ON THE SOLVABILITY OF A MULTIPPOINT BOUNDARY VALUE PROBLEM FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Necessary and sufficient conditions and effective sufficient conditions are given for the existence of solutions of the multipoint boundary value problem for a system of nonlinear generalized ordinary differential equations.

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Let \( \sigma_1, \ldots, \sigma_n \in \{-1, 1\} \); for \( m \in \{1, 2\} \) and \( i, k \in \{1, \ldots, n\} \), \( a_{mik} : [-a, a] \rightarrow \mathbb{R} \) be nondecreasing functions continuous at the points \(-a\) and \(a\);

\[
a_{ik}(t) = a_{1ik}(t) - a_{2ik}(t),
\]

\( A = (a_{ik})_{i,k=1}^n, \quad A_m = (a_{mik})_{i,k=1}^n \) \((m = 1, 2)\);

\( f = (f_k)_{k=1}^n : [-a, a] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a vector-function belonging to the Carathéodory class corresponding to the matrix-function \( A \), and \( \varphi_i : \text{BV}_i([-a, a], \mathbb{R}^n) \rightarrow \mathbb{R} \) \((i = 1, \ldots, n)\) be continuous functionals which are nonlinear in general.

For the system of generalized ordinary differential equations

\[
dx(t) = dA(t) \cdot f(t, x(t)),
\]

where \( x = (x_i)_{i=1}^n \), consider the multipoint boundary value problem

\[
x_i(-\sigma_i a) = \varphi_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n).
\]

The theory of generalized ordinary differential equations enables one to investigate ordinary differential, impulsive and difference equations from a common point of view (see [5]–[16]).

Throughout the paper the following notation and definitions will be used. Throughout the paper the following notation and definitions will be used.

\[ R = (-\infty, +\infty], \quad R_+ = [0, +\infty]; \quad [a, b] (a, b \in \mathbb{R}) \text{ is a closed segment.} \]

\[ R^{n \times m} \text{ is the space of all real } n \times m \text{-matrices } X = (x_{ij})_{i,j=1}^{n,m} \text{ with the norm} \]

\[ \|X\| = \max_{j=1,\ldots,m} \sum_{i=1}^{n} |x_{ij}|; \]

\[ R^n = R^{n \times 1} \text{ is the space of all real column } n \text{-vectors } x = (x_i)_{i=1}^{n}; \]

\[ R^n_+ = R^{n \times 1}_+. \]

\[ \text{diag}(\lambda_1, \ldots, \lambda_n) \text{ is the diagonal matrix with diagonal elements } \lambda_1, \ldots, \lambda_n; \]

\[ \delta_{ij} \text{ is the Kronecker symbol, i.e., } \delta_{ij} = 1 \text{ if } i = j \text{ and } \delta_{ij} = 0 \text{ if } i \neq j \text{ (} i, j = 1, \ldots, n). \]

\[ \nabla(X) \text{ is the total variation of the matrix-function } X : [a, b] \to R^{n \times m}, \]

\[ \text{i.e., the sum of total variations of the latter's components.} \]

\[ X(t-) \text{ and } X(t+) \text{ are the left and the right limits of the matrix-function} \]

\[ X : [a, b] \to R^{n \times m} \text{ at the point } t \text{ (we will assume } X(t) = X(a) \text{ for } t \leq a \text{ and } X(t) = X(b) \text{ for } t \geq b, \text{ if necessary}); \]

\[ d_1 X(t) = X(t) - X(t-), \quad d_2 X(t) = X(t+) - X(t); \]

\[ \|X\|_* = \sup \{\|X(t)\| : t \in [a, b]\}. \]

\[ BV([a, b], R^{n \times m}) \text{ is the set of all matrix-functions of bounded variation} \]

\[ X : [a, b] \to R^{n \times m} (\text{i.e., such that } \nabla(X) < +\infty); \]

\[ BV_*([a, b], R^n) \text{ is the normed space } (BV([a, b], R^n), \| \cdot \|_*); \]

\[ \text{If } B_1 \text{ and } B_2 \text{ are normed spaces, then an operator } g : B_1 \to B_2 \text{ (nonlinear, in general) is positive homogeneous if} \]

\[ g(\lambda x) = \lambda g(x) \]

for every } \lambda \in R_+ \text{ and } x \in B_1.\]

