Short Communications

M. ASHORDIA

ON THE EXISTENCE OF BOUNDED SOLUTIONS FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Sufficient conditions are given for the existence of bounded solutions for the systems of nonlinear generalized ordinary differential equations.

2000 Mathematics Subject Classification: 34K10.

Key words and phrases: Systems of nonlinear generalized ordinary differential equations, the Lebesgue–Stiltjes integral, existence of bounded solutions, sufficient conditions.

Let \( a_{mk} : \mathbb{R} \to \mathbb{R} \) \((m = 1, 2; i, k = 1, \ldots, n)\) be nondecreasing functions, \( a_{ik}(t) \equiv a_{1ik}(t) - a_{2ik}(t)\), \( A = (a_{ik})_{i,k=1}^{n} \), \( A_{m} = (a_{mk})_{i,k=1}^{n} \) \((m = 1, 2)\); \( f = (f_{k})_{k=1}^{n} : \mathbb{R} \times \mathbb{R}^{n} \to \mathbb{R}^{n} \) be a vector-function belonging to the Carathéodory class corresponding to the matrix-function \( A \).

In this paper we investigate the question of existence of solutions for the system of generalized ordinary differential equations
\[
dx(t) = dA(t) \cdot f(t, x(t)),
\]
where \( x = (x_{i})_{i=1}^{n} \), satisfying one of the following two conditions
\[
\sup \left\{ \sum_{i=1}^{n} |x_{i}(t)| : t \in \mathbb{R} \right\} < \infty \tag{2}
\]
and
\[
\sup \left\{ \sum_{i=1}^{n} |x_{i}(t)| : t \in \mathbb{R}_{+} \right\} < \infty \tag{3}
\]

We give sufficient conditions for the existence of solutions of the boundary value problems (1), (2) and (1), (3). Analogous results are contained in [9]–[14] for systems of ordinary differential and functional differential equations.

The theory of generalized ordinary differential equations enables one to investigate ordinary differential, impulsive and difference equations from a common point of view (see [1]–[8], [15]).

Throughout the paper the following notation and definitions will be used.

- $\mathbb{R} = [-\infty, +\infty[, \mathbb{R}_+ = [0, +\infty[, [a, b] (a, b \in \mathbb{R})$ is a closed segment.
- $\mathbb{R}^{n \times m}$ is the set all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$.
- $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the set all real column $n$-vectors $x = (x_i)_{i=1}^n$.
- $\mathbb{R}^{n \times m}_+ = \{(x_{ij}) \in \mathbb{R}^{n \times m} : x_{ij} \geq 0 \}$ is the set of all non-negative $n \times m$-matrices $X$.
- $\text{BV}([a, b], \mathbb{R}^{n \times m})$ is the set of all matrix-functions of bounded variation $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$, i.e., the sum of total variations of the latter’s components.
- $X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$ at the point $t$ (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary);
- $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\text{BV}([a, b], \mathbb{R}^{n \times m})$ is the set of all real column $n$-vectors $x = (x_i)_{i=1}^n$.

$\text{BV}^{loc}(\mathbb{R}, \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$ for which $\frac{b}{a} \text{BV}(X) < +\infty)$ for every $a, b \in \mathbb{R}$ ($a < b$).

$s_j : \text{BV}([a, b], \mathbb{R}) \rightarrow \text{BV}([a, b], \mathbb{R})$ ($j = 0, 1, 2$) are the operators defined, respectively, by

$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t)$ for $t \in [a, b]$.

If $g : [a, b] \rightarrow \mathbb{R}$ is a nondecreasing function, $x : [a, b] \rightarrow \mathbb{R}$ and $a \leq s < t \leq b$, then

$$\int_s^t x(\tau) \, dg(\tau) = \int_s^t x(\tau) \, ds_0(g)(\tau) + \sum_{s < \tau \leq t} x(\tau) d_1g(\tau) + \sum_{s \leq \tau < t} x(\tau) d_2g(\tau),$$

where $\int_s^t x(\tau) \, ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $[s, t[$ with respect to the measure $\mu(s_0(g))$ corresponding to the function $s_0(g)$.

