CRITERIA OF CORRECTNESS OF LINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF IMPULSIVE EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES ACTIONS

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Let $P \in L([a, b]; \mathbb{R}^{n \times n})$, $p \in L([a, b]; \mathbb{R}^n)$, $Q_j \in \mathbb{R}^{n \times n}$ ($j = 1, \ldots, m$), $q_j \in \mathbb{R}^n$ ($j = 1, \ldots, m$), $a = \tau_0 < \tau_1 < \cdots < \tau_m \leq \tau_{m+1} = b$, $e_0 \in \mathbb{R}^n$, and $\ell : \text{BVC}([a, b]; \tau_1, \ldots, \tau_m; \mathbb{R}^n) \to \mathbb{R}^n$ be a linear bounded operator such that the impulsive system

$$\frac{dx}{dt} = P(t)x + p(t), \quad (1)$$

$$x(\tau_j+) - x(\tau_j-) = Q_j x(\tau_j) + q_j \quad (j = 1, \ldots, m) \quad (2)$$

has a unique solution $x_0$ satisfying the boundary condition $\ell(x) = e_0$.

Consider sequences of matrix- and vector-functions $P_k \in L([a, b]; \mathbb{R}^{n \times n})$ ($k = 1, 2, \ldots$) and $p_k \in L([a, b]; \mathbb{R}^n)$ ($k = 1, 2, \ldots$), sequences of constant matrices $Q_{kj} \in \mathbb{R}^{n \times n}$ ($j = 1, \ldots, m$; $k = 1, 2, \ldots$) and constant vectors $q_{kj} \in \mathbb{R}^n$ ($j = 1, \ldots, m$; $k = 1, 2, \ldots$) and $e_{0k} \in \mathbb{R}^n$ ($k = 1, 2, \ldots$) and a sequence of linear bounded operators $\ell_k : \text{BVC}([a, b]; \tau_1, \ldots, \tau_m; \mathbb{R}^n) \to \mathbb{R}^n$ ($k = 1, 2, \ldots$).

In this paper necessary and sufficient conditions as well as effective sufficient conditions are established for a sequence of boundary value problems

$$\frac{dx}{dt} = P_k(t)x + p_k(t), \quad (3)$$

$$x(\tau_j+) - x(\tau_j-) = Q_{kj} x(\tau_j) + q_{kj} \quad (j = 1, \ldots, m), \quad (4)$$

$$\ell_k(x) = e_{0k} \quad (5)$$

$(k = 1, 2, \ldots)$ to have a unique solution $x_k$ for sufficiently large $k$ and

$$\lim_{k \to \infty} x_k(t) = x_0(t) \quad (6)$$

uniformly on $[a, b]$.

Analogous questions are investigated e.g. in [1], [2], [5], [6] (see the references therein, too) for systems of ordinary differential equations and in [3], [4] for systems of generalized ordinary differential equations.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} = ]-\infty, \infty[$, $\mathbb{R}^{n \times l}$ is the space of all real $n \times l$-matrices $X = (x_{ij})_{n \times l}$ with the norm $\|X\| = \max_{j=1, \ldots, n} \sum_{i=1}^{l} |x_{ij}|$, $O_{n \times l}$ is the zero $n \times l$-matrix.

$\det(X)$ is the determinant of a matrix $X \in \mathbb{R}^{n \times n}$. $I_n$ is the identity $n \times n$-matrix. $\delta_{ij}$ is the Kronecker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

$\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column $n$-vectors $x = (x_i)_{n \times 1}$.

$\text{BVC}([a, b]; \tau_1, \ldots, \tau_m; \mathbb{R}^{n \times l})$ is the Banach space of all continuous on the intervals $[a, \tau_1], [\tau_1, \tau_2], \ldots, [\tau_m, \tau_{m+1}]$ ($k = 1, \ldots, m$) matrix-functions of bounded variation $X : [a, b] \to \mathbb{R}^{n \times l}$ with the norm $\|X\|_S = \sup \{ \|X(t)\| : t \in [a, b] \}$.

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$L([a, b]; \mathbb{R}^{n \times l})$ is the set of all measurable and Lebesgue integrable on $[a, b]$ matrix-functions.

$C([a, b]; \mathbb{R}^{n \times l})$ is the set of all continuous on $[a, b]$ matrix-functions.

$\tilde{C}([a, b]; \mathbb{R}^{n \times l})$ is the set of all absolutely continuous on $[a, b]$ matrix-functions.

$\tilde{C}([a, b]; \{\tau_j\}_{j=1}^m; \mathbb{R}^{n \times l})$ is the set of all matrix-functions restrictions of which on every closed interval $[c, d]$ from $[a, b]$ belong to $\tilde{C}([c, d]; \mathbb{R}^{n \times l})$.

