Said Kouachi and Belgacem Rebiai

INARIANT REGIONS AND
THE GLOBAL EXISTENCE
FOR REACTION-DIFFUSION SYSTEMS
WITH A TRIDIAGONAL MATRIX
OF DIFFUSION COEFFICIENTS
Abstract. The aim of this study is to prove the global existence of solutions for reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions. In so doing, we make use of the appropriate techniques which are based on invariant domains and Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth. This result is a continuation of that by Kouachi [12].

2010 Mathematics Subject Classification. 35K45, 35K57.

Key words and phrases. reaction diffusion systems, invariant domains, Lyapunov functionals, global existence.
1. INTRODUCTION

We consider the reaction-diffusion system

\[
\begin{align*}
\frac{\partial u}{\partial t} - a_{11} \Delta u - a_{12} \Delta v - a_{23} \Delta w &= f(u, v, w) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \\
\frac{\partial v}{\partial t} - a_{21} \Delta u - a_{22} \Delta v - a_{23} \Delta w &= g(u, v, w) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \\
\frac{\partial w}{\partial t} - a_{21} \Delta u - a_{32} \Delta v - a_{11} \Delta w &= h(u, v, w) \quad \text{in} \quad \mathbb{R}^+ \times \Omega,
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
\lambda u + (1 - \lambda) \frac{\partial u}{\partial \eta} &= \beta_1, \\
\lambda v + (1 - \lambda) \frac{\partial v}{\partial \eta} &= \beta_2 \quad \text{on} \quad \mathbb{R}^+ \times \partial \Omega, \\
\lambda w + (1 - \lambda) \frac{\partial w}{\partial \eta} &= \beta_3,
\end{align*}
\]

and the initial data

\[
u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad w(0, x) = w_0(x) \quad \text{in} \quad \Omega,
\]

where:

(i) $0 < \lambda < 1$ and $\beta_i \in \mathbb{R}$, $i = 1, 2, 3$, for nonhomogeneous Robin boundary conditions.
(ii) $\lambda = \beta_i = 0$, $i = 1, 2, 3$, for homogeneous Neumann boundary conditions.
(iii) $1 - \lambda = \beta_i = 0$, $i = 1, 2, 3$, for homogeneous Dirichlet boundary conditions.

$\Omega$ is an open bounded domain of the class $C^1$ in $\mathbb{R}^N$ with boundary $\partial \Omega$, and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial \Omega$. The diffusion terms $a_{ij}$ ($i, j = 1, 2, 3$ and $(i, j) \neq (1, 3), (3, 1)$) are supposed to be positive constants with $a_{11} = a_{33}$ and $(a_{12} + a_{21})^2 + (a_{23} + a_{32})^2 < 4a_{11}a_{22}$, which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & 0 \\
    a_{21} & a_{22} & a_{23} \\
    0 & a_{32} & a_{11}
\end{pmatrix}
\]

is positive definite. The eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$ ($\lambda_1 < \lambda_2$, $\lambda_3 = a_{11}$) of $A$ are positive. If we put

\[
\underline{a} = \min \{a_{11}, a_{22}\} \quad \text{and} \quad \overline{a} = \max \{a_{11}, a_{22}\},
\]

then the positivity of $a_{ij}$’s implies that

\[
\lambda_1 < \underline{a} \leq \lambda_3 \leq \overline{a} < \lambda_2.
\]
The initial data are assumed to be in the domain

