ON PERIODIC TYPE BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER DIFFERENTIAL SYSTEMS

Abstract. For higher order systems of differential equations solvability and correctness conditions of periodic type boundary value problems are established.

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On the interval \( I = [a, b] \), let us consider the nonlinear differential system

\[
 u^{(n)} = f(t, u, \ldots, u^{(n-1)}) \tag{1}
\]

with the nonlinear boundary conditions

\[
 \sum_{k=1}^{n} \left( A_{ik}(u)u^{(k-1)}(a) + B_{ik}(u)u^{(k-1)}(b) \right) = c_i(u) \quad (i = 1, \ldots, n). \tag{2}
\]

Here \( f : I \times \mathbb{R}^{nl} \to \mathbb{R}^l \) is a vector function from the Carathéodory class, and

\[
 A_{ik} : C^{n-1}(I; \mathbb{R}^l) \to \mathbb{R}^{l \times l}, \quad B_{ik} : C^{n-1}(I; \mathbb{R}^l) \to \mathbb{R}^{l \times l}, \quad c_i : C^{n-1}(I; \mathbb{R}^l) \to \mathbb{R}^l
\]

are nonlinear continuous operators satisfying some additional conditions.

These conditions are rather general, and, in particular, they are satisfied for a periodic problem, the Dirichlet problem and the Neumann problem

\[
 u^{(i-1)}(a) = c_i \quad (i = 1, \ldots, n); \tag{3_1}
\]

\[
 u^{(i-1)}(a) = c_i \quad (i = 1, \ldots, m), \quad u^{(i-1)}(b) = c_{m+i} \quad (i = 1, \ldots, n-m); \tag{3_2}
\]

\[
 u^{(n-i)}(a) = c_i \quad (i = 1, \ldots, n-m), \quad u^{(i-1)}(b) = c_{n-m+i} \quad (i = 1, \ldots, m); \tag{3_3}
\]

\[
 u^{(n-i)}(a) = c_i \quad (i = 1, \ldots, n-m), \quad u^{(n-i)}(b) = c_{n-m+i} \quad (i = 1, \ldots, n-m), \tag{3_4}
\]

where \( m \) is an integer part of the number \( \frac{n}{2} \), \( c_i \in \mathbb{R}^l \) (i = 1, \ldots, n).

In the case of even $n = 2m$, the above-mentioned conditions hold for the problem

$$u^{(2i-2)}(a) = c_i, \quad u^{(2i-2)}(b) = c_{m+i} \quad (i = 1, \ldots, m),$$

which is sometimes said to be the Lidston problem.

In the case where $l = 1$, these classical problems has been intensively investigated, and they are the subject of numerous investigations (see, e.g., [1]–[12] and the references therein).

We single out a class of boundary value problems of the type (1), (2) to which belong the above-mentioned two-point boundary value problems (1), (3) $(k = 1, 2, 3, 4, 5)$, and we propose a unique method for their investigation.

By the way, we are trying to find general properties which unite the problems from that class.

Before formulating the main results, we introduce the notation.

$R_l$ is the $l$-dimensional real Euclidean space with the norm $\| \cdot \|_{R_l}$.

$R_l \times l$ is the space of real $l \times l$ matrices with the norm $\| \cdot \|_{R_l \times l}$.

$x \cdot y$ is the scalar product of the vectors $x, y \in R_l$.

$C^{n-1}(I; R_l)$ is the Banach space of $(n-1)$-times continuously differentiable vector functions $u : I \rightarrow R_l$ with the norm $\| u \|_{C^{n-1}} = \max \left\{ \sum_{k=1}^{n} \| u^{(k-1)}(t) \|_{R_l} : t \in I \right\}$.

$\tilde{C}^{n-1}(I; R_l)$ is the space of all vector functions $u \in C^{n-1}(I; R_l)$ for which $u^{(n-1)}$ is absolutely continuous.

Of course, we will seek for a solution of the problem (1), (2) in the space $\tilde{C}^{n-1}(I; R_l)$.

For arbitrary $x_k$ and $y_k \in R_l$ $(k = 1, \ldots, n)$, we set

$$\nu(x_1, \ldots, x_n; y_1, \ldots, y_n) =$$

$$= \begin{cases} \sum_{k=1}^{m} (-1)^k (x_k \cdot x_{n-k+1} - y_k \cdot y_{n-k+1}) & \text{for } n = 2m, \\ -\frac{(-1)^m}{2} (\| x_{m+1} \|_{R_l}^2 - \| y_{m+1} \|_{R_l}^2) & \text{for } n = 2m + 1. \end{cases}$$

