(x)u(x)
\] (9)
and
\[
\left| u^{(m)}(x) \right|^2 = |p_0(x)u(x)|^2.
\] (10)
Multiplying (9) by \( u \), integrating over \( \Omega \), by Lemma 1, we get
\[
\int_{\Omega} |p_0(x)||u(x)|^2 dx = \int_{\Omega} \left| u^{(m)}(x) \right|^2 dx.
\] (11)
Integrating (10) over \( \Omega \) and assuming that \( u(x) \neq 0 \), by condition (8), we get
\[
\int_{\Omega} \left| u^{(m)}(x) \right|^2 dx = \int_{\Omega} |p_0(x)u(x)|^2 dx < \frac{(2\pi)^{\frac{m}{2}}}{\omega_1 \cdots \omega_n} \int_{\Omega} |p_0(x)||u(x)|^2 dx.
\] (12)
On the other hand, from (8) and (11) we get the inequality
\[
\int_{\Omega} |p_0(x)||u(x)|^2 dx \leq \frac{\omega_1 \cdots \omega_n}{(2\pi)^{\frac{m}{2}}} \int_{\Omega} \left| u^{(m)}(x) \right|^2 dx,
\]
which contradicts to (12). The obtained contradiction completes the proof of the theorem.

Remark 1. In Theorem 3 condition (7) is optimal and it cannot be weakened: strict inequality cannot be replaced by an unstrict one. Indeed, consider the equation
\[
u^{(m)} = l u,
\] (13)
where \( l \) is a constant. If
\[
0 < l < (-1)^m \frac{(2\pi)^{\frac{m}{2}}}{\omega_1 \cdots \omega_n},
\]
then by Theorem 3 equation (13) has only a trivial solution. However, if
\[
l = (-1)^m \frac{(2\pi)^{\frac{m}{2}}}{\omega_1 \cdots \omega_n} \quad (l = 0),
\]
then it is obvious that the function
\[
u(x) = \sin \left( \frac{2\pi}{\omega_1} x_1 \right) \cdots \sin \left( \frac{2\pi}{\omega_n} x_n \right) \quad (u(x) = 1)
\]
is a nontrivial \( \omega \)-solution of equation (13).

Below we formulate existence theorems.
Theorem 4. Let $m$ be even, and let along with (3) the inequalities
\[ (-1)^{|m_1|+|m_2|} \int_{\Omega_2} p_\alpha(x_\alpha) \, dx_\alpha < 0 \quad \text{for} \quad \alpha \in S^m, \] (14)
\[ \int_{\Omega} p_0(x) \, dx \neq 0 \]
hold. Then equation (1) has one and only one $\omega$-periodic solution.

Theorem 5. Let $m_1$ be the only odd component of the the multindex $m$, and let there exist $j \in \{1, 2\}$ such that along with (5) the inequalities
\[ (-1)^{j+\frac{m_2}{2}} \int_{\Omega_2} p_\alpha(x_\alpha) \, dx_\alpha < 0 \quad \text{for} \quad \alpha \in S^m, \] (15)
\[ (-1)^j \int_0^{\omega_2} \ldots \int_0^{\omega_n} p_0(x_1, x_2, \ldots, x_n) \, dx_2 \ldots dx_n < 0 \quad \text{for} \quad x_1 \in \mathbb{R} \]
hold. Then equation (1) has one and only one $\omega$-periodic solution.

Remark 2. In Theorems 4 (Theorem 5) condition (14) (condition (15)) is essential and it cannot be weakened. If for at least one $\alpha \in S^m$ $p_\alpha(x_\alpha) \equiv 0$, then equation (1) may not have an $\omega$-periodic solution. To verify this, consider the equation
\[ u^{(2, 2, 2)} = u^{(2, 2, 0)} + u^{(2, 0, 2)} + u^{(0, 2, 0)} - u^{(0, 0, 2)} + \sin^2(x_1) \, u - 1. \] (16)
In the case, where $n = 3$, $m_1 = m_2 = m_3 = 2$ and $\omega_1 = \omega_2 = \omega_3 = \pi$, this equation satisfies all of the conditions of Theorem 4, except condition (14). For $\alpha = (2, 0, 0)$ we have $p_{\alpha}(x_2, x_3) \equiv 0$. As a result equation (16) has no $(\pi, \pi, \pi)$-periodic solution. Assume the contrary: let equation (16) have a $(\pi, \pi, \pi)$-periodic solution $u$. By Theorem 1, it is unique, and therefore is independent of $x_2$ and $x_3$. Hence $u$ satisfies the equation
\[ \sin^2(x_1) \, u - 1 = 0. \]
But the latter equation has only a discontinuous solution. The obtained contradiction proves that equation (16) has no $(\pi, \pi, \pi)$-periodic solution.

Theorem 6. Let $m$ be even, and let
\[ 0 < (-1)^{|m_1|} p_0(x) < \frac{(2\pi)^{|m_1|}}{\omega_1^{m_1} \ldots \omega_n^{m_n}}. \] (17)
Moreover, let $p_0$ and $q \in C^m$. Then equation (2) has one and only one $\omega$-periodic solution.

Theorem 7. Let $m$ be even, and let
\[ (-1)^{|m_1|} p_0(x) < 0 \quad \text{for} \quad x \in \mathbb{R}^n. \] (18)
Moreover, let $p_0$ and $q \in C^m$. Then equation (2) has one and only one $\omega$-periodic solution.

Theorem 8. Let $m$ be odd, and let there exist a number $j \in \{1, 2\}$ such that
\[ (-1)^j p_0(x) < 0 \quad \text{for} \quad x \in \mathbb{R}^n. \] (19)
Moreover, let $p_0$ and $q \in C^m$. Then equation (2) has one and only one $\omega$-periodic solution.

Remark 3. In Theorems 6, 7 and 8 the requirement of additional regularity of functions $p_0$ and $q$ is sharp. If this condition is violated, then equation (2) may not have a $\omega$-periodic classical solution. Indeed, consider the equation
\[ u^{(m)} = p_0(x_2, \ldots, x_n) \, u - p_0^2(x_2, \ldots, x_n), \]
where $m$ is even, and $p_0(x_2, \ldots, x_n)$ is an arbitrary continuous $(\omega_2, \ldots, \omega_n)$–periodic function satisfying (18). By Theorem 3, this equation has at most one solution. Hence

$$u(x) = p_0(x_2, \ldots, x_n).$$

But $u$ is a classical solution if and only if $p_0 \in C^m$.

**Remark 4.** In Theorems 6, 7 and 8, respectively, the strict inequalities (17), (18) and (19) cannot be replaced by unstrict ones. To verify this, consider the equation

$$u^{(m)} = p_0(x_2, \ldots, x_n) u - 1,$$

where $m$ is odd and $p_0(x_2, \ldots, x_n)$ is a smooth $(\omega_2, \ldots, \omega_n)$–periodic function such that $p_0(x_2, \ldots, x_n) \geq 0$, $p_0(x_2, \ldots, x_n) \neq 0$. By Theorem 2, this equation has at most one solution. Therefore $u$ is a solution of the equation

$$p_0(x_2, \ldots, x_n) u - 1 = 0.$$

But the latter equation has a continuous solution if and only if

$$p_0(x_2, \ldots, x_n) > 0 \text{ for } (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}.$$

**References**


**Author’s address:**

T. Kiguradze
Florida Institute of Technology
Department of Mathematical Sciences
150 W. University Blvd.
Melbourne, FL 32901
USA
E-mail: tkigurad@fit.edu