Short Communications

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ON SOLVABILITY AND WELL-POSEDNESS OF INITIAL–BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER NONLINEAR HYPERBOLIC EQUATIONS

Abstract. The sufficient conditions for unique local solvability, global solvability and of well-posedness of initial-boundary value problems for higher order nonlinear hyperbolic equations are studied.

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Let $b > 0$, $I$ be a compact interval containing zero, $\Omega = I \times [0, b]$, $m$ and $n$ be natural numbers and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ be a continuous function. In the rectangle $\Omega$ consider the nonlinear hyperbolic equation

$$u^{(m,n)} = f(x, y, u^{(m,0)}, \ldots, u^{(m,n-1)}, u^{(0,n)}, \ldots, u^{(m-1,n)}, \mathcal{D}^{m-1,n-1}[u])$$  \hspace{1cm}(1)$$

with the initial–boundary conditions

$$u^{(j,0)}(0, y) = \varphi_j(y) \quad (j = 0, \ldots, m - 1),$$

$$h_k(u^{(m,0)}(x, \cdot))(x) = \psi_k(x) \quad (k = 1, \ldots, n).$$  \hspace{1cm}(2)$$

Here for any $j$ and $k$

$$u^{(j,k)}(x, y) = \frac{\partial^{j+k}u(x, y)}{\partial x^j \partial y^k}, \quad \mathcal{D}^{m-1,n-1}[u](x, y) = \left(u^{(j-1,k-1)}(x, y)\right)_{1,1}^{m,n},$$

$\varphi_j \in C^n([0, b])$, $\psi_k \in C(I)$ and $h_k : C^{m-1}([0, b]) \to C(I)$ is a linear bounded operator.

The linear case of problem (1),(2), i.e., the linear hyperbolic equation

\[ u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} \sum_{k=0}^{n} p_{jk}(x,y)u^{(j,k)} + q(x,y) \quad (3) \]

with conditions (2) is studied in [3] and [4]. In [3] necessary and sufficient conditions of well-posedness and so-called \( \mu \)-well-posedness of problem (3),(2) are established. In [4] a complete description of problem (3),(2) in the ill-posed case is given.

For the history of the matter see [2–5] and the references quoted therein. The general initial-boundary value problem (1),(2) has been little investigated. Namely this problem is investigated in the present paper.

Throughout the paper we will use the following notations.

\( \mathbb{R} \) is the set of real numbers; \( \mathbb{R}^{m \times n} \) is the space of real \( m \times n \) matrices

\[ Z = (z_{ij})_{1,1}^{m,n} = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{m1} & \cdots & z_{mn} \end{pmatrix} \]

with the norm \( \|Z\| = \sum_{i=1}^{m} \sum_{j=1}^{n} |z_{ij}| \).

\( C(I) \) and \( C(\Omega) \), respectively, are the Banach spaces of continuous functions \( z : I \to \mathbb{R} \) and \( u : \Omega \to \mathbb{R} \), with the norms

\[ \|z\|_{C(I)} = \max\{|z(x)| : x \in I\}, \quad \|u\|_{C(\Omega)} = \max\{|u(x,y)| : (x,y) \in \Omega\}. \]

\( C(I; \mathbb{R}^{m \times n}) \) is the Banach space of continuous matrix functions \( Z : I \to \mathbb{R}^{m \times n} \) with the norm \( \|Z\|_{C(I; \mathbb{R}^{m \times n})} = \max\{|Z(x)| : x \in I\} \).

