ON LIDSTONE BOUNDARY VALUE PROBLEM FOR HIGHER ORDER NONLINEAR HYPERBOLIC EQUATIONS WITH TWO INDEPENDENT VARIABLES

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Let $m$ and $n$ be positive integers, $a > 0$, $b > 0$ and $D = [0, a] \times [0, b]$. In the rectangle $D$ consider the nonlinear hyperbolic equation

$$u^{(2m, 2n)} = f(x, y, u, \ldots, u^{(2m-1,0)}, \ldots, u^{(0,2n-1)}, \ldots, u^{(2m-1,2n-1)})$$

(1)

with the boundary conditions

$$u^{(2i,0)}(0, y) = \varphi_{1i}(y), \quad u^{(2i,0)}(b, y) = \varphi_{2i}(y) \quad (i = 0, \ldots, m - 1),$$

$$u^{(2m, 2k)}(x, 0) = \psi_{1k}(x), \quad u^{(2m, 2k)}(x, b) = \psi_{2k}(x) \quad (i = 0, \ldots, n - 1),$$

(2)

where

$$u^{(i,k)}(x, y) = \frac{\partial^{i+k} u(x, y)}{\partial x^i \partial y^k} \quad (i = 0, \ldots, 2m; \quad k = 0, \ldots, 2n).$$

Moreover, below it will be assumed that the function $f : D \times \mathbb{R}^{4mn} \to \mathbb{R}$ is continuous, the functions $\varphi_{2i} : [0, b] \to \mathbb{R}$, $\varphi_{2i} : [0, b] \to \mathbb{R}$ (i = 0, ..., m - 1) are 2m-times continuously differentiable, and the functions $\psi_{2k} : [0, a] \to \mathbb{R}$, $\psi_{2k} : [0, a] \to \mathbb{R}$ (i = 0, ..., n - 1) are continuous.

By $C^{2m, 2n}(D)$ denote the space of continuous functions $u : D \to \mathbb{R}$ having the continuous partial derivatives $u^{(j,k)}$ (j = 0, ..., 2m; k = 0, ..., 2n). By a solution of problem (1), (2) we will understand a classical solution, i.e., a function $u \in C^{2m, 2n}(D)$ satisfying equation (1) and boundary conditions (2) everywhere in $D$.

By analogy with the problem

$$z^{(2m)} = g(x, z, \ldots, z^{(2m-1)}),$$

$$z^{(2i)}(0) = c_{1i}, \quad z^{(2i)}(a) = c_{2i} \quad (i = 1, \ldots, n),$$

(3)

problem (1), (2) will be called the Lidstone problem.

Problem (3), (4) and its various generalizations were investigated by many authors (see, e.g., [1–8], [12]). As for the problem (1), (2), it was studied in the case, where $m = n = 1$ and (1) is a linear equation (see [9–11]).

The given below sufficient conditions of solvability and unique solvability of problem (1), (2) concern the case, where on the set $D \times \mathbb{R}^{n}$ the function $f$ on satisfies either of the conditions

$$|f(x, y, z_{00}, \ldots, z_{2m-10}, \ldots, z_{02n-1}, \ldots, z_{2m-12n-1})| \leq \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} p_{ik}(x, y)|z_{ik}| + q(x, y, \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} |z_{ik}|)$$

(5)

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Then by \( (10) \) we have

\[
|f(x, y; z_0, \ldots, z_{2m-12n-1}) - f(x, y; \overline{z}_0, \ldots, \overline{z}_{2m-12n-1})| \\
\leq \sum_{i=0}^{2m-1} \sum_{k=0}^{2n-1} p_{ik}(x, y) |z_i - \overline{z}_i|,
\]

where \( p_{ik} : D \to [0, +\infty) \) \((i = 0, \ldots, 2m-1; k = 0, \ldots, 2n-1)\) are continuous functions, and \( q : D \times [0, +\infty) \to [0, +\infty) \) is a continuous function that is nondecreasing in the second argument and

\[
\lim_{\rho \to +\infty} \frac{1}{\rho} \int_0^a \int_0^b q(x, y, \rho) \, dx \, dy = 0.
\]