If $a = b$, then we assume

$$\int_a^b x(t) \, dg(t) = 0.$$
If \( g(t) \equiv g_1(t) - g_2(t) \), where \( g_1 \) and \( g_2 \) are nondecreasing functions, then

\[
\int_s^t x(\tau) \, dg(\tau) = \int_s^t x(\tau) \, dg_1(\tau) - \int_s^t x(\tau) \, dg_2(\tau) \quad \text{for} \quad s \leq t.
\]

\( L([a, b], \mathbb{R}; g) \) is the set of all functions \( x : [a, b] \rightarrow \mathbb{R} \) measurable and integrable with respect to the measures \( \mu(g_i) \) \( i = 1, 2 \), i.e. such that

\[
\int_a^b |x(t)| \, dg_i(t) < +\infty \quad (i = 1, 2).
\]

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If \( G = (g_{ik})_{i,k=1}^{l,n} : [a, b] \rightarrow \mathbb{R}^{l \times n} \) is a nondecreasing matrix-function and \( D \subset \mathbb{R}^{n \times m} \), then \( L([a, b], D; G) \) is the set of all matrix-functions \( X = (x_{kj})_{k,j=1}^{m,l} : [a, b] \rightarrow D \) such that \( x_{kj} \in L([a, b], \mathbb{R}; g_{ik}) \) \( i = 1, \ldots, l; k = 1, \ldots, m; j = 1, \ldots, n \):

\[
\int_s^t dG(\tau) \cdot X(\tau) = \left( \sum_{k=1}^n \int_s^t x_{kj}(\tau) dg_{ik}(\tau) \right)_{i,j=1}^{l,m} \quad \text{for} \quad a \leq s \leq t \leq b,
\]

\[
S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 0, 1, 2).
\]

If \( D_1 \subset \mathbb{R}^n \) and \( D_2 \subset \mathbb{R}^{n \times m} \), then \( K([a, b] \times D_1, D_2; G) \) is the Carathéodory class, i.e., the set of all mappings \( F = (f_{kj})_{k,j=1}^{m,n} : [a, b] \times D_1 \rightarrow D_2 \) such that for each \( i \in \{1, \ldots, l\} \), \( j \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, n\} \): a) the function \( f_{kj}(\cdot, x) : [a, b] \rightarrow D_2 \) is \( \mu(g_{ik}) \)-measurable for every \( x \in D_1 \); b) the function \( f_{kj}(t, \cdot) : D_1 \rightarrow D_2 \) is continuous for \( \mu(g_{ik}) \)-almost every \( t \in [a, b] \), and \( \sup \{ |f_{kj}(\cdot, x)| : x \in D_0 \} \in L([a, b], \mathbb{R}; g_{ik}) \) for every compact \( D_0 \subset D_1 \).

If \( G_j : [a, b] \rightarrow \mathbb{R}^{l \times n} \) \( j = 1, 2 \) are nondecreasing matrix-functions, \( G = G_1 - G_2 \) and \( X : [a, b] \rightarrow \mathbb{R}^{n \times m} \), then

\[
\int_s^t dG(\tau) \cdot X(\tau) = \int_s^t dG_1(\tau) \cdot X(\tau) - \int_s^t dG_2(\tau) \cdot X(\tau) \quad \text{for} \quad s \leq t,
\]

\[
S_k(G) = S_k(G_1) - S_k(G_2) \quad (k = 0, 1, 2),
\]

\[
L([a, b], D; G) = \bigcap_{j=1}^{2} L([a, b], D; G_j),
\]

\[
K([a, b] \times D_1, D_2; G) = \bigcap_{j=1}^{2} K([a, b] \times D_1, D_2; G_j).
\]
Let \( L_{loc}(\mathbb{R}, D; G) \) be the set of all matrix-functions \( X = \mathbb{R} \rightarrow D \) such that its restriction on \([a, b]\) belongs to \( L([a, b], D; G) \) for every \( a \) and \( b \) from \( \mathbb{R} \) (\( a < b \)).

\( K_{loc}(\mathbb{R} \times D_1, D_2; G) \) is the set of all matrix-functions \( F = (f_{kj})^n_{k,j=1} : \mathbb{R} \times D_1 \rightarrow D_2 \) such that its restriction on \([a, b]\) belongs to \( K([a, b], D; G) \) for every \( a \) and \( b \) from \( \mathbb{R} \) (\( a < b \)).