\( C^k(I) \) is the Banach space of \( k \)-times continuously differentiable functions \( z : I \to \mathbb{R} \), with the norm

\[ \|z\|_{C^k(I)} = \sum_{i=0}^{k} \|z^{(i)}\|_{C(I)}. \]

\( C^{m,n}(\Omega) \) is the Banach space of functions \( u : \Omega \to \mathbb{R} \), having continuous partial derivatives \( u^{(j,k)} \) \( (j = 0, \ldots, m; \ k = 0, \ldots, n) \), with the norm

\[ \|u\|_{C^{m,n}(\Omega)} = \sum_{j=0}^{m} \sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega)}. \]

\( \tilde{C}^{m,n}(\Omega) \) is the Banach space of functions \( u : \Omega \to \mathbb{R} \), having continuous partial derivatives \( u^{(j,k)} \) \( (j = 0, \ldots, m; \ k = 0, \ldots, n; \ j + k < m + n) \), with the norm

\[ \|u\|_{\tilde{C}^{m,n}(\Omega)} = \sum_{k=0}^{n-1} \|u^{(m,k)}\|_{C(\Omega)} + \sum_{j=0}^{m-1} \sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega)}. \]

If \( u \in \tilde{C}^{m,n}(\Omega) \) and \( r_0 > 0 \), then \( \tilde{B}^{m,n}(z; \Omega, r_0) = \{\zeta \in \tilde{C}^{m,n}(\Omega) : \|\zeta - z\|_{\tilde{C}^{m,n}} \leq r_0\} \).
It will be assumed that \((x, y, z_1, \ldots, z_{n+m}, Z) \rightarrow f(x, y, z_1, \ldots, z_{n+m}, Z)\) is continuous in \(\Omega \times \mathbb{R}^{n+m} \times \mathbb{R}^{m \times n}\) and \emph{continuously differentiable} with respect to \(z_1, \ldots, z_{n+m}\).

Let \(I_0 \subset I\) be an arbitrary (not necessarily compact) set containing zero. By a \emph{solution of problem} \((1),(2)\) in the rectangle \(\Omega_0 = I_0 \times [0, b]\) we understand a classical solution, i.e., a function \(u : \Omega_0 \rightarrow \mathbb{R}\) having the continuous partial derivatives \(u^{(i,k)}\) \((i = 0, \ldots, m; k = 0, \ldots, n)\) and satisfying \((1)\) and \((2)\) at every point of \(\Omega_0\).

**Definition 1.** A solution \(u\) of problem \((1),(2)\) defined on \(\Omega_0 = I_0 \times [0, b]\) is called \emph{continuable to the right} (to the left), if there exists an interval \(I_1 \supset I_0\) and a solution \(u_1\) of this problem in \(\Omega_1 = I_1 \times [0, b]\) such that \(\sup I_1 > \sup I_0\) (\(\inf I_1 < \inf I_0\)) and

\[
u_1(x, y) = u(x, y) \quad \text{for} \quad (x, y) \in \Omega_0.
\]

\(u\) is called \emph{non-continuable} if it is non-continuable both to the right and to the left.

**Definition 2.** A solution \(u\) of problem \((1),(2)\) defined on \(I_0 \times [0, b]\) is a called \emph{global solution} (local solution) if \(I_0 = I\) \((I_0 \neq I)\) is a compact interval such that \([-\varepsilon, \varepsilon] \cap I \subset I_0\) for any sufficiently small \(\varepsilon > 0\). Problem \((1),(2)\) is called globally solvable (locally solvable), if it has a global (local) solution.

Along with \((1),(2)\) consider the perturbed problem

\[
v^{(m,n)} = f(x, y, v^{(m,0)}, \ldots, v^{(m,n-1)}), v^{(0,n)}, \ldots, v^{(m-1,n)}, D^{m-1,n-1}[v] + q(x, y),
\]

\[
v^{(j,0)}(0, y) = \varphi_j(y) + \tilde{\varphi}_j(y) \quad (j = 0, \ldots, m - 1),
\]

\[
h_k(v^{(m,0)}(x, \cdot))(x) = \psi_k(x) + \tilde{\psi}_k(x) \quad (k = 1, \ldots, n).
\]

Let \(I_0 \subset I\) be a compact interval containing zero, \(u\) be a solution of problem \((1),(2)\) in \(\Omega_0 = I_0 \times [0, b]\), and let \(r_0\) be a positive constant. Introduce the following