An operator } \varphi : BV([a, b], R^n) \to R^n \text{ is called nondecreasing if for every }\]

\[ x, y \in BV([a, b], R^n) \text{ such that } x(t) \leq y(t) \text{ for } t \in [a, b] \text{ the inequality } \varphi(x)(t) \leq \varphi(y)(t) \text{ holds for } t \in [a, b]. \]
\[ s_j : \text{BV}([a, b], R) \to \text{BV}([a, b], R) \quad (j = 0, 1, 2) \]
are the operators defined, respectively, by
\[ s_1(x)(a) = s_2(x)(a) = 0, \]
\[ s_1(x)(t) = \sum_{a < \tau \leq t} d_1 x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a \leq \tau < t} d_2 x(\tau) \quad \text{for} \quad a < t \leq b, \]
and
\[ s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \quad \text{for} \quad t \in [a, b]. \]

If \( g : [a, b] \to R \) is a nondecreasing function, \( x : [a, b] \to R \) and \( a \leq s < t \leq b \), then
\[ \int_s^t x(\tau) \, dg(\tau) = \int_s^t x(\tau) \, dS_0(g)(\tau) + \sum_{s < \tau \leq t} x(\tau) d_1 g(\tau) + \sum_{s \leq \tau < t} x(\tau) d_2 g(\tau), \]
where \( \int_s^t x(\tau) \, dS_0(g)(\tau) \) is the Lebesgue–Stieltjes integral over the open interval \([s, t] \) with respect to the measure \( \mu_0(s_0(g)) \) corresponding to the function \( S_0(g) \).

If \( a = b \), then we assume
\[ \int_a^b x(t) \, dg(t) = 0, \]
and if \( a > b \), then we assume
\[ \int_a^b x(t) \, dg(t) = -\int_b^a x(t) \, dg(t). \]

If \( g(t) \equiv g_1(t) - g_2(t) \), where \( g_1 \) and \( g_2 \) are nondecreasing functions, then
\[ \int_s^t x(\tau) \, dg(\tau) = \int_s^t x(\tau) \, dg_1(\tau) - \int_s^t x(\tau) \, dg_2(\tau) \quad \text{for} \quad s \leq t. \]

\( L([a, b], R; g) \) is the set of all functions \( x : [a, b] \to R \) measurable and integrable with respect to the measures \( \mu_0(g_i) \) \((i = 1, 2)\), i.e. such that
\[ \int_a^b |x(t)| \, dg_i(t) < +\infty \quad (i = 1, 2). \]

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If \( G = (g_{ik})_{i,k=1}^{l,n} : [a, b] \to R^{l \times n} \) is a nondecreasing matrix-function and \( D \subset R^{n \times m} \), then \( L([a, b], D; G) \) is the set of all matrix-functions \( X = \)
(x_{kj})^{n,m}_{k,j=1} : [a, b] \to D \text{ such that } x_{kj} \in L([a, b], R; g_{ik}) \text{ (} i = 1, \ldots, l; k = 1, \ldots, n; j = 1, \ldots, m\text{);} \\
\int_s^t dG(\tau) \cdot X(\tau) = \left( \sum_{k=1}^n \int_s^t x_{kj}(\tau)dg_{ik}(\tau) \right)^{i,m}_{i,j=1} \text{ for } a \leq s \leq t \leq b, \\
S_j(G)(t) \equiv (s_j(g_{ik})(t))^{i,n}_{i,k=1} \text{ (} j = 0, 1, 2\text{).} \\
\text{If } D_1 \subset R^n \text{ and } D_2 \subset R^{n \times m}, \text{ then } K([a, b] \times D_1, D_2; G) \text{ is the Carathéodory class, i.e., the set of all mappings } F = (f_{kj})^{n,m}_{k,j=1} : [a, b] \times D_1 \to D_2 \text{ such that for each } i \in \{1, \ldots, l\}, j \in \{1, \ldots, m\} \text{ and } k \in \{1, \ldots, n\}: \\
a) \text{ the function } f_{kj}(\cdot, x) : [a, b] \to D_2 \text{ is } \mu(g_{ik})\text{-measurable for every } x \in D_1; \\
b) \text{ the function } f_{kj}(t, \cdot) : D_1 \to D_2 \text{ is continuous for } \mu(g_{ik})\text{-almost every } t \in [a, b], \text{ and} \\
\sup \{|f_{kj}(\cdot, x)| : x \in D_0\} \in L([a, b], R; g_{ik})\text{ for every compact } D_0 \subset D_1, \\
\text{If } G_j : [a, b] \to R^{l \times n} \text{ (} j = 1, 2\text{) are nondecreasing matrix-functions, } G = G_1 - G_2 \text{ and } X : [a, b] \to R^{n \times m}, \text{ then} \\
\int_s^t dG(\tau) \cdot X(\tau) = \int_s^t dG_1(\tau) \cdot X(\tau) - \int_s^t dG_2(\tau) \cdot X(\tau) \text{ for } s \leq t, \\
S_k(G) = S_k(G_1) - S_k(G_2) \text{ (} k = 0, 1, 2\text{),} \\
L([a, b], D; G) = \bigcap_{j=1}^2 L([a, b], D; G_j), \\
K([a, b] \times D_1, D_2; G) = \bigcap_{j=1}^2 K([a, b] \times D_1, D_2; G_j). \\
\text{If } G(t) \equiv \text{diag}(t, \ldots, t), \text{ then we omit } G \text{ in the notation containing } G, \\
The inequalities between the vectors and between the matrices are understood componentwise. \\
A vector-function } x \in \text{BV}([-a, a], R^n) \text{ is said to be a solution of the system (1) if} \\
x(t) = x(s) + \int_s^t dA(\tau) \cdot f(\tau, x(\tau)) \text{ for } -a \leq s \leq t \leq a. \\
\text{By a solution of the system of generalized ordinary differential inequalities} \\
dx(t) \leq dA(t) \cdot f(t, x(t)) \text{ (} \geq \text{)}
we mean a vector-function $x \in \text{BV}([-a, a], R^n)$ such that