$$
\Sigma = \begin{cases}
\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \mu_2 v_0 \leq a_{21} u_0 + a_{23} w_0 \leq \mu_1 v_0, \ a_{32} w_0 \leq a_{12} w_0 \} & \text{if } \mu_2 \beta_2 \leq a_{21} \beta_1 + a_{23} \beta_3 \leq \mu_1 \beta_2, \ a_{32} \beta_1 \leq a_{12} \beta_3, \\
\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \mu_2 v_0 \leq a_{21} u_0 + a_{23} w_0 \leq \mu_1 v_0, \ a_{12} w_0 \leq a_{32} w_0 \} & \text{if } \mu_2 \beta_2 \leq a_{21} \beta_1 + a_{23} \beta_3 \leq \mu_1 \beta_2, \ a_{12} \beta_3 \leq a_{32} \beta_1, \\
\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \frac{1}{\mu_2} (a_{21} u_0 + a_{23} w_0) \leq v_0 \leq \frac{1}{\mu_1} (a_{21} u_0 + a_{23} w_0), \ a_{32} w_0 \leq a_{12} w_0 \} & \text{if } \frac{1}{\mu_2} (a_{21} \beta_1 + a_{23} \beta_3) \leq \beta_2 \leq \frac{1}{\mu_1} (a_{21} \beta_1 + a_{23} \beta_3), \ a_{32} \beta_1 \leq a_{12} \beta_3, \\
\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \frac{1}{\mu_2} (a_{21} u_0 + a_{23} w_0) \leq v_0 \leq \frac{1}{\mu_1} (a_{21} u_0 + a_{23} w_0), \ a_{32} w_0 \geq a_{12} w_0 \} & \text{if } \frac{1}{\mu_2} (a_{21} \beta_1 + a_{23} \beta_3) \leq \beta_2 \leq \frac{1}{\mu_1} (a_{21} \beta_1 + a_{23} \beta_3), \ a_{32} \beta_1 \geq a_{12} \beta_3,
\end{cases}
$$

where

$$
\mu_1 = \bar{a} - \lambda_1 > 0 > \mu_2 = \bar{a} - \lambda_2.
$$

Since we use the same methods to treat all the cases, we will tackle only with the first one. We suppose that the reaction terms \( f, g \) and \( h \) are continuously differentiable, polynomially bounded on \( \Sigma \),

$$
\left( f (r_1, r_2, r_3), g (r_1, r_2, r_3), h (r_1, r_2, r_3) \right)
$$

is in \( \Sigma \) for all \((r_1, r_2, r_3)\) in \( \partial \Sigma \) (we say that \((f, g, h)\) points into \( \Sigma \) on \( \partial \Sigma \)), i.e.,

$$
a_{21} f (r_1, r_2, r_3) + a_{23} h (r_1, r_2, r_3) \leq \mu_1 g (r_1, r_2, r_3) \tag{1.6}
$$

for all \( r_1, r_2 \) and \( r_3 \) such that \( \mu_2 r_2 \leq a_{21} r_1 + a_{23} r_3 = \mu_1 r_2 \) and \( a_{32} r_1 \leq a_{12} r_3 \), and

$$
\mu_2 g (r_1, r_2, r_3) \leq a_{21} f (r_1, r_2, r_3) + a_{23} h (r_1, r_2, r_3) \tag{1.6a}
$$

for all \( r_1, r_2 \) and \( r_3 \) such that \( \mu_2 r_2 = a_{21} r_1 + a_{23} r_3 \leq \mu_1 r_2 \) and \( a_{32} r_1 \leq a_{12} r_3 \), and

$$
a_{32} f (r_1, r_2, r_3) \leq a_{12} h (r_1, r_2, r_3) \tag{1.6b}
$$

for all \( r_1, r_2 \) and \( r_3 \) such that \( \mu_2 r_2 \leq a_{21} r_1 + a_{23} r_3 \leq \mu_1 r_2 \) and \( a_{32} r_1 = a_{12} r_3 \), and for positive constants \( E \) and \( D \), we have

$$
(E f + D g + h) (u, v, w) \leq C_1 (u + v + w + 1), \tag{1.7}
$$

for all \((u, v, w)\) in \( \Sigma \), where \( C_1 \) is a positive constant.

In the two-component case, where \( a_{12} = 0 \), Kouachi and Youkana [13] generalized the method of Haraux and Youkana [4] with the reaction terms \( f (u, v) = -\lambda F (u, v) \) and \( g (u, v) = \mu F (u, v) \) with \( F (u, v) \geq 0 \), requiring the condition