We study the problem (1), (2) in the case where there exists a positive number $\mu$ such that for any $x_k, y_k \in R_l$ and $v \in C^{n-1}$ the operators $A_{ik}$ and $B_{ik}$ satisfy the inequality

$$(-1)^m \nu(x_1, \ldots, x_n; y_1, \ldots, y_n) \leq$$

$$\leq \mu \sum_{k=1}^{m} (\| x_k \|_{R_l} + \| y_k \|_{R_l}) \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} (A_{ik}(v)x_k + B_{ik}(v)y_k) \right\|_{R_l}. \quad (4)$$
As for the vector function f, it satisfies the condition
\[ p_s(t, \|x_1\|_{R^n}) \leq (-1)^{m+1} f(t, x_1, \ldots, x_n) \cdot x_1 \leq p^*(t, \|x_1\|_{R^n}) \]  
(5)
on the set \( I \times R^n \), where \( p_s \) and \( p^* : I \times [0, +\infty] \rightarrow R \) are non-decreasing with respect to a second argument Carathéodory functions such that
\[ \lim_{\rho \to +\infty} \int_a^b p_s(t, \rho) \, dt = +\infty. \]  
(6)

We will also assume that \( A_{ik}, B_{ik} \) and \( c_i \) \((i, k = 1, \ldots, n)\) are bounded in \( C^{n-1}(I; R^l) \).  
(7)

**Theorem 1.** *If the conditions (4)–(7) hold, then the problem (1), (2) has at least one solution.*

As it is already said above, for the periodic problem, the Dirichlet problem, the Neumann and Lidston problems, and the mixed problem, the condition (4) is automatically satisfied. Therefore it is clear that if the conditions (5)–(7) hold, then each of the above-mentioned problems (1), (3k), \( k \in \{1, 2, 3, 4, 5\} \), has at least one solution.

We will give also examples of another well-known boundary value problems, for which the condition (4) is satisfied. More precisely, in the case of even \( n = 2m \), we consider the problems
\[ \alpha_i u^{(i-1)}(a) + \alpha_{m+i} u^{(n-i)}(a) = c_i, \]
\[ \beta_i u^{(i-1)}(b) + \beta_{m+i} u^{(n-i)}(b) = c_{m+i}, \quad (i = 1, \ldots, m) \]  
(82m)
and
\[ u^{(i-1)}(a) = \eta_i u^{(i-1)}(b) + c_i, \]
\[ u^{(n-i)}(b) = \eta_i u^{(n-i)}(a) + c_{m+i}, \quad (i = 1, \ldots, m), \]  
(92m)
while in the case of \( n = 2m + 1 \), we consider the problems
\[ \alpha_i u^{(i-1)}(a) + \alpha_{m+i} u^{(n-i)}(a) = c_i, \quad \beta_i u^{(i-1)}(b) + \beta_{m+i} u^{(n-i)}(b) = c_{m+i}, \quad (i = 1, \ldots, m), \]
\[ u^{(m)}(a) = \eta u^{(m)}(b) + c_n, \]  
(82m+1)
and
\[ u^{(i-1)}(a) = \eta_i u^{(i-1)}(b) + c_i, \quad u^{(n-i)}(b) = \eta_i u^{(n-i)}(a) + c_{m+i}, \quad (i = 1, \ldots, m), \]
\[ u^{(m)}(a) = \eta u^{(m)}(b) + c_n, \]  
(92m+1)
where \( \alpha_i, \beta_i, \eta_i, \) and \( \eta \) are real numbers, \( c_i \in R^l \). Moreover,
\[ (-1)^{m+i} \alpha_i \alpha_{m+i} \geq 0, \quad |\alpha_i| + |\alpha_{m+i}| > 0, \]
\[ (-1)^{m+i} \beta_i \beta_{m+i} \leq 0, \quad |\beta_i| + |\beta_{m+i}| > 0, \quad \eta_i \neq 0, \quad |\eta| \leq 1. \]  
(10)

From Theorem 1 it follows the following

**Corollary 1.** *If the conditions (5)–(7) and (10) hold, then both problems (1), (8n) and (1), (9n) have at least one solution.*
Under the conditions of Theorem 1, the components of the vector function 
\( (t, x_1, \ldots, x_n) \rightarrow f(t, x_1, \ldots, x_n) \) may have an arbitrary growth order with respect to \( x_1 \). As an example, we consider the nonlinear differential system

\[
\begin{align*}
  u_k^{(n)} &= (-1)^{m+1} \mu_k + h_k(t, u_1, \ldots, u_l) \\
  &\quad + g_k(t, u_1, \ldots, u_l) \quad (k = 1, \ldots, l)
\end{align*}
\]

with the periodic boundary conditions

\[
  u_k^{(i-1)}(a) = u_k^{(i-1)}(b) + c_{ik} \quad (i = 1, \ldots, n; \quad k = 1, \ldots, l),
\]

or with the Neumann conditions

\[
  u_k^{(i-1)}(a) = c_{ik} \quad (i = 1, \ldots, m; \quad k = 1, \ldots, l),
\]

\[
  u_k^{(i-1)}(b) = c'_{ik} \quad (i = 1, \ldots, n - m; \quad k = 1, \ldots, l),
\]

where \( \mu_k > 0 \), \( c_{ik}, c'_{ik} \) are constants, and \( g_k: I \times R^2 \rightarrow R \) and \( h_k: I \times R^l \rightarrow R^l \) are Carathéodory functions.