The inequalities between the matrices are understood componentwise.

A vector-function \( x \in \text{BV}_{loc}(\mathbb{R}, \mathbb{R}^n) \) is said to be a solution of the system (1) if

\[
x(t) = x(s) + \int_s^t dA(\tau) \cdot f(\tau, x(\tau)) \quad \text{for} \quad s \leq t \quad (s, t \in \mathbb{R}).
\]

**Theorem 1.** Let there exist numbers \( \sigma_i \in \{-1, 1\} \) \((i = 1, \ldots, n)\), vector-functions \( \alpha_m = (\alpha_{mi})^n_{i=1} \in \text{BV}_{loc}(\mathbb{R}, \mathbb{R}^n) \) \((m = 1, 2)\) and matrix-functions \( (\beta_{mik})^n_{i,k=1}, \beta_{mik} \in L_{loc}(\mathbb{R}, \mathbb{R}; a_{jik}) \) \((m, j = 1, 2; i, k = 1, \ldots, n)\) such that

\[
\alpha_{mi}(t) = \alpha_{mi}(0) + \sum_{k=1}^n \left( \int_0^t \beta_{mik}(\tau) \, da_{1ik}(\tau) - \int_0^t \beta_{3-mik}(\tau) \, da_{2ik}(\tau) \right) \quad (m = 1, 2; i = 1, \ldots, n), \quad (4)
\]

\[
\alpha_1(t) \leq \alpha_2(t) \quad \text{for} \quad t \in \mathbb{R}, \quad (5)
\]

\[
(1)^m \sigma_i (f_k(t, x_1, \ldots, x_i-1, \alpha_{ji}(t), x_{i+1}, \ldots, x_n) - \beta_{mik}(t)) \leq 0
\]

\[
\text{for} \quad \mu(a_{1+[m-j]ik})\text{-almost all} \quad t \in \mathbb{R} \quad \text{and} \quad \alpha_1(t) \leq (x_i)^n_{i=1} \leq \alpha_2(t) \quad (m, j = 1, 2; i, k = 1, \ldots, n),
\]

\[
(1)^m \left( x_i - (1)^m \sum_{k=1}^n f_k(t, x_1, \ldots, x_n) \, da_{ik}(t) - \alpha_{mi}(t) - (1)^m \, d_j a_{mi}(t) \right) \leq 0
\]

\[
\text{for} \quad t \in \mathbb{R}, \quad \alpha_1(t) \leq (x_i)^n_{i=1} \leq \alpha_2(t) \quad \text{and} \quad (1)^m \sigma_i > 0 \quad (m, j = 1, 2; i = 1, \ldots, n) \quad (6)
\]

and

\[
\sup \{ |\alpha_{mi}(t)| : t \in \mathbb{R} \} < \infty \quad (m = 1, 2; i = 1, \ldots, n).
\]

Then the problem (1), (2) is solvable.

**Corollary 1.** Let the matrix-function \( A(t) = (a_{ik})^n_{i,k=1} \) be nondecreasing on \( \mathbb{R} \) and let there exist numbers \( \sigma_i \in \{-1, 1\} \) \((i = 1, \ldots, n)\), vector-functions \( \alpha_m = (\alpha_{mi})^n_{i=1} \in \text{BV}_{loc}(\mathbb{R}, \mathbb{R}^n) \) \((m = 1, 2)\) and matrix-functions \( (\beta_{mik})^n_{i,k=1}, \beta_{mik} \in L_{loc}(\mathbb{R}, \mathbb{R}; a_{jik}) \) \((m, j = 1, 2; i, k = 1, \ldots, n)\) such that

\[
\alpha_{mi}(t) = \alpha_{mi}(0) + \sum_{k=1}^n \left( \int_0^t \beta_{mik}(\tau) \, da_{1ik}(\tau) \right) \quad (m = 1, 2; i, k = 1, \ldots, n), \quad (8)
\]
the conditions (5) – (7) hold, and the inequalities

\((-1)^m \sigma_i (f_k(t,x_1,\ldots,x_{i-1},\alpha_{ji}(t),x_{i+1},\ldots,x_n) - \beta_{jik}(t)) \leq 0\)

\((j = 1, 2; \ i, k = 1, \ldots, n)\)

are fulfilled for \(\mu(a_{ik})\)-almost all \(t \in \mathbb{R}\) and \(\alpha_1(t) \leq (x_i)^{n}_{i=1} \leq \alpha_2(t)\). Then the problem (1), (2) is solvable.

