BOUNDARY-VALUE PROBLEMS OF PIEZOELECTRICITY IN DOMAINS WITH CUTS

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In a domain with cut static boundary-value problems of piezoelectricity are investigated. The existence and uniqueness of solutions of the considered boundary value problems are proved and an asymptotic expansion of solutions near the edges of the cut are obtained.

Let $\Omega$ and $\Omega_1 (\overline{\Omega}_1 \subset \Omega)$ be a bounded domains in the three-dimensional Euclidean space $\mathbb{R}^3$ with infinitely smooth boundaries $\partial \Omega$ and $\partial \Omega_1$, respectively. We assume that the boundary $\partial \Omega_1$ of the domain $\Omega_1$ is the union of two surfaces: $\partial \Omega_1 = \overline{S} \cup S_0$, and that the boundary $\partial S = \partial S_0$ is an infinitely smooth curve. Denote $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.

We suppose that the domain $\Omega \setminus \overline{S}$ is filled with an anisotropic homogeneous piezoelectric material having a cut at $\overline{S}$.

In the domain $\Omega \setminus \overline{S}$, let us consider the system of static equations of piezoelectricity for a homogeneous anisotropic medium [1]:

$$A(D_x)u + F = 0,$$

(1)

where $u = (u_1, u_2, u_3, u_4)$; $u_1, u_2, u_3$ are the components of the displacement vector, $u_4$ is the electric potential, $F_i$, $i = 1, 2, 3$, are the components of the mass force, $F_4$ is the electric charge density, $A(D_x)$ is the differential operator of the form

$$A(D_x) = \|A_{jk}(D_x)\|_{5 \times 5},$$

(2)

$$A_{jk}(D_x) = c_{ijlk}\partial_i \partial_j,$$

$$A_{jk}(D_x) = c_{ijk}\partial_k,$$

$$A_{4k}(D_x) = -e_{ijk}\partial_k,$$

$$A_{4k}(D_x) = e_{ijkl},$$

where $c_{ijlk}, e_{ijk}, e_{ik}$ are respectively the elastic, piezoelectric and dielectric constants.

In the equalities (1) it is assumed that summation is carried out over the repeated indices. We will follow this rule in the sequel.

The constants in (1) satisfy the symmetry conditions

$$c_{ijlk} = c_{jilk}, \quad e_{ijkl} = e_{klij}, \quad e_{ik} = e_{ki},$$

(3)

and the condition of positiveness of the internal energy:

$$\forall (\xi_{ij}), (\eta_{ij}): \quad \xi_{ij} = \xi_{ji}, \quad \exists \gamma > 0 \quad c_{ijkl} \xi_{ij}\xi_{kl} \geq c_0 \xi_{ij}\xi_{ij}, \quad e_{ij}\eta_{ij} \geq e_0 \eta_{ij} \eta_{ij}.$$  

(4)

Note that the operator $A(D_x)$ is a strongly elliptic operator, but is not formally self-adjoint or positive definite.

We introduce the following electromechanical stress operator $T(D_y, n)$:

$$T(D_y, n) = \|T_{jk}(D_y, n)\|_{4 \times 4},$$

$$T_{jk}(D_y, n) = c_{ijlk}n_l \partial_i,$$

$$T_{jk}(D_y, n) = e_{ijk}n_k \partial_j,$$

$$T_{jk}(D_y, n) = -e_{ijk}n_k \partial_i,$$

$$T_{jk}(D_y, n) = e_{ijkl},$$

(4)

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where \( n(y) = (n_1(y), n_2(y), n_3(y)) \) is the unit normal at the point \( y \in \partial \Omega_2 \), directed outward from \( \Omega_2 \).

Let us suppose that the considered piezoelectric body is grounded and mechanically clamped over the part \( S_1 \) of the boundary \( \partial \Omega \), whereas the remaining part \( S_2 \) of the boundary \( \partial \Omega \) is mechanically free and electrically isolated. A metal foil is located at the cut at \( S \) and connected to the electric source with a known potential. Then the corresponding boundary value problem can be formulated as follows.