$$
\lim_{s \to +\infty} \left[ \frac{\ln (1 + F (r, s))}{s} \right] < \alpha^* \text{ for any } r \geq 0,
$$
with
\[ \alpha^* = \frac{2a_{11}a_{22}}{n(a_{11} - a_{22})^2 \|u_0\|_\infty} \min \left\{ \frac{\lambda}{\mu}, \frac{a_{11} - a_{22}}{a_{21}} \right\}, \]
where the positive diffusion coefficients \(a_{11}, a_{22}\) satisfy \(a_{11} > a_{22}\), and \(a_{21}, \lambda, \mu\) are positive constants. This condition reflects the weak exponential growth of the reaction term \(F\). Kanel and Kirane [6] proved the global existence in the case where \(g(u,v) = -f(u,v) = uv^n\) and \(n\) is an odd integer, under the embarrassing condition
\[ |a_{12} - a_{21}| < C_p, \]
where \(C_p\) contains a constant from Solonnikov’s estimate [18]. Later they improved their results in [7] to obtain the global existence under the restrictions
\[
\begin{align*}
H_1. \quad & a_{22} < a_{11} + a_{21}, \\
H_2. \quad & a_{12} < \varepsilon_0 = \frac{a_{11}a_{22}(a_{11} + a_{21} - a_{22})}{a_{11}a_{22} + a_{21}(a_{11} + a_{21} - a_{22})} \quad \text{if} \quad a_{11} \leq a_{22} < a_{11} + a_{21}, \\
H_3. \quad & a_{12} < \min \left\{ \frac{1}{2}(a_{11} + a_{21}), \varepsilon_0 \right\} \quad \text{if} \quad a_{22} < a_{11},
\end{align*}
\]
and
\[ |F(v)| \leq C_F \left( 1 + |v|^{1-\varepsilon} \right), \quad vF(v) \geq 0 \quad \text{for all} \quad v \in \mathbb{R}, \]
where \(\varepsilon\) and \(C_F\) are positive constants with \(\varepsilon < 1\) and
\[ g(u,v) = -f(u,v) = uF(v). \]
Kouachi [12] has proved global existence for solutions of two-component reaction-diffusion systems with a general full matrix of diffusion coefficients and nonhomogeneous boundary conditions.

Many chemical and biological operations are described by reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients. The components \(u(t,x), v(t,x)\) and \(w(t,x)\) can represent either chemical concentrations or biological population densities (see, e.g., Cussler [1] and [2]).

We note that the case of strongly coupled systems which are not triangular in the diffusion part is more difficult. As a consequence of the blow-up of the solutions found in [16], we can indeed prove that there is a blow-up of the solutions in finite time for such nontriangular systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case. For this purpose, we construct invariant domains in which we can demonstrate that for any initial data in these domains, the problem (1.1)–(1.5) is equivalent to the problem for which the global existence follows from the usual techniques based on Lyapunov functionals (see Kirane and Kouachi [8], Kouachi and Youkana [13] and Kouachi [12]).
2. Local Existence and Invariant Regions

This section is devoted to proving that if \((f, g, h)\) points into \(\Sigma\) on \(\partial\Sigma\), then \(\Sigma\) is an invariant domain for the problem (1.1)-(1.5), i.e., the solution remains in \(\Sigma\) for any initial data in \(\Sigma\). Once the invariant domains are constructed, both problems of the local and global existence become easier to be established. For the first problem we demonstrate that the system (1.1)-(1.3) with the boundary conditions (1.4) and the initial data in \(\Sigma\) is equivalent to a problem for which the local existence throughout the time interval \([0, T^*]\) can be obtained by the known procedure, and for the second one we need invariant domains as explained in the preceding section.

The main result of this section is

**Proposition 1.** Suppose that \((f, g, h)\) points into \(\Sigma\) on \(\partial\Sigma\). Then for any \((u_0, v_0, w_0)\) in \(\Sigma\) the solution \((u, v, w)\) of the problem (1.1)-(1.5) remains in \(\Sigma\) for all \(t\)'s in \([0, T^*]\).

**Proof.** Let \((x_{i1}, x_{i2}, x_{i3})\), \(i = 1, 2, 3\), be the eigenvectors of the matrix \(A^t\) associated with its eigenvalues \(\lambda_i\), \(i = 1, 2, 3\) (\(\lambda_1 < \lambda_3 < \lambda_2\)). Multiplying the equations (1.1), (1.2) and (1.3) of the given reaction-diffusion system by \(x_{i1}\), \(x_{i2}\) and \(x_{i3}\), respectively, and summing the resulting equations, we get

\[
\frac{\partial}{\partial t} z_i - \lambda_i \Delta z_i = F_i (z_1, z_2, z_3) \quad \text{in } [0, T^*] \times \Omega, \quad (2.1)
\]

\[
\frac{\partial}{\partial t} z_2 - \lambda_2 \Delta z_2 = F_2 (z_1, z_2, z_3) \quad \text{in } [0, T^*] \times \Omega, \quad (2.2)
\]

\[
\frac{\partial}{\partial t} z_3 - \lambda_3 \Delta z_3 = F_3 (z_1, z_2, z_3) \quad \text{in } [0, T^*] \times \Omega, \quad (2.3)
\]