$$x(t) \leq x(s) + \int_s^t dA(\tau) \cdot f(\tau, x(\tau)) \quad (\geq) \quad \text{for} \quad -a \leq s \leq t \leq a.$$ 

If $s \in R$ and $\beta \in \text{BV}[a, b], R)$ are such that

$$1 + (-1)^j d_j \beta(t) \neq 0 \quad \text{for} \quad (-1)^j (t - s) < 0 \quad (j = 1, 2),$$

then by $\gamma_\beta(\cdot, s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t) \, d\beta(t), \quad \gamma(s) = 1.$$ 

It is known (see [6], [8]) that

$$\gamma_\beta(t, s) = \begin{cases} 
\exp(s_\beta(\beta)(t) - s_\beta(\beta)(s)) \prod_{s < \tau \leq t} (1 - d_1 \beta(\tau))^{-1} \times 
\prod_{s \leq \tau < t} (1 + d_2 \beta(\tau)) & \text{for} \quad t > s, \\
\exp(s_\beta(\beta)(t) - s_\beta(\beta)(s)) \prod_{t < \tau \leq s} (1 - d_1 \beta(\tau)) \times 
\prod_{t \leq \tau < s} (1 + d_2 \beta(\tau))^{-1} & \text{for} \quad t < s, \\
1 & \text{for} \quad t = s.
\end{cases} \quad (3)$$

**Definition 1.** Let $\sigma_1, \ldots, \sigma_n \in \{-1, 1\}$. We say that the pair $((c_{il})_{i,l=1}^n; \varphi_{il})_{i=1}^n \in \text{BV}([a, b], R^{n \times n})$ and a positive homogeneous nondecreasing operator $(\varphi_{il})_{i=1}^n: \text{BV}([a, b], R^n) \rightarrow R^n$ belongs to the set $U^{\sigma_1, \ldots, \sigma_n}$ if the functions $c_{il}$ ($i \neq l; i, l = 1, \ldots, n$) are nondecreasing on $[a, b]$ and continuous at the point $t_i = -\sigma_i a$,

$$d_j c_{li}(t) \geq 0 \quad \text{for} \quad t \in [-a, a] \quad (j = 1, 2; \quad i = 1, \ldots, n)$$

and the problem

$$\sigma_i dx_i(t) \leq \sum_{l=1}^n x_i(t) dc_{il}(t) \quad \text{for} \quad t \in [-a, a] \setminus \{-\sigma_i a\} \quad (i = 1, \ldots, n),$$

$$(-1)^j d_j x_i(-\sigma_i a) \leq x_i(-\sigma_i a) d_j c_{ii}(-\sigma_i a) \quad (j = 1, 2; \quad i = 1, \ldots, n);$$

$$x_i(-\sigma_i a) \leq \varphi_{ii}(|x_1|, \ldots, |x_n|) \quad (i = 1, \ldots, n)$$

has no nontrivial non-negative solution.