**Definition 3.** Problem \((1),(2)\) is called \((u; r_0)\) \emph{well-posed} if there exist positive constants \(\delta\) and \(r\) such that for any \(\varphi_j \in C^n([0, b]) \ (j = 0, \ldots, m - 1)\), \(\tilde{\varphi}_j \in C(I) \ (k = 1, \ldots, n)\), and \(q \in C(\Omega_0)\) satisfying the inequality

\[
\sum_{j=0}^{m-1} \|\varphi_j\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\tilde{\varphi}_k\|_{C(I)} + \|q\|_{C(\Omega_0)} \leq \delta,
\]

problem \((4),(5)\) in the ball \(\overline{B}^{m,n}(u; \Omega_0, r_0)\) has a unique solution \(v\) and the inequality

\[
\|u - v\|_{C^{m,n}(J, [0, b])} \leq r \left(\sum_{j=0}^{m-1} \|\varphi_j\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\tilde{\varphi}_k\|_{C(I)} + \|q\|_{C(J \times [0, b])}\right)
\]

holds for every compact subinterval \(J \subset I_0\) containing zero.
**Definition 4.** Problem (1),(2) is called *well-posed* if there exist positive constants $\delta$ and $r$ such that for any $\tilde{\varphi}_j \in C^m([0,b])$ ($j = 0, \ldots, m-1$), $\tilde{\psi}_k \in C(I_0)$ ($k = 1, \ldots, n$), and $q \in C(\Omega_0)$ satisfying (6) problem (4),(5) has a unique solution $v$ in $\Omega$ and estimate (7) is valid for every compact subset $J \subset I$ containing zero.

The proposed method of investigation of problem (1),(2) is based on the theory of boundary value problems for ordinary differential equations (see, e.g. [1]). For the boundary value problem

$$z^{(n)} = p(y, z, \ldots, z^{(n-1)}); \quad l_k(z) = c_k \quad (k = 1, \ldots, n), \quad (8)$$

where $l_k : C^{n-1}([0,b]) \to \mathbb{R}$ ($k = 1, \ldots, n$) are linear bounded functionals and $p : [0,b] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function having continuous partial derivatives

$$p_k(y, z_1, \ldots, z_n) = \frac{\partial p(y, z_1, \ldots, z_n)}{\partial z_k} \quad (k = 1, \ldots, n),$$

we introduce a definition of a strongly isolated solution, which is a modification of the definition from [1].

**Definition 5.** A solution $z$ of problem (8) is called *strongly isolated* if the problem

$$\zeta^{(n)} = \sum_{j=1}^{n} p_j(y, z(y), \ldots, z^{(n-1)}(y))\zeta^{(j-1)}; \quad l_k(\zeta) = 0 \quad (k = 1, \ldots, n)$$

has only a trivial solution.

Set

$$\Phi(y) = (\varphi_{j-1}^{k-1}(y))_{1,1}^{m,n},$$

$$p_0(y, z_1, \ldots, z_n) = f(0, y, z_1, \ldots, z_n, \varphi_0^{(n)}(y), \ldots, \varphi_{m-1}^{(n)}(y), \Phi(y)), \quad (9)$$

$$p[u](x, y, z_1, \ldots, z_n) = f(x, y, z_1, \ldots, z_n, u^{(0,n)}(x,y), \ldots, u^{(m-1,n)}(x,y), \mathcal{D}^{n-1,n-1}[u](x,y)).$$

**Theorem 1.** Let $z_0$ be a strongly isolated solution of the problem

$$z^{(n)} = p_0(y, z, \ldots, z^{(n-1)}), \quad h_k(z)(0) = \psi_k(0) \quad (k = 1, \ldots, n). \quad (9)$$

Then problem (1), (2) has a local solution $u$ satisfying the condition

$$u^{(m,0)}(0,y) = z_0(y) \quad \text{for} \quad y \in [0,b].$$

Furthermore, if $f(x,y,z_1,\ldots,z_{n+m},Z)$ is locally Lipschitz continuous with respect to $Z$, then problem (1), (2) is $(u;r_0)$–well–posed for some sufficiently small $r_0 > 0$.  