with the boundary conditions

\[
\lambda z_i + (1 - \lambda) \frac{\partial z_i}{\partial \eta} = \rho_i, \quad i = 1, 2, 3, \quad \text{on } [0, T^*] \times \partial \Omega, \quad (2.4)
\]

and the initial data

\[
z_i(0, x) = z_i^0(x), \quad i = 1, 2, 3, \quad \text{in } \Omega, \quad (2.5)
\]

where

\[
z_i = x_{i1} u + x_{i2} v + x_{i3} w, \quad i = 1, 2, 3, \quad \text{in } [0, T^*] \times \Omega, \quad (2.6)
\]

\[
\rho_i = x_{i1} \beta_1 + x_{i2} \beta_2 + x_{i3} \beta_3, \quad i = 1, 2, 3,
\]

and

\[
F_i (z_1, z_2, z_3) = x_{i1} f + x_{i2} g + x_{i3} h, \quad i = 1, 2, 3, \quad (2.7)
\]

for all \((u, v, w)\) in \(\Sigma\).

First, as has been mentioned above, note that the condition of the parabolicity of the system (1.1)-(1.3) implies the parabolicity of the system
Now, to prove that the domain $(2.9)$ is proved and $(2.6)$ can be written as

$$
(a_{12} + a_{21})^2 + (a_{23} + a_{32})^2 < 4a_{11}a_{22} \implies \text{ (det } A > 0 \text{ and } a_{11}a_{22} - a_{23}a_{32} > 0).
$$

Since $\lambda_1, \lambda_2$ and $\lambda_3$ ($\lambda_1 < \lambda_3 < \lambda_2$) are the eigenvalues of the matrix $A^t$, the problem (1.1)–(1.5) is equivalent to the problem (2.1)–(2.5) and to prove that $\Sigma$ is an invariant domain for the system (1.1)–(1.3) it suffices to prove that the domain

$$
\{ (z_1^0, z_2^0, z_3^0) \in \mathbb{R}^3 : z_i^0 \geq 0, \ i = 1, 2, 3 \} = (\mathbb{R}^+)^3
$$

is invariant for the system (2.1)–(2.3) and that

$$
\Sigma = \{ (u_0, v_0, w_0) \in \mathbb{R}^3 : z_i^0 = x_{i1}u_0 + x_{i2}v_0 + x_{i3}w_0 \geq 0, \ i = 1, 2, 3 \}.
$$

If we assume, without loss of generality, that $a_{11} \leq a_{22}$ and choose $x_{12} = \mu_1$, $x_{22} = -\mu_2$ and $x_{33} = a_{12}$, then we have

$$
x_{i1}u_0 + x_{i2}v_0 + x_{i3}w_0 \geq 0, \ i = 1, 2, 3 \iff \begin{cases} -a_{21}u_0 + \mu_1v_0 - a_{23}w_0 \geq 0, \\ a_{21}u_0 - \mu_2v_0 + a_{23}w_0 \geq 0, \\ -a_{32}u_0 + a_{12}w_0 \geq 0. \end{cases} \iff \mu_2v_0 \leq a_{21}u_0 + a_{23}w_0 \leq \mu_1v_0, \ a_{32}u_0 \leq a_{12}w_0.
$$

Thus (2.9) is proved and (2.6) can be written as

$$
\begin{align*}
z_1 &= -a_{21}u + \mu_1v - a_{23}w, \\
z_2 &= a_{21}u - \mu_2v + a_{23}w, \\
z_3 &= -a_{32}u + a_{12}w.
\end{align*}
$$

Now, to prove that the domain $(\mathbb{R}^+)^3$ is invariant for the system (2.1)–(2.3), it suffices to show that $F_i(\Sigma)$ is positive for all $(z_1, z_2, z_3)$ such that $z_i = 0$ and $z_j \geq 0$, $j = 1, 2, 3$ ($j \neq i$), $i = 1, 2, 3$, thanks to the invariant domain method (see Smoller [17]). Using the expressions (2.7), we get

$$
\begin{align*}
F_1 &= -a_{21}f + \mu_1g - a_{23}h, \\
F_2 &= a_{21}f - \mu_2g + a_{23}h, \\
F_3 &= -a_{32}f + a_{12}h
\end{align*}
$$