The set $U^{\sigma_1, \ldots, \sigma_n}$ has been introduced by I. Kiguradze for ordinary differential equations (see [1], [2]).

**Theorem 1.** The problem (1), (2) is solvable if and only if there exist vector-functions $\alpha_m = (\alpha_{mi})_{i=1}^n \in \text{BV}([-a, a], R^n)$ ($m = 1, 2$) and
matrix-functions \((\beta_{mik})_{i,k=1}^n : [-a, a] \to \mathbb{R}^{n \times n}\) such that \(\beta_{mik} \in L([-a, a], R; a_{ik})\) \((m, j = 1, 2; i, k = 1, \ldots, n)\),

\[
\alpha_{mi}(t) = \alpha_{mi}(-\sigma_i a) + \sum_{k=1}^n \left( \int_{-\sigma_i a}^t \beta_{mik}(\tau) d\alpha_{1ik}(\tau) - \int_{-\sigma_i a}^t \beta_{mik}(\tau) d\alpha_{2ik}(\tau) \right) \quad (m = 1, 2; i = 1, \ldots, n),
\]

\[
(4)
\]

\[
(1)^m \sigma_i (f_k(t, x_1, \ldots, x_i-1, \alpha_j(t), x_{i+1}, \ldots, x_n) - \beta_{mik}(t)) \leq 0
\]

for \(\mu(a_{1+i|m-j|ik})\)-almost every \(t \in [-a, a]\),

\[
\alpha_{1}(t) \leq \alpha_{2}(t) \quad (m, j = 1, 2; i, k = 1, \ldots, n),
\]

\[
(5)
\]

and the inequalities

\[
\alpha_{1i}(-\sigma_i a) \leq \varphi_i(x_1, \ldots, x_n) \leq \alpha_{2i}(-\sigma_i a) \quad (i = 1, \ldots, n)
\]

are fulfilled on the set \(\{(x_i)_{i=1}^n \in BV([a, b], R^n); \alpha_1(t) \leq (x_i)_{i=1}^n \leq \alpha_2(t)\} \text{ for } t \in [-a, a]\}.

**Corollary 1.** Let the matrix-function \(A(t) = (a_{ik})_{i,k=1}^n\) be nondecreasing on \([-a, a]\). Then the problem (1), (2) is solvable if and only if there exist vector-functions \(\alpha_m = (\alpha_{mi})_{i=1}^n \in BV([-a, a], R^n)\) \((m = 1, 2)\) and matrix-functions \((\beta_{mik})_{i,k=1}^n : [-a, a] \to \mathbb{R}^{n \times n}\) \((m = 1, 2; i, k = 1, \ldots, n)\) such that \(\beta_{mik} \in L([-a, a], R; a_{ik})\) \((m = 1, 2; i, k = 1, \ldots, n)\),

\[
\alpha_{mi}(t) = \alpha_{mi}(-\sigma_i a) + \sum_{l=1}^n \left( \int_{-\sigma_i a}^t \beta_{mik}(\tau) d\alpha_{1lk}(\tau) \right) \quad (m = 1, 2; i, k = 1, \ldots, n),
\]

the conditions (4)–(6) hold, and the inequalities

\[
(1)^m \sigma_i (f_k(t, x_1, \ldots, x_i-1, \alpha_j(t), x_{i+1}, \ldots, x_n) - \beta_{jik}(t)) \leq 0
\]

\((j = 1, 2; i, k = 1, \ldots, n)\)

are fulfilled for \(\mu(a_{ik})\)-almost every \(t \in [-a, a]\) and \(\alpha_1(t) \leq (x_i)_{i=1}^n \leq \alpha_2(t)\).

