We consider the linear system
\[ \dot{x} = A(t)x, \quad x \in \mathbb{R}^3, \quad t \geq 0, \]
with a continuous piecewise differentiable skew-symmetric matrix \( A(t) = (a_{ij})_{i,j=1}^{3} \) for all \( t \geq 0 \). Such systems coincide with kinematic equations of the rigid body mechanics, in particular, they are applied in robotics \([1]\) in modelling automatized production based on automatic holonomic systems for parametric construction of programmed motions of executive devices in a three dimensional physical space. Four-dimensional systems with skew-symmetric coefficient matrix are also applied in the gyroscope theory \([2]\).

Following \([3,4]\), for the elements \( a_{ij}(t) \) of the skew-symmetric matrix \( A(t) \) we define: the function vector
\[ a(t) \equiv (a_{23}(t), -a_{13}(t), a_{12}(t)) \in \mathbb{R}^3, \quad t \geq 0, \]
the scalar functions
\[ C(\eta) \equiv \cos \int_0^\eta \|a(\tau)\|d\tau, \quad S(\eta) \equiv \sin \int_0^\eta \|a(\tau)\|d\tau, \quad t \geq 0, \]
and the vector function of two-variables
\[ v(t, \eta) \equiv \begin{pmatrix} -a_{12}(t) + a_{13}(t)C(\eta) \\ -a_{12}(t)a_{23}(t)C(\eta) + a_{11}(t)a(t)S(\eta) \\ a_{12}(t)a_{23}(t)C(\eta) + a_{13}(t)a(t)S(\eta) \end{pmatrix}, \quad t, \eta \in [0, +\infty). \]

In the above-mentioned works, for the quasi-integrals
\[ L_1(x(t), t) \equiv (x(t), a(t)) - (x(0), a(0)), \]
\[ L_2(x(t), t) \equiv (x(t), v(t, t)) - (x(0), v(0, 0)), \]

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of the non-stationary system \((1_A)\) on its solutions \(x(\cdot) : [0, +\infty) \to \mathbb{R}^3\), which are ordinary integrals in the stationary case and identically vanish, the estimates

\[
|L_1(x(t), t)| \leq \|x(0)\| \int_0^t \|\dot{a}(\tau)\| d\tau, \ t \geq 0, \quad (21)
\]

\[
|L_2(x(t), t)| \leq c_2 \|x(0)\| \int_0^t \|a(\tau)\| \|\dot{a}(\tau)\| d\tau, \ t \geq 0, \quad (22)
\]

are obtained with the constant \(c_2 = 2\sqrt{3}\). In those papers it is also proved that the first estimate may turn into equality (be efficient), whereas for the second one this was established for \(c_2 = 1\).

The authors of the present paper improved the estimate \((22)\) up to the one with \(c_2 = 2\) and proved that the latter is unimprovable. It should be noted that the efficiency of both estimates (the estimate \((21)\) and the estimate \((22)\) with the constant \(c_2 = 2\)) is realized for different three-dimensional systems \((1_A)\). In this connection, we have the following two problems on the simultaneous efficiency of the estimates \((21)\) and \((22)\) with \(c_2 = 2\): 1) efficiency of these estimates for the common system \((1_A)\) but, probably, for its different its solutions; 2) simultaneous efficiency of both estimates for one nontrivial solution \(x(t)\) of the same system \((1_A)\).

The aim of this paper is to prove that the estimates \((21)\) and \((22)\) with \(c_2 = 2\) cannot be efficient simultaneously for one nontrivial solution \(x(t)\) of the system \((1_A)\) at the same moment of time \(t = t_0 > 0\) such that \(\dot{a}(\tau) \neq 0\) for \(\tau \in [0, t_0]\).

The following theorem establishes this.

**Theorem.** Let \(x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^3 \setminus \{0\}\) be an arbitrary solution of any three-dimensional system \((1_A)\), and \(h\) be a fixed constant, \(h \in (0.9; 1]\). If for some \(t_0 > 0\) the estimate

\[
|L_1(x(t_0), t_0)| \geq h\|x(0)\| \int_0^{t_0} \|\dot{a}(\tau)\| d\tau 
\]

is fulfilled, then the inequality

\[
|L_2(x(t_0), t_0)| \leq 2 \left[1 - (2 - \sqrt{2}) \frac{(h - 0.8)(h - 0.9)}{2 + h}\right] \|x(0)\| \int_0^{t_0} \|a(\tau)\| \|\dot{a}(\tau)\| d\tau
\]

is valid.