for all $(u, v, w)$ in $\Sigma$. Since from (1.6), (1.6a) and (1.6b) we have $F_i(\Sigma)$ is positive for all $(z_1, z_2, z_3)$ such that $z_i = 0$ and $z_j \geq 0$, $j = 1, 2, 3$ ($j \neq i$), $i = 1, 2, 3$, we obtain $z_i(t, x) \geq 0, i = 1, 2, 3$, for all $(t, x) \in [0, T^*] \times \Omega$. Then $\Sigma$ is an invariant domain for the system (1.1)–(1.3). $\square$
In addition, the system (1.1)–(1.3) with the boundary conditions (1.4) and initial data in Σ is equivalent to the system (2.1)–(2.3) with the boundary conditions (2.4) and positive initial data (2.5). As has been mentioned at the beginning of this section and since
\[ \rho_i, \quad i = 1, 2, 3, \]
given by
\[ \begin{align*}
\rho_1 &= -a_{21}\beta_1 + \mu_1\beta_2 - a_{23}\beta_3, \\
\rho_2 &= a_{21}\beta_1 - \mu_2\beta_2 + a_{23}\beta_3, \\
\rho_3 &= -a_{32}\beta_1 + a_{12}\beta_3,
\end{align*} \]
are positive, we have for any initial data in \( C(\Omega) \) or \( L^p(\Omega) \), \( p \in [1, +\infty] \), the local existence and uniqueness of solutions to the initial value problem (2.1)–(2.5) and consequently those of the problem (1.1)–(1.5) follow from the basic existence theory for abstract semilinear differential equations (see Friedman [3], Henry [5] and Pazy [15]). These solutions are classical on \([0, T^*] \times \Omega\), where \( T^* \) denotes the eventual blow up time in \( L^\infty(\Omega) \). A local solution is continued globally by a priori estimates.

Once invariant domains are constructed, one can apply the Lyapunov technique and establish the global existence of unique solutions for (1.1)–(1.5).

3. Global Existence

As the determinant of the linear algebraic system (2.6), with respect to the variables \( u, v \) and \( w \), is different from zero, to prove the global existence of solutions of the problem (1.1)–(1.5) one needs to prove it for the problem (2.1)–(2.5). To this end, it suffices (see Henry [5]) to derive a uniform estimate of \( \| F_i(z_1, z_2, z_3) \|_p, \quad i = 1, 2, 3 \) on \([0, T]\), \( T < T^* \), for some \( p > N/2 \), where \( \| \cdot \|_p \) denotes the usual norms in spaces \( L^p(\Omega) \) defined by
\[ \| u \|_p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p \, dx, \quad 1 \leq p < \infty, \quad \text{and} \quad \| u \|_\infty = \sup_{x \in \Omega} |u(x)|. \]

Let \( \theta \) and \( \sigma \) be two positive constants such that
\[ \begin{align*}
\theta &> A_{12}, \\
(\theta^2 - A_{12}^2) (\sigma^2 - A_{23}^2) &> (A_{13} - A_{12}A_{23})^2,
\end{align*} \]
where
\[ A_{ij} = \frac{\lambda_i + \lambda_j}{2\sqrt{\lambda_i\lambda_j}}, \quad i, j = 1, 2, 3 \quad (i < j), \]
and let
\[ \theta_q = \theta^{(p-q+1)^2} \text{ and } \sigma_p = \sigma^{p^2}, \quad \text{for } q = 0, 1, \ldots, p \text{ and } p = 0, 1, \ldots, n, \]
where \( n \) is a positive integer. The main result of this section is
Theorem 1. Let \((z_1(t, \cdot), z_2(t, \cdot), z_3(t, \cdot))\) be any positive solution of (2.1)–(2.5). Introduce the functional

\[
t \mapsto L(t) = \int_\Omega H_n (z_1(t, x), z_2(t, x), z_3(t, x)) \, dx,
\]

(3.4)

where

\[
H_n (z_1, z_2, z_3) = \sum_{p=0}^{n} \sum_{q=0}^{p} C_n^p C_q^p \theta q \sigma p q \cdot z_1^{p-q} z_2^{q} z_3^{n-p},
\]

(3.5)

with \(n\) being a positive integer and \(C_n^p = \frac{n!}{(n-p)!p!} \).

Then the functional \(L\) is uniformly bounded on the interval \([0, T], T < T^*\).

For the proof of Theorem 1 we need some preparatory Lemmas.