**Theorem 2.** Let the condition

\[
(1)^{m+1} \sigma_i f_k(t, x_1, \ldots, x_n) \operatorname{sgn} x_i \leq \sum_{l=1}^n p_{milk}(t)|x_l| + q_k(t)
\]

for \(\mu(a_{mik})\)-almost every \(t \in [-a, a]\) \((m = 1, 2; i, k = 1, \ldots, n)\) \((7)\)
be fulfilled on \( R^n \), and let the inequalities
\[
|\varphi_i(x_1, \ldots, x_n)| \leq \varphi_0(|x_1|, \ldots, |x_n|) + \zeta_i \quad (i = 1, \ldots, n)
\]
be fulfilled on \( BV([-a, a], R^n) \), where \((p_{mik})_{k=1}^n \in L([-a, a], R_{+}^{n \times n}; A_m) \) \((m=1, 2; i = 1, \ldots, n)\), \(q_k = (q_{ki})_{i=1}^n \in L([-a, a], R_+^n; A_m) \) \((m = 1, 2)\), \(\zeta_i \in R_+ \) \((i = 1, \ldots, n)\). Let, moreover, there exist a matrix-function \((c_{il})_{i=1}^n \in BV([-a, a], R_+^{n \times n})\) such that
\[
((c_{il})_{l=1}^n: (c_{il})_{i=1}^n) \in \mathcal{U}_{\alpha^{1}, \ldots, \alpha^{n}}
\]
and
\[
\sum_{m=1}^2 \sum_{k=1}^n \int_{s}^t p_{mik}(\tau)da_{mik}(\tau) \leq c_{il}(t) - c_{il}(s)
\]
for \(-a \leq s < t \leq a \) \((i, l = 1, \ldots, n)\).

Then the problem \((1), (2)\) is solvable.

**Corollary 2.** Let there exist \(m, m_1 \in \{1, 2\}\) such that \(m + m_1 = 3\) and the conditions \((7)\) and
\[
(-1)^{m_1+1}\mu_j f_k(t, x_1, \ldots, x_n) \operatorname{sgn} x_i \leq \sum_{i=1}^n \eta_{il} |x_l| + q_k(t)
\]
for \(\mu(a_{i, i})\)-almost every \(t \in [-a, a] \) \((i, k = 1, \ldots, n)\)
are fulfilled on \( R^n \), the inequalities
\[
|\varphi_i(x_1, \ldots, x_n)| \leq \mu_i |x_i(s_i)| + \zeta_i \quad (i = 1, \ldots, n)
\]
be fulfilled on \( BV_s([-a, b], R^n) \), and let
\[
0 \leq d_j \alpha_i(t) < |\eta_{il}|^{-1} \quad \text{for} \quad (-1)^j (t + \sigma_i a) > 0 \quad (j = 1, 2; \; i = 1, \ldots, n)
\]
and
\[
\mu_i \gamma_i(s_i, -\sigma_i a) < 1 \quad (i = 1, \ldots, n),
\]
where \((p_{mik})_{k=1}^n \in L([-a, a], R_+^{n \times n}; A_m) \) \((i = 1, \ldots, n)\), \(\eta_{il} \in R_+ \) \((i \neq l; i, l = 1, \ldots, n)\), \(\eta_{ii} < 0 \) \((i = 1, \ldots, n)\), \(q_k = (q_{ki})_{i=1}^n \in L([-a, a], R_+^n; A_m) \) \((m = 1, 2)\), \(\zeta_i \in R_+ \) \((i = 1, \ldots, n)\), \(\mu_i \in R_+\) and \(s_i \in [-a, a] \) \((i = 1, \ldots, n)\), \(s_i \neq -\sigma_i a \) \((i = 1, \ldots, n)\),
\[
\alpha_i(t) \equiv \sum_{k=1}^n a_{m_ik}(t) \quad (i = 1, \ldots, n),
\]
\[
\gamma_i(t, s) \equiv \gamma_{ai}(t, s) \quad (i = 1, \ldots, n),
\]
\[
\alpha_i(t) \equiv \eta_i \sigma_i (\alpha_i(t) - \alpha_i(-\sigma_i a)) \quad (i = 1, \ldots, n),
\]
and the functions \(\gamma_{ai} \) \((i = 1, \ldots, n)\) are defined according to \((3)\). Let, moreover,
\[
g_{ii} < 1 \quad (i = 1, \ldots, n)
\]
and the real part of every characteristic value of the matrix \((\xi_{it})_{i,t=1}^n\) be negative, where
\[
\xi_{it} = \eta_{it}(\delta_{it} + (1 - \delta_{it})h_i) - \eta_{it}g_{it} \quad (i, l = 1, \ldots, n),
\]
\[
g_{it} = \mu_i(1 - \mu_i\gamma_i(s_i, -\sigma_i a))^{-1}\gamma_i(s_i) +
+ \max \{\gamma_i(-a), \gamma_i(a)\} \quad (i, l = 1, \ldots, n),
\]
\[
\gamma_i(-\sigma_i a) = 0, \quad \gamma_i(t) = |\beta_{il}(t) - \beta_{il}(-\sigma_i a)| - (1 - \delta_{il})d_j^i\beta_{il}(-\sigma_i a)
\]
for \((-1)^{j}(t + \sigma_i a) > 0 \quad (j = 1, 2; \quad i, l = 1, \ldots, n),
\]
\[
\beta_{il}(t) \equiv \sum_{k=1-a}^n \int p_{mikl}(\tau)da_{mk}(\tau) \quad (i = 1, \ldots, n),
\]
\[
h_i = 1 + (\mu_i - 1)(1 - \mu_i\gamma_i(s_i, -\sigma_i a))^{-1} \quad \text{for} \quad \mu_i > 1 \quad (i = 1, \ldots, n).
\]