**Proof.** The statement of the theorem is evident if \(\dot{a}(\tau) \equiv 0\) for all \(\tau \in [0, t_0]\). Thus let us consider the opposite case. We introduce the vectors

\[
e_1 := (1, 0, 0) \in \mathbb{R}^3, \quad w(t) := e_1 \times a(t), \quad f(t) := \|a(t)\| e_1, \ t \geq 0.
\]
Then the vector function $v(t, \eta)$ satisfies the equality

$$v(t, \eta) = (a(t) \times w(t))C(\eta) - \|a(t)\|w(t)S(\eta), \ t, \eta \geq 0. \quad (5)$$

According to Lemma 2 in [1], the equality

$$L_2(x(t), t) = \int_0^t (\sigma(\tau, t), \frac{\partial v(\tau, t)}{\partial t}) \, d\tau, \ t \geq 0, \quad (6)$$

is valid. We now estimate the absolute value of the scalar product under the integral sign in (6) by using the inequality $\|a(\tau)^\prime\| \leq \|\dot{a}(\tau)\|$ (see [3]) and the pairwise orthogonality of the vectors $f(\tau)$ and $w(\tau)$, as well as $e_1$ and $e_1 \times \dot{a}(\tau)$:

$$0 = \left| \left( x(\tau), \left\{ C(t)w(\tau) \times a(\tau) \right\} \right) \right|$$

$$= \left| \left( x(\tau), \left\{ [C(t)w(\tau) - S(t)f(\tau)] \times a(\tau) \right\} \right) \right|$$

$$\leq \left| \left( x(\tau), \left\{ [C(t)w(\tau) - S(t)f(\tau)]_\tau \times a(\tau) \right\} \right) \right| +$$

$$+ \left| \left( x(\tau), \left\{ [C(t)w(\tau) - S(t)f(\tau)] \times \dot{a}(\tau) \right\} \right) \right|$$

$$\leq \|x(0)\| \|a(\tau)\| \|C(t)(e_1 \times \dot{a}(\tau)) - S(t)a(\tau)^\prime e_1\| +$$

$$+ \|x(\tau) \times \dot{a}(\tau)\| \left\{ C^2(t)\|w(\tau)\|^2 + S^2(t)\|f(\tau)\|^2 \right\}^{1/2} \leq$$

(here the use is made of the equality $\|f(\tau)\| = \|a(\tau)\|$ and the estimate $\|w(\tau)\| \leq \|a(\tau)\|$ for all $\tau \geq 0$)

$$\leq \|x(0)\| \|a(\tau)\| \left[ \|\dot{a}(\tau)\| + \left\| \frac{x(\tau)}{\|x(\tau)\|} \times \dot{a}(\tau)\right\| \right], \ 0 \leq \tau \leq t.$$  

Thus, by virtue of the above inequality, from the equality (6) we obtain the following estimate for all $t \geq 0$:

$$|L_2(x(t), t)| \leq \|x(0)\| \int_0^t \|a(\tau)\| \left[ \|\dot{a}(\tau)\| + \left\| \frac{x(\tau)}{\|x(\tau)\|} \times \dot{a}(\tau)\right\| \right] \, d\tau. \quad (7)$$

Suppose now that the estimate (3) is fulfilled for some $t = t_0 > 0$. Let

$$s(\tau) := \sin \angle \{x(\tau), \dot{a}(\tau)\}, \ c(\tau) := \sqrt{1 - s^2(\tau)}, \ I_0 := \int_0^{t_0} \|\dot{a}(\tau)\| \, d\tau.$$

Define also the set $T_0 := \{ \tau \in [0, t_0] : \ s(\tau) \leq 1/\sqrt{2} \}$ and its complement $CT_0 := [0, t_0] \setminus T_0$ in $[0, t_0]$. Since every solution of the system (1.4) satisfies for all $t \geq 0$ the equality $\|x(\tau)\| \equiv \|x(0)\|$, without loss of generality we can
assume that in the estimates (3) and (4) the equality \(|\|x(\tau)\| \equiv 1, \tau \geq 0,\) is identically fulfilled.

Lemma 2 in [3] implies the estimates

\[
\begin{align*}
\frac{h}{4}I_0 & \leq \left| L_1(x(t_0), t_0) \right| = \left| \int_0^{t_0} (x(\tau), \dot{a}(\tau)) \, d\tau \right| \leq \\
& \leq \int_0^{t_0} \left| (x(\tau), \dot{a}(\tau)) \right| \, d\tau \leq \int_0^{t_0} \|\dot{a}(\tau)\| \left| \cos \angle (x(\tau), \dot{a}(\tau)) \right| \, d\tau \leq \\
& \leq \int_0^{t_0} \|\dot{a}(\tau)\| \, d\tau + \int_{CT_0} \|\dot{a}(\tau)\| |c(\tau)| \, d\tau \leq \\
& \leq \max_{\tau \in CT_0} \{1 - 2^{-1}\sin^2 \alpha\} \int_0^{t_0} \|\dot{a}(\tau)\| \, d\tau + \int_{CT_0} \|\dot{a}(\tau)\| \, d\tau.
\end{align*}
\]