Remark 1. In Theorem 1 the requirement of strong isolation of a solution z to problem (9) is essential and it cannot be replaced by the requirement of uniqueness of a solution. Indeed, consider the problem
\[ u^{(1,1)} = (u^{(1,0)})^2 + x^2; \quad u(0, y) = 0, \quad u^{(1,0)}(x, 0) = u^{(1,0)}(x, b), \] (10)
for which problem (9) has the form
\[ z' = z^2; \quad z(0) = z(b). \] (11)
It is clear that problem (10) has no solution. On the other hand problem (11) has only a trivial solution which is not strongly isolated.

Remark 2. Under the conditions of Theorem 1 problem (1),(2) may have an infinite set of solutions even for smooth \( f \). Indeed, consider the problem
\[ u^{(1,1)} = \sin(u^{(1,0)}) + x f_0(x, y, u^{(1,0)}, u^{(0,1)}, u), \]
\[ u(0, y) = 0, \quad u^{(1,0)}(x, 0) = u^{(1,0)}(x, b), \] (12)
where \( f_0 : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) is a continuously differentiable function. For (12) problem (9) has the form
\[ z' = \sin z; \quad z(0) = z(b). \]
The latter problem has a countable set of strongly isolated solutions \( z_k = k \pi \) \((k = 0, \pm 1, \ldots)\). By Theorem 1, for every integer \( k \) there exists positive \( \varepsilon_k \) such that in \( \Omega_k = I_k \times [0, b] \), where \( I_k = [-\varepsilon_k, \varepsilon_k] \cap I \), problem (12) has a unique solution \( u_k \) satisfying the condition
\[ u_k^{(1,0)}(0, y) = k \pi \quad \text{for} \quad y \in [0, b]. \]

**Theorem 2.** Let \( u \) be a a non-continuable solution of problem (1),(2) defined in \( \Omega_0 = I_0 \times [0, b] \). Furthermore, let for any \( x_0 \in I_0 \) the function \( z(y) = u^{(m,0)}(x_0, y) \) be a strongly isolated solution of the problem
\[ z^{(n)} = p[u](x_0, y, z, z', \ldots, z^{(n-1)}), \]
\[ h_k(z)(x_0) = \psi_k(x_0) \quad (k = 1, \ldots, n). \] (13)
Then \( I_0 \) is an open set in \( I \). Moreover, if \( a^* = \sup I_0 \not\in I_0 \), then
\[ \limsup_{x \to a^*} \left( \|u^{(m,0)}(x, \cdot)\|_{C^{n-1}([0, b])} + \sum_{j=0}^{m-1} \|u^{(j,0)}(x, \cdot)\|_{C^n([0, b])} \right) = +\infty, \] (14)
and if \( a_* = \inf I_0 \not\in I_0 \), then
\[ \liminf_{x \to a_*} \left( \|u^{(m,0)}(x, \cdot)\|_{C^{n-1}([0, b])} + \sum_{j=0}^{m-1} \|u^{(j,0)}(x, \cdot)\|_{C^n([0, b])} \right) = +\infty. \] (15)

Remark 3. In Theorem 2 the requirement of strong isolation of the solution \( z(y) = u^{(m,0)}(x_0, y) \) of problem (13) for every \( x_0 \in I_0 \) is essential and it cannot be weakened. As an example in the rectangle \([-2, 2] \times [0, b] \) consider the problem
\[ u^{(1,1)} = |u|u^{(1,0)} + u, \quad u(0, y) = 1, \quad u^{(1,0)}(x, 0) = u^{(1,0)}(x, b). \]
This problem has a non-continuable solution $u(x, y) = 1 - x$ defined on the set $[-2, 1] \times [0, b]$. Indeed, supposing that $u$ can be continued to the right, by continuity of $u$ and $u^{(1,0)}$ we will have

$$u^{(1,0)}(x, y) < 0, \quad u(x, y) < 0 \quad \text{for} \quad (x, y) \in (1, 1 + \delta] \times [0, b]$$

for some sufficiently small $\delta > 0$. Consequently

$$u^{(1,1)}(x, y) = |u(x, y)|u^{(1,0)}(x, y) + u(x, y) < 0 \quad \text{for} \quad (x, y) \in (1, 1 + \delta] \times [0, b].$$

But the latter inequality contradicts to the periodicity of $u^{(1,0)}$ with respect to the second argument. Consequently (14) does not hold for $u$. The reason for this is that problem (13) has the form

$$z' = |1 - x_0| z + 1 - x_0, \quad v(0) = v(b),$$

and $z(y) = -1$ is a strongly isolated solution of this problem for every $x_0 < 1$, but not for $x_0 = 1$.