Along with \((1), (2)\) we will consider the differential inequality

\[
|u^{(2m, 2n)}(x, y)| \leq \sum_{i=0}^{2m-1} \sum_{k=0}^{2n-1} p_{ik}(x, y) |u^{(i, k)}(x, y)|
\]

with the homogeneous boundary conditions

\[
u^{(2i, 0)}(0, y) = 0, \quad u^{(2i, 0)}(a, y) = 0 \quad (i = 0, \ldots, m-1),
\]

\[
u^{(2m, 2k)}(x, 0) = 0, \quad u^{(2m, 2k)}(x, b) = 0 \quad (i = 0, \ldots, n-1).
\]

By a solution of problem \((8), (9)\) we will understand a function \( u \in C^{2m, 2n}(D) \) satisfying inequality \((8)\) and boundary conditions \((9)\) everywhere in \( D \).

**Theorem 1.** Let conditions \((5)\) and \((7)\) (condition \((6)\)) hold, and let problem \((8), (9)\) have only a trivial solution. Then problem \((1), (2)\) has at least one (one and only one) solution.

For arbitrary \( s_0 > 0, s \in [0, s_0] \) and a positive integer \( j \) set

\[
\lambda_1(s; s_0) = \frac{1}{s_0}, \quad \lambda_{2j+1}(s; s_0) = \frac{s(s_0 - s)}{2s_0} \left(\frac{s_0}{8}\right)^{j-1},
\]

\[
\lambda_2(s; s_0) = \frac{s(s_0 - s)}{s_0}, \quad \lambda_{2j+2}(s; s_0) = \frac{s^2(s_0 - s)^2}{2s_0} \left(\frac{s_0}{8}\right)^{j-1}.
\]

**Theorem 2.** Let conditions \((5)\) and \((7)\) (condition \((6)\)) hold, and

\[
\sum_{i=0}^{2m-1} \sum_{k=0}^{2n-1} \int_0^a \int_0^b p_{ik}(x, y) \lambda_{2m-i}(x; a) \lambda_{2n-k}(y; b) \, dx \, dy \leq 1.
\]

Then problem \((1), (2)\) has at least one (one and only one) solution.

Let

\[
\mu_{2j-1}(s_0) = \left(\frac{s_0}{8}\right)^{j-1}, \quad \mu_{2j}(s_0) = 2 \left(\frac{s_0}{8}\right)^j \quad (j = 1, 2, \ldots).
\]

Then by \((10)\) we have

\[
\lambda_k(s; s_0) \leq \frac{1}{s_0} \mu_k(s_0) \quad \text{for} \quad 0 \leq s \leq s \quad (k = 1, 2, \ldots).
\]

Therefore Theorem 2 implies the

**Corollary 1.** Let conditions \((5)\) and \((7)\) (condition \((6)\)) hold, and let

\[
\sum_{i=0}^{2m-1} \sum_{k=0}^{2n-1} \mu_{2m-i}(a) \mu_{2n-k}(b) \int_0^a \int_0^b p_{ik}(x, y) \, dx \, dy \leq ab.
\]