**Theorem 2.** Let there exist numbers \(\sigma_i \in \{-1, 1\} (i = 1, \ldots, n)\), vector-functions \(\alpha_m = (\alpha_{mi})^{n}_{i=1} \in BV_{loc}(\mathbb{R}, \mathbb{R}^{n}) (m = 1, 2)\) and matrix-functions \((\beta_{mik})^{n}_{i,k=1}\), \(\beta_{mik} \in L_{loc}(\mathbb{R}, \mathbb{R}; a_{jik}) (m, j = 1, 2; \ i, k = 1, \ldots, n)\) such that

\[\alpha_1(t) \leq \alpha_2(t) \quad \text{for} \quad t \in \mathbb{R}^+,\]

\[\begin{aligned}
\sigma_i (f_k(t,x_1,\ldots,x_{i-1},\alpha_{ji}(t),x_{i+1},\ldots,x_n) - \beta_{mik}(t)) &\leq 0 \\
&\quad \text{for} \quad \mu(\alpha_{1+[m-j]ik})\text{-almost all} \quad t \in \mathbb{R}^+ \quad \text{and} \\
\alpha_1(t) &\leq (x_i)^{n}_{i=1} \leq \alpha_2(t) \quad \text{for} \quad t \in \mathbb{R}^+, \quad \alpha_1(t) \leq (x_i)^{n}_{i=1} \leq \alpha_2(t) \quad \text{and} \\
&\quad (-1)^{2d} \sigma_i > 0 \quad (m, j = 1, 2; \ i, k = 1, \ldots, n) \\
\end{aligned}\]

\[\sup \left\{ |\alpha_{mi}(t)| : t \in \mathbb{R}^+ \right\} < \infty \quad (m = 1, 2; \ i = 1, \ldots, n).\]

Then the problem (1), (3) is solvable.

**Corollary 2.** Let the matrix-function \(A(t) = (a_{ik})^{n}_{i,k=1}\) be nondecreasing on \(\mathbb{R}^+\) and let there exist numbers \(\sigma_i \in \{-1, 1\} (i = 1, \ldots, n)\), vector-functions \(\alpha_m = (\alpha_{mi})^{n}_{i=1} \in BV_{loc}(\mathbb{R}^+, \mathbb{R}^{n}) (m = 1, 2)\) and matrix-functions \((\beta_{mik})^{n}_{i,k=1}\), \(\beta_{mik} \in L_{loc}(\mathbb{R}^+, \mathbb{R}; a_{jik}) (m, j = 1, 2; \ i, k = 1, \ldots, n)\) such that the conditions (8) – (11) hold, and the inequalities

\[\begin{aligned}
\sigma_i (f_k(t,x_1,\ldots,x_{i-1},\alpha_{ji}(t),x_{i+1},\ldots,x_n) - \beta_{jik}(t)) &\leq 0 \\
&\quad \text{for} \quad t \in \mathbb{R}^+, \quad \alpha_1(t) \leq (x_i)^{n}_{i=1} \leq \alpha_2(t) \quad \text{and} \\
&\quad (-1)^{2d} \sigma_i > 0 \quad (m, j = 1, 2; \ i, k = 1, \ldots, n) \\
\end{aligned}\]

are fulfilled for \(\mu(a_{ik})\)-almost all \(t \in \mathbb{R}^+\) and \(\alpha_1(t) \leq (x_i)^{n}_{i=1} \leq \alpha_2(t)\). Then the problem (1), (3) is solvable.

**Acknowledgement**

This work is supported by the Georgian National Science Foundation (Grant No. GNSF/ST06/3-002)

**References**


(Received 14.04.2008)

Author’s addresses:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193, Georgia

Sukhumi State University
12, Jikia St., Tbilisi 0186, Georgia

E-mail: ashord@rmi.acnet.ge