On the set $C([a, b]; \mathbb{R}^{n \times l}) \times \mathbb{R}^{n \times l} \times \cdots \times \mathbb{R}^{n \times l} \times L([a, b]; \mathbb{R}^{l \times k})$ we introduce the operator

$$B_0(\Phi, G_1, \ldots, G_m, X)(t) \equiv \int_a^t \Phi(s)X(s) \, ds + \sum_{j=0, \tau_j \in [a, b]} G_j \int_{\tau_j}^t X(s) \, ds,$$

where $G_0 = O_{n \times n}$.

Under a solution of the system (1), (2) we understand a continuous from the left vector-function $x \in C([a, b] \setminus \{\tau_j\}_{j=1}^m; \mathbb{R}^{n \times l}) \cap \mathcal{BVC}([a, b]; R^1 \times \ldots \times R^m)$ satisfying the system (1) for a.e. $t \in [a, b]$ and the equality (2) for every $j \in \{1, \ldots, n\}$.

We assume everywhere that $\det(I_n + Q_j) \neq 0 \ (j = 1, \ldots, m)$.

Note that this condition guarantees the unique solvability of the system (1), (2) under the Cauchy condition $x(\theta_0) = c_0$.

**Definition 1.** We say that a sequence $(P_k, p_k; \{Q_{kj}\}_{j=1}^m, k) (k = 1, 2, \ldots)$ belongs to the set $S(P, p; \{Q_{kj}\}_{j=1}^m, k)$ if for every $c_0 \in \mathbb{R}^n$ and $c_k \in \mathbb{R}^n (k = 1, 2, \ldots)$ satisfying the condition $\lim_{k \to \infty} c_k = c_0$ the problem (3)–(5) has a unique solution $x_k$ for any sufficiently large $k$ and the condition (6) holds uniformly on $[a, b]$.

**Theorem 1.** Let

$$\lim_{k \to \infty} \ell_k(y) = \ell(y) \quad \text{for} \quad y \in \mathcal{BVC}([a, b]; R^m).$$

Then

$$((P_k, p_k; \{Q_{kj}\}_{j=1}^m, k)_{k=1}^\infty) \in S(P, p; \{Q_{kj}\}_{j=1}^m, k)$$

if and only if there exist sequences of matrix-functions $\Phi_k, \Phi_k \in C([a, b]; \mathbb{R}^{n \times n}) (k = 1, 2, \ldots)$ and constant matrices $G_j, G_{kj} \in \mathbb{R}^{n \times n}$, $G_0 = G_{k0} = O_{n \times n} (j = 0, \ldots, m; k = 1, 2, \ldots)$ such that

$$\lim_{k \to \infty} \sup_{j=0}^m \int_{\tau_j}^{\tau_{j+1}} \|\Phi_k'(t) + \left(\Phi_k(t) + \sum_{i=0}^j Q_{ki}(t)\right)P_k(t)\| \, dt < \infty,$$

$$\inf \left\{ \det \left( \Phi(t) + \sum_{i=0}^j G_i \right) : t \in [\tau_j, \tau_{j+1}] \right\} > 0 \quad (j = 0, \ldots, m),$$

$$\lim_{k \to \infty} G_{kj} = G_j \quad (j = 1, \ldots, m),$$

$$\lim_{k \to \infty} Q_{kj} = Q_j, \quad \lim_{k \to \infty} q_{kj} = q_j \quad (j = 1, \ldots, m),$$

and the conditions

$$\lim_{k \to \infty} \Phi_k(t) = \Phi(t),$$

$$\lim_{k \to \infty} B_0(\Phi_k, G_{k1}, \ldots, G_{km}, P_k)(t) = B_0(\Phi, G_1, \ldots, G_m, P)(t),$$

$$\lim_{k \to \infty} B_0(\Phi_k, G_{k1}, \ldots, G_{km}, P_k)(t) = B_0(\Phi, G_1, \ldots, G_m, P)(t)$$

are fulfilled uniformly on $[a, b]$. 
Remark 1. The conditions (14) and (15) are fulfilled uniformly on \([a, b]\) if and only if the conditions
\[
\lim_{k \to \infty} \int_{t_j}^{t} \left( \Phi_k(s) + \sum_{i=0}^{j} G_{ki} \right) P_k(s) \, ds = \int_{t_j}^{t} \left( \Phi(s) + \sum_{i=0}^{j} G_i \right) P(s) \, ds,
\]
\[
\lim_{k \to \infty} \int_{t_j}^{t} \left( \Phi_k(s) + \sum_{i=0}^{j} G_{ki} \right) p_k(s) \, ds = \int_{t_j}^{t} \left( \Phi(s) + \sum_{i=0}^{j} G_i \right) p(s) \, ds,
\]
respectively, are fulfilled uniformly on \([\tau_j, \tau_{j+1}]\) for every \(j \in \{0, \ldots, m\}\).