Then the problem (1), (2) is solvable.

Remark 1. In Corollary 2 as the matrix-function \(C = (c_{il})_{i,t=1}^n\) we take
\[
c_{il}(-\sigma_i a) = 0 \quad (i, l = 1, \ldots, n),
\]
\[
c_{il}(t) = \eta_{il}(\alpha_i(t) - \alpha_i(-\sigma_i a) - (-1)^{j}d_j^i\alpha_i(-\sigma_i a)) +
+ \beta_{il}(t) - \beta_{il}(-\sigma_i a) - (-1)^{j}d_j^i\beta_{il}(-\sigma_i a)
\]
for \((-1)^{j}(t + \sigma_i a) > 0 \quad (j = 1, 2; \quad i, l = 1, \ldots, n).

If the matrix-function \(A = (a_{ik})_{i,k=1}^n:\ [-a, a] \to R^{n \times n}\) is nondecreasing, then Corollary 2 has the following form.

**Corollary 3.** Let the matrix-function \(A = (a_{ik})_{i,k=1}^n:\ [-a, a] \to R^{n \times n}\) be nondecreasing, the conditions (8), (10) hold, the condition
\[
\sigma_i f_k(t, x_1, \ldots, x_n) \text{sgn} x_i \leq \sum_{i=1}^n \eta_{il}|x_i| + q_k(t)
\]
for \(\mu(a_{ik})\text{-almost every} t \in [-a, a] \quad (i, k = 1, \ldots, n) \quad \text{(11)}\)
be fulfilled on \(R^n\) and let the real part of every characteristic value of the matrix \((\eta_{il}(\delta_{il} + (1 - \delta_{il})h_i))_{i,t=1}^n\) be negative, where
\[
\alpha_i(t) \equiv \sum_{k=1}^n a_{ik}(t) \quad (i = 1, \ldots, n),
\]
and the functions \(\gamma_i(t, s) \quad (i = 1, \ldots, n)\) and \(\alpha_i(t) \quad (i = 1, \ldots, n)\) and the numbers \(h_i \quad (i = 1, \ldots, n)\) are defined as in Corollary 2. Then the problem (1), (2) is solvable.

**Corollary 4.** Let the matrix-function \(A = (a_{ik})_{i,k=1}^n:\ [-a, a] \to R^{n \times n}\) be nondecreasing and continuous from the left, the conditions (8), (10), (11)
and

\[ 0 \leq d_2 \alpha_i(t) < |\eta_{ii}|^{-1} \quad \text{for } t \in ]-a, a[ \quad (i = 1, \ldots, n) \]

hold and let the real part of every characteristic value of the matrix \( (\eta_{il}(\delta_{il} + (1 - \delta_{il})h_i))_{i,l=1}^n \) be negative, where the functions \( \alpha_i(t) \) \( (i = 1, \ldots, n) \), \( \gamma_i(t, s) \) \( (i = 1, \ldots, n) \) and \( a_i(t) \) \( (i = 1, \ldots, n) \) are defined as in Corollary 3. Then the problem (1), (2) is solvable.

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