Lemma 1. Let \(H_n\) be the homogeneous polynomial defined by (3.5). Then

\[
\frac{\partial H_n}{\partial z_1} = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_n^{p-1} C_{p-1}^q \theta q \sigma p q \cdot z_1^{p-q} z_2^{q} z_3^{n-p-1},
\]

(3.6)

\[
\frac{\partial H_n}{\partial z_2} = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_n^{p} C_{p}^q \theta q \sigma p q \cdot z_1^{p-q} z_2^{q} z_3^{n-p},
\]

(3.7)

\[
\frac{\partial H_n}{\partial z_3} = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_n^{p-1} C_{p-1}^q \theta q \sigma p q \cdot z_1^{p-q} z_2^{q} z_3^{n-p-1}.
\]

(3.8)

Proof. Differentiating \(H_n\) with respect to \(z_1\) and using the fact that

\[
q C^q_p = p C^{q-1}_{p-1} \text{ and } p C^q_n = n C^{q-1}_{n-1}
\]

(3.9)

for \(q = 1, 2, \ldots, p, p = 1, 2, \ldots, n\), we get

\[
\frac{\partial H_n}{\partial z_1} = n \sum_{p=0}^{n} \sum_{q=0}^{p} C^{p} C_{p-1}^{q} \theta q \sigma p q \cdot z_1^{p-q} z_2^{q} z_3^{n-p}.
\]

Replacing in the sums the indexes \(q - 1\) by \(q\) and \(p - 1\) by \(p\), we deduce (3.6). For the formula (3.7), differentiating \(H_n\) with respect to \(z_2\), taking into account

\[
C^q_p = C^{q-p}_{p}, \quad q = 0, 1, \ldots, p - 1 \text{ and } p = 1, 2, \ldots, n,
\]

(3.10)

using (3.9) and replacing the index \(p - 1\) by \(p\), we get (3.7).

Finally, we have

\[
\frac{\partial H_n}{\partial z_3} = \sum_{p=0}^{n-1} \sum_{q=0}^{p} (n-p) C^{p} C_{p}^{q} \theta q \sigma p q \cdot z_1^{p-q} z_2^{q} z_3^{n-p-1}.
\]

Since \((n-p) C^{p}_n = (n-p) C^{n-p}_n = n C^{n-p-1}_{n-1} = n C^p_{n-1}\), we get (3.8). □
Lemma 2. The second partial derivatives of $H_n$ are given by

\[
\frac{\partial^2 H_n}{\partial z_1^2} = n (n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p \theta_p q + 2 \sigma_p + 1 z_1 q \cdot p - q (n-2) - p, \tag{3.11}
\]

\[
\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n (n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p \theta_p q + 1 \sigma_p + 2 z_1 q \cdot p - q (n-2) - p, \tag{3.12}
\]

\[
\frac{\partial^2 H_n}{\partial z_2^2} = n (n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p \theta_p q + 1 \sigma_p + 1 z_2 q \cdot p - q (n-2) - p, \tag{3.13}
\]

\[
\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n (n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p \theta_p q + 2 \sigma_p + 1 z_1 z_2 q \cdot p - q (n-2) - p, \tag{3.14}
\]

\[
\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n (n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p \theta_p q \sigma_p + 1 z_1 z_2 q \cdot p - q (n-2) - p, \tag{3.15}
\]

\[
\frac{\partial^2 H_n}{\partial z_3^2} = n (n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p \theta_p q \sigma_p + 2 z_1 z_2 q \cdot p - q (n-2) - p. \tag{3.16}
\]

Proof. Differentiating $\frac{\partial H_n}{\partial z_1}$ given by (3.6) with respect to $z_1$ yields

\[
\frac{\partial^2 H_n}{\partial z_1^2} = n \sum_{p=1}^{n-1} \sum_{q=1}^{p} q C_{n-1}^p \theta_p q + 1 \sigma_p + 1 z_1 q \cdot p - q (n-1) - p.
\]

Using (3.9), we get (3.11).

\[
\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n \sum_{p=1}^{n-1} \sum_{q=0}^{p-1} (p - q) C_{n-1}^p \theta_p q + 1 \sigma_p + 1 z_1 q \cdot p - q (n-1) - p.
\]

Applying (3.10) and then (3.9), we get (3.12).