Then problem \((1), (2)\) has at least one (one and only one) solution.
Let us show that in Theorem 2 and Corollary 1, respectively, conditions (11) and (12) are unimprovable from the viewpoint that they cannot be replaced by the conditions
\[ \sum_{k=0}^{2n-1} \sum_{i=0}^{2m-1} \int_0^a \int_0^b p_{ik}(x,y) \lambda_{2m-i}(x) \lambda_{2n-k}(y) \, dx \, dy \leq (1+\varepsilon) \] (11ε)
and
\[ \sum_{k=0}^{2n-1} \sum_{i=0}^{2m-1} \mu_{2m-i}(a) \mu_{2n-k}(b) \int_0^a \int_0^b p_{ik}(x,y) \, dx \, dy \leq (1+\varepsilon)ab, \] (12ε)
no matter how small \( \varepsilon > 0 \) is. Indeed, as it was shown in [6] (see Example 1.1), for an arbitrary \( \varepsilon > 0 \) there exist continuous functions \( g_1 : [0,a] \to [0,\infty) \) and \( g_2 : [0,b] \to [0,\infty) \) such that
\[ 4 < a \int_0^a g_1(x) \, dx < 4\sqrt{1+\varepsilon}, \quad 4 < b \int_0^b g_2(y) \, dy < 4\sqrt{1+\varepsilon}, \]
and the boundary value problems
\[ w'' = -g_1(x)w, \quad w(0) = w(a) = 0 \]
and
\[ w'' = -g_2(y)w, \quad w(0) = w(b) = 0 \]
have nontrivial solutions \( w_1 \) and \( w_2 \). If \( m > 1 \) (\( n > 1 \)), then by \( v_1 \) (by \( v_2 \)) denote the solution of the problem
\[ v^{(2m-2)}(x) = w_1(x), \quad v^{(2)}(0) = v^{(2)}(a) = 0 \quad (i = 0, \ldots, m-1) \]
\[ v^{(2n-2)}(y) = w_2(y), \quad v^{(2)}(0) = v^{(2)}(b) = 0 \quad (k = 0, \ldots, n-1). \]
For \( m = 1 \) (\( n = 1 \)) set \( v_1(x) = w_1(x) \) (\( v_2(y) = w_2(y) \)). Then the function
\[ u(x,y) = v_1(x)v_2(y) \]
is a nontrivial solution of the homogeneous equation
\[ u^{(2m,2n)} = g(x,y)u^{(2m-2,2n-2)} \]
subject to the boundary conditions (9), where
\[ g(x,y) = g_1(x)g_2(y) \]
and
\[ 16 < ab \int_0^a \int_0^b g(x,y) \, dx \, dy < 16(1+\varepsilon). \] (13)
On the other hand, the function
\[ f(x,y,z_0,\ldots,z_{2m-12n-1}) = g(x,y)z_{2m-22n-2} \]
satisfies condition (6), where
\[ p_{ik}(x,y) \equiv 0 \quad \text{for} \quad i \neq 2m-2 \quad \text{or} \quad k \neq 2n-2, \]
\[ p_{ik}(x,y) \equiv g(x,y) \quad \text{for} \quad i = 2m-2, \quad k = 2n-2. \]
Moreover, as it follows from inequality (13), conditions (11) and (12) are violated, while conditions (11ε) and (12ε) hold.

**Theorem 3.** Let conditions (5) and (7) (condition (6) hold, where
\[ p_{ik}(x,y) \equiv p_{ik} \quad (i = 0, \ldots, 2m-1; \quad k = 0, \ldots, 2n-1) \]
are nonnegative constants satisfying the inequality
\[ \sum_{k=0}^{2n-1} \sum_{i=0}^{2m-1} \left( \frac{a}{\pi} \right)^{2m-i} \left( \frac{b}{\pi} \right)^{2n-k} p_{ik} < 1. \] (14)
Then problem (1), (2) has at least one (one and only one) solution.
Let \( i \in \{0, \ldots, m-1 \} \), \( k \in \{0, \ldots, n-1 \} \). Then the differential equation
\[
 u^{(2m,2n)}(x,y) = (-1)^{m+n+i+k} \left( \frac{\pi}{a} \right)^{2m-2i} \left( \frac{\pi}{b} \right)^{2n-2k} u^{(2i,2k)}(x,y)
\]
has a nontrivial solution
\[
 u(x,y) = \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right).
\]
Consequently, in Theorem 3 the strict inequality (14) cannot be replaced by the unstrict inequality
\[
 \sum_{k=0}^{2n-1} \sum_{i=0}^{2m-1} \left( \frac{a}{\pi} \right)^{2m-i} \left( \frac{b}{\pi} \right)^{2n-k} p_{ik} \leq 1.
\]

References


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