**Definition 6.** We say that the function $f$ belongs to the set $S_{h_1,...,h_n}$ if there exist functions $p_{ik} \in C(\Omega)$ ($i = 1, 2; \ k = 1,\ldots, n$) such that:

(i) $p_{1i}(x, y) \leq f_i(x, y, z_1,\ldots, z_{n+m}, Z) \leq p_{2i}(x, y)$ for $(x, y) \in \Omega$ ($i = 1,\ldots, n$);

(ii) for any $x \in I$ and measurable functions $p_i : [0, b] \to \mathbb{R}$ ($i = 1,\ldots, n$) satisfying inequalities $p_{1i}(x, y) \leq p_i(y) \leq p_{2i}(x, y)$ for $(x, y) \in \Omega$ ($i = 1,\ldots, n$) the problem

$$\zeta^{(n)} = \sum_{j=1}^{n} f_j(y)\zeta^{(j-1)}; \quad h_k(\zeta)(x) = 0 \ (k = 1,\ldots, n)$$

has only a trivial solution.

**Theorem 3.** Let there exist a positive constant $l_0$ such that

$$f \in S_{h_1,...,h_n},$$

$$|f(x, y, z_1,\ldots, z_{n+m}, Z)| \leq l_0 \left(1 + \sum_{k=1}^{n+m} |z_k| + \|Z\|\right).$$

Then problem (1), (2) is globally solvable. Furthermore, if $f(x, y, z_1,\ldots, z_{n+m}, Z)$ is locally Lipschitz continuous with respect to $Z$, then problem (1), (2) is well–posed.

**Remark 4.** In Theorem 3 condition (16) is optimal and it cannot be weakened. Indeed, in the rectangle $[-\pi, \pi] \times [0, b]$ consider the problem

$$u^{(1,1)} = \arctan(u^{(1,0)}) - \arctan(1 + u^2);$$

$$u(0, y) = 0, \quad u^{(1,0)}(x, 0) = u^{(1,0)}(x, b),$$

$$u^{(1,0)}(x, y) < 0, \quad u(x, y) < 0 \quad \text{for} \quad (x, y) \in (1, 1 + \delta] \times [0, b]$$
for which condition (17) holds but condition (16) is violated. As a result problem (18) has a unique solution \( u(x, y) \equiv \tan(x) \), which cannot be continued outside the rectangle \((-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, b] \).

Below separately consider the case, where the righthand side of equation (1.1) does not contain the derivatives \( u^{(m,k)} \) \((k = 1, \ldots, n - 1)\), i.e., where equation (1.1) has the form

\[
 u^{(m,n)} = g(x, y, u^{(m,0)}, \ldots, u^{(m-1,n)}, D^{m-1,n-1}[u]),
\]

where \((x, y, z_1, \ldots, z_{m+1}, Z) \rightarrow g(x, y, z_1, \ldots, z_{m+1}, Z)\) is continuous in \(\Omega \times \mathbb{R}^{m+1} \times \mathbb{R}^{m \times n}\) and continuously differentiable with respect to \(z_1, \ldots, z_{m+1}\). We also assume that the function \(g\) is sublinear, i.e., for some constant \(l_0 > 0\) \(g\) satisfies the inequality

\[
 |g(x, y, z_1, \ldots, z_{m+1}, Z)| \leq l_0 \left( 1 + \sum_{k=1}^{m+1} |z_k| + ||Z|| \right)
\]
in \(\Omega \times \mathbb{R}^{m+1} \times \mathbb{R}^{m \times n}\).