**Corollary 1.** Let the conditions (7) and (12) hold. Let, moreover, there exist matrixfunctions \(\Phi, \Phi_k \in C([a, b]; \mathbb{R}^{n \times n})\) \((k = 1, 2, \ldots)\) such that the conditions (9) and
\[
\inf \left\{ \left| \det \left( \Phi(t) + (1 - \delta_{0j}) J_i \right) \right| : t \in [\tau_j, \tau_{j+1}] \right\} > 0 \quad (j \in \{0, \ldots, m\})
\]
hold and the conditions (13),
\[
\lim_{k \to \infty} \int_{t_j}^{t} \left( \Phi_k(s) + (1 - \delta_{0j}) J_i \right) P_k(s) \, ds = \int_{t_j}^{t} \left( \Phi(s) + (1 - \delta_{0j}) J_i \right) P(s) \, ds
\]
and
\[
\lim_{k \to \infty} \int_{t_j}^{t} \left( \Phi_k(s) + (1 - \delta_{0j}) J_i \right) p_k(s) \, ds = \int_{t_j}^{t} \left( \Phi(s) + (1 - \delta_{0j}) J_i \right) p(s) \, ds
\]
be fulfilled uniformly on \([\tau_j, \tau_{j+1}]\) for every \(j \in \{0, \ldots, m\}\). Then the condition (8) holds.

**Corollary 2.** Let the conditions (7) and (12) hold. Let, moreover, there exist matrixfunctions \(\Phi, \Phi_k \in C([a, b]; \mathbb{R}^{n \times n})\) \((k = 1, 2, \ldots)\) such that
\[
\lim_{k \to \infty} \int_{a}^{b} \left\| \Phi_k(t) + \Phi(t) P_k(t) \right\| \, dt < \infty, \quad \inf \left\{ \left| \det(\Phi(t)) \right| : t \in [a, b] \right\} > 0
\]
and the conditions (13) and
\[
\lim_{k \to \infty} \int_{a}^{b} \Phi_k(s) P_k(s) \, ds = \int_{a}^{b} \Phi(s) P(s) \, ds, \quad \lim_{k \to \infty} \int_{a}^{b} \Phi_k(s) p_k(s) \, ds = \int_{a}^{b} \Phi(s) p(s) \, ds
\]
are fulfilled uniformly on \([a, b]\). Then the condition (8) holds.

**Corollary 3.** Let the conditions (7), (11) and (12) hold. Let, moreover, there exist constant matrices \(G_j, G_{ki} \in \mathbb{R}^{n \times n}\), \(G_0 = G_{k0} = O_{n \times n}\) \((j = 0, \ldots, m; k = 1, 2, \ldots)\) such that
\[
\lim_{k \to \infty} \sup_{j = 0}^{m} \int_{t_j}^{t_{j+1}} \left\| \left( I_n + \sum_{i=0}^{j} Q_{ki} \right) P_k(t) \right\| \, dt < \infty, \quad (j = 1, \ldots, m)
\]
and the conditions
\[
\lim_{k \to \infty} \int_{t_j}^{t} \left( I_n + \sum_{i=0}^{j} G_{ki} \right) P_k(s) \, ds = \int_{t_j}^{t} \left( I_n + \sum_{i=0}^{j} G_i \right) P(s) \, ds,
\]
\[
\lim_{k \to \infty} \int_{t_j}^{t} \left( I_n + \sum_{i=0}^{j} G_{ki} \right) p_k(s) \, ds = \int_{t_j}^{t} \left( I_n + \sum_{i=0}^{j} G_i \right) p(s) \, ds
\]
are fulfilled uniformly on \([\tau_j, \tau_{j+1}]\) for every \(j \in \{0, \ldots, m\}\). Then the condition (8) holds.
Corollary 4. Let the conditions (7), (12) and (16) hold and the conditions
\[ \lim_{k \to \infty} \int_{a}^{t} P_k(s) \, ds = \int_{a}^{t} P(s) \, ds, \quad \lim_{k \to \infty} \int_{a}^{t} p_k(s) \, ds = \int_{a}^{t} p(s) \, ds \] (17)
be fulfilled uniformly on \([a, b]\). Then the condition (8) holds.

Corollary 5. Let the conditions (7), (12), and (16) hold and the condition (17) be fulfilled uniformly on \([a, b]\). Then the condition (8) holds.

Remark 2. In Theorem 1 and Corollaries 1–5 we can assume without loss of generality that \(\Phi(t) \equiv I_n\) and \(G_j = O_{n \times n} \, (j = 1, \ldots, m)\) everywhere they appear. So that the condition (10) in Theorem 1 as well as the analogous conditions in the corollaries are valid automatically.

These results follow from analogous results for a system of so-called generalized ordinary differential equations contained in [4] because the system (1), (2) is its particular case.

REFERENCES


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