We look for a vector-function
\[
\mathbf{u} = (u_1, u_2, u_3, u_4)^T : \Omega \rightarrow \mathbb{R}^4
\]
belonging to the space \([W^1_p(\Omega)]^4\) and satisfying the conditions
\[
\begin{align*}
A(D\mathbf{u}) &= 0 \quad \text{in} \quad \Omega \setminus S, \\
rs_i\{[T(D\mathbf{u}, n)\mathbf{u}]_j^\pm\} &= \varphi, \quad j = 1, 2, 3; \quad \text{on} \quad S, \\
rS_i\{u_i\}^\pm &= 0, \quad i = 1, 2, \quad \text{on} \quad S_1, \\
rS_2\{[T(D\mathbf{u}, n)\mathbf{u}]_j^+\} &= 0, \quad \text{on} \quad S_2,
\end{align*}
\]
where \( \{f\}^+ \) denotes the trace of \( f \) on \( \partial \Omega_2 \) from \( \Omega_2 \) and \( \{f\}^- \) denotes the trace of the function \( f \) on \( \partial \Omega_1 \) from \( \Omega_1 \).

Using the potential theory and the general theory of pseudodifferential equations on manifolds with boundary we have proved the existence and uniqueness of a solution of the considered problem.

Employing an asymptotic expansion of solutions of strongly elliptic pseudodifferential equations and an asymptotic expansion of potential-type functions obtained in \([2, 3]\), for sufficiently smooth data of the problem we have obtained an explicit asymptotic expansion of the solution near cut’s edge \( \partial S \) as well as near the curve \( \partial S_1 \), where the type of the boundary conditions changes. It turned out that the exponent of the first term of an asymptotic expansion of solution near the cut’s edge \( \partial S \) is \( \frac{1}{2} \).

We have found an important class of anisotropic piezoelectric media for which the oscillation of solutions vanishes near the curve \( \partial S_1 \), at least the first three terms of the asymptotic do not contain logarithms and the singularity of solutions is calculated by a simple formula.

Denote by \( V_0(h) \) the single layer potential:
\[
V_0(h)(x) = \int_{\partial \Omega} T(\partial \mathbf{s}, n(x))H(x - y)h(y)dy, \quad x \in \Omega,
\]
where \( H(x) \) is the fundamental matrix-function of the differential operator \( A(\partial \mathbf{s}) \), and let \( \beta_k, k = 1, 4 \), be the eigenvalues of the matrix \( \sigma_{V_0}(x_1, 0, +1) \), where \( \sigma_{V_0}(x_1, \xi) \) is the principal homogeneous symbol of the operator \( V_0 \).

In the case where the conditions
\[
\beta_{1, 2} = \pm \alpha(x_1), \quad \beta_{3, 4} = \pm ib(x_1) \quad (b(x_1) > 0, \ x_1 \in \partial S_1)
\]
are fulfilled, solutions in the neighborhood of the curve \( \partial S_1 \) possess the following properties:

1) The solutions near the \( \partial S \) admit the asymptotic expansion
\[
c_1 \rho^{\gamma_1} + c_2 \rho^{\frac{1}{2} \pm i\theta} + c_3 \rho^{\gamma_2} + \cdots,
\]
where
\[
\gamma_1 = \frac{1}{2} - \frac{1}{\pi} \arctg 2b, \quad \gamma_2 = \frac{1}{2} + \frac{1}{\pi} \arctg 2b.
\]
The singularity of the solutions is characterized by the quantity

$$\gamma_1 = \frac{1}{2} - \frac{1}{\pi} \sup_{\partial S_1} \arctg 2b,$$

which depends on the elastic, piezoelectric and dielectric constants and on the geometry of $\partial S_1$ and can take any value from the interval $(0; 1/2)$.

2) Since $\gamma_1 < \frac{1}{2}$, the oscillation vanishes in some neighborhood of $\partial S_1$.

The class of piezoelectric media for which the conditions (6) are fulfilled is not empty (for example $TeO_2$ belongs to this class). On the other hand, for media that do not possess piezoelectric properties (i.e. are pure elastic) we have $\gamma_1 = \frac{1}{2}$ and the solutions oscillate. Thus, in the case of piezoelectricity we reveal effects which are not observed in the case of classical elasticity.

For some class of piezoelectric media the singularity of solution in the neighborhood of $\partial S_1$ can also reach the value $\frac{1}{2}$. For example, such property is possessed by piezoelectric media with cubic anisotropy.

Finally note that the boundary value problems of piezoelectricity of different type in domains with cuts were considered in [4].

References


Authors’ address:
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mails: t_buchukuri@yahoo.com
chkadua@rmi.acnet.ge