Corollaries 1—3 concern the case, where (2) is either the initial–Dirichlet

\[
\begin{align*}
 u^{(j,0)}(0, y) &= \varphi_j(y) \quad (j = 0, \ldots, m - 1), \\
 u^{(m,i-1)}(x, y_1(x)) &= \psi_{1i}(x) \quad (i = 1, \ldots, n^*), \\
 u^{(m,k-1)}(x, y_2(x)) &= \psi_{2k}(x) \quad (k = 1, \ldots, n - n^*),
\end{align*}
\]

or the initial–periodic conditions

\[
\begin{align*}
 u^{(j,0)}(0, y) &= \varphi_j(y) \quad (j = 0, \ldots, m - 1), \\
 u^{(m,k-1)}(x, y_1(x)) &= u^{(m,k-1)}(x, y_2(x)) + \psi_k(x) \quad (k = 1, \ldots, n),
\end{align*}
\]

where \(n^*\) is the integer part of \(n/2\), \(\varphi_j \in C^n([0, b])\), \(\psi_k \in C(I)\), \(\psi_{1k}, \psi_{2k} \in C(I)\), \(y_1, y_2 \in C(I)\), \(0 \leq y_1(x) < y_2(x) \leq b\) for \(x \in I\).

**Corollary 1.** Let there exist a nonnegative function \(p_0 \in C(\Omega)\) and a positive number \(\varepsilon\) such the condition

\[
-p_0(x, y) \leq (-1)^{n-n^*} (y - y_1(x))^{n-2n^*} g_{z_1}(x, y, z_1, \ldots, z_{m+1}, Z) \leq \\
\leq \frac{\alpha_n - \varepsilon}{4} \left( \frac{2\pi}{y_2(x) - y_1(x)} \right)^{2n^*}
\]

holds in \(\Omega \times \mathbb{R}^{m+1} \times \mathbb{R}^{m \times n}\), where \(\alpha_n = 1\) for \(n = 2n^*\) and \(\alpha_n = n/2\) for \(n = 2n^* + 1\). Then problem (19), (20) is globally solvable. Furthermore, if \(f(x, y, z_1, \ldots, z_{m+1}, Z)\) is locally Lipschitz continuous with respect to \(Z\), then problem (19), (20) is well–posed.

**Corollary 2.** Let there exist nonnegative functions \(p_i \in C(\Omega)\) \((i = 0, 1)\) such that

\[
\int_{y_1(x)}^{y_2(x)} p_1(x, y) \, dy > 0 \quad \text{for } x \in I,
\]
and the condition
\[-p_0(x,y) \leq \sigma g_{z_1}(x,y,z_1,\ldots,z_{m+1},Z) \leq -p_1(x,y),\]
holds in $\Omega \times \mathbb{R}^{m+1} \times \mathbb{R}^{m \times n}$, where
\[
\sigma = (-1)^n \quad \text{for} \quad n = 2n^*, \quad \text{and} \quad \sigma \in \{-1,1\} \quad \text{for} \quad n = 2n^* + 1.
\]
Then problem (19), (21) is globally solvable. Furthermore, if $g(x,y,z_1,\ldots,z_{m+1},Z)$ is locally Lipschitz continuous with respect to $Z$, then problem (19), (21) is well-posed.

**Corollary 3.** Let $n = 2n^*$, and let there exist a positive number $\varepsilon$ and a nonnegative function $p_1 \in C(\Omega)$ satisfying inequality (1.41) such the condition
\[
p_1(x,y) \leq (-1)^n g_{z_1}(x,y,z_1,\ldots,z_{m+1},Z) \leq \left(\frac{2\pi - \varepsilon}{g_2(x) - y_1(x)}\right)^n,
\]
holds in $\Omega \times \mathbb{R}^{m+1} \times \mathbb{R}^{m \times n}$. Then problem (19), (21) is globally solvable. Furthermore, if $g(x,y,z_1,\ldots,z_{m+1},Z)$ is locally Lipschitz continuous with respect to $Z$, then problem (19), (21) is well-posed.

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