\[
\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n \sum_{p=0}^{n-2} \sum_{q=0}^{p} ((n - 1) - p) C_{n-1}^p \theta_p q + 1 \sigma_p + 1 z_1 z_2 q \cdot p - q (n-2) - p.
\]

Applying successively (3.10), (3.9) and (3.10) for the second time, we deduce (3.13).

\[
\frac{\partial^2 H_n}{\partial z_2^2} = n \sum_{p=1}^{n-1} \sum_{q=0}^{p-1} (p - q) C_{n-1}^p \theta_p q \sigma_p + 1 z_1 z_2 q \cdot p - q (n-1) - p.
\]

The application of (3.10) and then of (3.9) yields (3.14).

\[
\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n \sum_{p=0}^{n-2} \sum_{q=0}^{p} ((n - 1) - p) C_{n-1}^p \theta_p q \sigma_p + 2 z_1 z_2 q \cdot p - q (n-2) - p.
\]
Applying (3.10) and then (3.9) yields (3.15). Finally we get (3.16) by differentiating \( \frac{\partial H_n}{\partial z_3} \) with respect to \( z_3 \) and applying successively (3.10), (3.9) and (3.10) for the second time.

**Proof of Theorem 1.** Differentiating \( L \) with respect to \( t \) yields

\[
L'(t) = \int_{\Omega} \left( \frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial t} + \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial t} + \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial t} \right) \, dx =
\]

\[
= \int_{\Omega} \left( \lambda_1 \frac{\partial H_n}{\partial z_1} \Delta z_1 + \lambda_2 \frac{\partial H_n}{\partial z_2} \Delta z_2 + \lambda_3 \frac{\partial H_n}{\partial z_3} \Delta z_3 \right) \, dx +
\]

\[
+ \int_{\Omega} \left( \frac{\partial H_n}{\partial z_1} F_1 + \frac{\partial H_n}{\partial z_2} F_2 + \frac{\partial H_n}{\partial z_3} F_3 \right) \, dx =: I + J.
\]

Using Green’s formula in \( I \), we get

\[
I_1 = \int_{\partial \Omega} \left( \lambda_1 \frac{\partial H_n}{\partial z_1} \frac{\partial z_1}{\partial \eta} + \lambda_2 \frac{\partial H_n}{\partial z_2} \frac{\partial z_2}{\partial \eta} + \lambda_3 \frac{\partial H_n}{\partial z_3} \frac{\partial z_3}{\partial \eta} \right) \, ds,
\]

where \( ds \) denotes the \((n-1)\)-dimensional surface element, and

\[
I_2 = - \int_{\Omega} \left[ \lambda_1 \frac{\partial^2 H_n}{\partial z_1^2} |\nabla z_1|^2 + (\lambda_1 + \lambda_2) \frac{\partial^2 H_n}{\partial z_1 \partial z_2} \nabla z_1 \nabla z_2 
\]

\[
+ (\lambda_1 + \lambda_3) \frac{\partial^2 H_n}{\partial z_1 \partial z_3} \nabla z_1 \nabla z_3 + \lambda_2 \frac{\partial^2 H_n}{\partial z_2^2} |\nabla z_2|^2 
\]

\[
+ (\lambda_2 + \lambda_3) \frac{\partial^2 H_n}{\partial z_2 \partial z_3} \nabla z_2 \nabla z_3 + \lambda_3 \frac{\partial^2 H_n}{\partial z_3^2} |\nabla z_3|^2 \right] \, dx.
\]

We prove that there exists a positive constant \( C_2 \) independent of \( t \in [0, T^*] \) such that

\[
I_1 \leq C_2 \text{ for all } t \in [0, T^*], \quad \text{(3.17)}
\]

and that

\[
I_2 \leq 0. \quad \text{(3.18)}
\]

To see this, we follow the same reasoning as in [11].