From Theorem 1 for these problems we have

**Corollary 2.** Let there exist an integrable with respect to the first argument and non-decreasing with respect to the second argument function \( g_0: I \times [0, +\infty] \rightarrow [0, +\infty] \), and a positive constant \( r \) such that on the set \( I \times R^l \) the inequalities

\[
  g_k(t, u_1, \ldots, u_l) \geq g_0(t, \sum_{i=1}^{l} |u_i|),
\]

\[
  |h_k(t, u_1, \ldots, u_l)| \geq r g_0(t, \sum_{i=1}^{l} |u_i|) \quad (k = 1, \ldots, l)
\]

hold. Then both problems (11), (12) and (11), (13) have at least one solution.

It is evident that under the conditions of Corollary 2, the numbers \( \mu_k > 0 \) 
\((k = 1, \ldots, l)\) may be arbitrarily large, and \((t, u_1, \ldots, u_l) \rightarrow g_k(t, u_1, \ldots, u_l) \)
\((k = 1, \ldots, l)\) may be arbitrary functions, increasing rapidly with respect to \( u_1, \ldots, u_l \).

We can prove the unique solvability of the problem (1), (2) and its stability with respect to small perturbations of the right-hand side of the system (1) and boundary data only in the case where

\[
  f(t, x_1, \ldots, x_n) \equiv f(t, x_1),
\]

\[
  A_{ik} = \Lambda_{ik}, \quad B_{ik} = \Lambda_{ik}, \quad c_i = c_i \quad (i, k = 1, \ldots, n),
\]

where \( A_{ik} \) and \( B_{ik} \) are constant \( l \times l \) matrices, and \( c_i \) is a constant \( l \)-dimensional vector.
Thus let us consider the boundary value problem
\[ u^{(n)} = f(t, u), \quad (15) \]
\[ \sum_{k=1}^{n} \left( A_{ik} u^{(k-1)}(a) + B_{ik} u^{(k-1)}(b) \right) = c_i \quad (i = 1, \ldots, n) \quad (16) \]
and the corresponding perturbed problem
\[ v^{(n)} = f(t, v) + h(t), \quad (15') \]
\[ \sum_{k=1}^{n} \left( A_{ik} v^{(k-1)}(a) + B_{ik} v^{(k-1)}(b) \right) = c'_i \quad (i = 1, \ldots, n). \quad (16') \]

We will use the following definition of the well-posedness.

The problem (15), (16) is said to be well-posed if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \), such that the problem (15'), (16') is uniquely solvable for an arbitrary integrable vector function \( h : I \to \mathbb{R}^l \) and vectors \( c'_i \in \mathbb{R}^l \) (\( i = 1, \ldots, n \)) satisfying the condition
\[ \sum_{i=1}^{n} \| c_i - c'_i \|_{\mathbb{R}^l} + \left\| \int_a^t h(s) \, ds \right\|_{\mathbb{R}^l} < \delta \quad \text{for} \quad t \in I, \]
and the inequality
\[ \| v - u \|_{C^{n-1}} < \varepsilon \]
holds, where \( u \) and \( v \) are solutions of the problems (15), (16) and (15'), (16'), respectively.

**Theorem 2.** Let there exist \( \mu > 0 \) such that for arbitrary \( x_k \) and \( y_k \in \mathbb{R}^l \) (\( k = 1, \ldots, n \)) the inequality
\[ (-1)^{m+1} \nu(x_1, \ldots, x_n; y_1, \ldots, y_n) \leq \sum_{k=1}^{n} \| x_k \|_{l} + \| y_k \|_{l} \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} (A_{ik} x_k + B_{ik} y_k) \right\|_{\mathbb{R}^l} \] \quad (17)
holds. Moreover, the vector function \( f \) satisfies the condition
\[ (-1)^{m+1} (f(t, x) - f(t, y)) \cdot (x - y) \geq p(t) \| x - y \|_{\mathbb{R}^l}^2, \quad (18) \]
where \( p : I \to [0, +\infty) \) is an integrable function positive on the set of positive measure. Then the problem (15),(16) is well-posed.

From this theorem it follows, in particular, that if the condition (18) is satisfied, then the above-mentioned classical two-point boundary value problems for the system (16) are well-posed.

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REFERENCES


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