(we now use the evident equality \(|\cos \alpha| \leq 1 - 2^{-1}\sin^2 \alpha\))

\[
\frac{h}{4}I_0 \leq I_0 - \frac{1}{4} \int_{CT_0} \|\dot{a}(\tau)\| \, d\tau,
\]

whence \( \int_{CT_0} \|\dot{a}(\tau)\| \, d\tau \leq 4(1 - h)I_0. \) The last estimate yields

\[
\int_{CT_0} \|\dot{a}(\tau)\| \, d\tau \geq (4h - 3)I_0. \tag{8}
\]

Consider now the case \(\|a(t_0)\| \geq \|a(0)\|\) (the opposite case can be treated analogously). Under this assumption, the estimate (3) implies that

\[
\|a(t_0)\| = \max \left\{ \|a(t_0)\|, \|a(0)\| \right\} \geq \max \left\{ \|a(t_0)\|, \|a(0)\| \right\} \geq \\
\geq 2^{-1} \left| (x(t_0), a(t_0)) - (x(0), a(0)) \right| \geq hI_0/2. \tag{9}
\]

Next we define the set

\[
T \equiv \left\{ t \in [0, t_0] : \int_t^{t_0} \|\dot{a}(\tau)\| \, d\tau \leq 0.4I_0 \right\}
\]

for which the equality \( \int_T \|\dot{a}(\tau)\| \, d\tau = 0.4I_0 \) is obviously fulfilled. Using the estimate (8), we obtain the inequalities

\[
I_0 \geq \int_{T_0 \cap T} \|\dot{a}(\tau)\| \, d\tau = \int_{T_0} \ldots \, d\tau + \int_{T} \ldots \, d\tau - \int_{T_0 \cap T} \ldots \, d\tau \geq
\]
\[ \geq (4h - 2.6)I_0 - \int_{T_0 \cap T} \| \dot{a} (\tau) \| \, d\tau. \]

These inequalities result in the estimate \( \int_{T_0 \cap T} \| \dot{a} (\tau) \| \, d\tau \geq (4h - 3.6)I_0. \)

Moreover, the inequality
\[
\min_{t \in T} \| a(t) \| \geq \| a(t_0) \| - \max_{t \in T} \int_{t_0}^t \| \dot{a} (\tau) \| \, d\tau = \| a(t_0) \| - 0.4I_0
\]
implies the estimates
\[
\int_{T_0} \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau \geq \min_{\tau \in T} \| a(\tau) \| \int_{T_0 \cap T} \| \dot{a} (\tau) \| \, d\tau \geq (4h - 3.6)(\| a(t_0) \| - 0.4I_0)I_0.
\]

Thus, by virtue of the inequality (11), the following estimates are valid:
\[
J_0 \equiv \int_{0}^{t_0} s(\tau) \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau \leq \\
\leq \int_{\mathcal{C}T_0} \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau + \max_{\tau \in T_0} \int_{T_0} \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau \leq \\
\leq \int_{0}^{t_0} \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau - \frac{\sqrt{2} - 1}{\sqrt{2}} \int_{T_0} \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau \leq \\
\leq \int_{0}^{t_0} \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau - 2(2 - \sqrt{2})(h - 0.9)(\| a(t_0) \| - 0.4I_0)I_0. \quad (10)
\]

Moreover, the inequalities
\[
J_1 \equiv \int_{0}^{t_0} \| a(\tau) \| \| \dot{a} (\tau) \| \, d\tau \leq \\
\leq \max_{\tau \in [0, t_0]} \| a(\tau) \| \int_{t}^{t_0} \| \dot{a} (\tau) \| \, d\tau \leq (\| a(t_0) \| + I_0)I_0 \quad (11)
\]
are also fulfilled. Obviously, the inequality (9) is equivalent to the estimate
\[
\| a(t_0) \| - 0.4I_0 \geq b(\| a(t_0) \| + I_0),
\]
where \( b \equiv (h - 0.8)/(2 + h). \)

Using (7), (10) and (11), we get the relations
\[
|L_2(x(t_0), t_0)| \leq (J_1 + J_0) \leq
\]
\[ \leq 2J_1 - 2(2 - \sqrt{2})(h - 0.9)(\|a(t_0)\| - 0.4J_0)I_0 \leq \]
\[ \leq 2J_1 - 2b(2 - \sqrt{2})(h - 0.9)(\|a(t_0)\| + I_0)I_0 \leq \]
\[ \leq 2\left[1 - (2 - \sqrt{2}) \frac{(h - 0.8)(h - 0.9)}{2 + h}\right]J_1. \]

By virtue of Lemma 2 in [3], the latter inequalities imply the desired inequality (4).

Thus the theorem is proved. \(\square\)

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References


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