(i) If \( 0 < \lambda < 1 \), using the boundary conditions (2.4) we get

\[
I_1 = \int_{\partial \Omega} \left( \lambda_1 \frac{\partial H_n}{\partial z_1} (\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2} (\gamma_2 - \alpha z_2) + \lambda_3 \frac{\partial H_n}{\partial z_3} (\gamma_3 - \alpha z_3) \right) \, ds,
\]

where \( \alpha = \frac{\omega_0}{\gamma_i} \) and \( \gamma_i = \frac{\phi_i}{\gamma_i}, i = 1, 2, 3 \). Since

\[
H(z_1, z_2, z_3) = \lambda_1 \frac{\partial H_n}{\partial z_1} (\gamma_1 - \alpha z_1) + \lambda_2 \frac{\partial H_n}{\partial z_2} (\gamma_2 - \alpha z_2) + \lambda_3 \frac{\partial H_n}{\partial z_3} (\gamma_3 - \alpha z_3)
\]

\[
= P_n(z_1, z_2, z_3) - Q_n(z_1, z_2, z_3),
\]
where $P_{n-1}$ and $Q_n$ are polynomials with positive coefficients and respective
degrees $n-1$ and $n$, and since the solution is positive, we obtain
\[
\limsup_{(|z_1|+|z_2|+|z_3|)\to+\infty} H(z_1, z_2, z_3) = -\infty,
\]
which proves that $H$ is uniformly bounded on $(\mathbb{R}^+)^3$, and consequently
(3.17).

(ii) If $\lambda = 0$, then $I_1 = 0$ on $[0, T^*]$. 

(iii) The case of the homogeneous Dirichlet conditions is trivial since the
positivity of the solution on $[0, T^*] \times \Omega$ implies $\frac{\partial z_1}{\partial n} \leq 0$, $\frac{\partial z_2}{\partial n} \leq 0$ and $\frac{\partial z_3}{\partial n} \leq 0$
on $[0, T^*] \times \partial \Omega$. Consequently, one again gets (3.17) with $C_2 = 0$.

Now, we prove (3.18). Applying Lemma 1 and Lemma 2, we get
\[
I_2 = -n(n-1) \int_\Omega \sum_{p=0}^{n-2} \sum_{q=0}^p C_{pq}^p C_{pq}^q [(B_{pq}z) \cdot z] \, dx,
\]
where
\[
B_{pq} = \begin{pmatrix}
\frac{\lambda_1 \theta_{q+2} \sigma_{p+2}}{2} & \frac{\lambda_1 + \lambda_2}{2} \theta_{q+1} \sigma_{p+2} & \frac{\lambda_1 + \lambda_3}{2} \theta_{q+1} \sigma_{p+1} \\
\frac{\lambda_1 + \lambda_2}{2} \theta_{q+1} \sigma_{p+2} & \lambda_2 \theta_q \sigma_{p+2} & \frac{\lambda_2 + \lambda_3}{2} \theta_q \sigma_{p+1} \\
\frac{\lambda_1 + \lambda_3}{2} \theta_{q+1} \sigma_{p+1} & \frac{\lambda_2 + \lambda_3}{2} \theta_q \sigma_{p+1} & \lambda_3 \theta_q \sigma_p
\end{pmatrix},
\]
for $q = 0, 1, \ldots, p, p = 0, 1, \ldots, n - 2$ and $z = (\nabla z_1, \nabla z_2, \nabla z_3)^t$.
The quadratic forms (with respect to $\nabla z_1, \nabla z_2$ and $\nabla z_3$) associated with
the matrices $B_{pq}$, $q = 0, 1, \ldots, p, p = 0, 1, \ldots, n - 2$, are positive since
their main determinants $\Delta_1$, $\Delta_2$ and $\Delta_3$ are positive too, according to
the Sylvester criterion. To see this, we have

1. $\Delta_1 = \lambda_1 \theta_{q+2} \sigma_{p+2} > 0$ for $q = 0, 1, \ldots, p$ and $p = 0, 1, \ldots, n - 2$.

2. $\Delta_2 = \left| \begin{array}{cc}
\frac{\lambda_1 \theta_{q+2} \sigma_{p+2}}{2} & \frac{\lambda_1 + \lambda_2}{2} \theta_{q+1} \sigma_{p+2} \\
\frac{\lambda_1 + \lambda_2}{2} \theta_{q+1} \sigma_{p+2} & \lambda_2 \theta_q \sigma_{p+2}
\end{array} \right|$

$= \lambda_1 \lambda_2 \theta_{q+1}^2 \sigma_{p+2}^2 (\theta^2 - A_{12}^2),
$
for $q = 0, 1, \ldots, p$ and $p = 0, 1, \ldots, n - 2$. 

Using (3.1), we get $\Delta_2 > 0$.

3. $\Delta_3 = \left| \begin{array}{ccc}
\frac{\lambda_1 \theta_{q+2} \sigma_{p+2}}{2} & \frac{\lambda_1 + \lambda_2}{2} \theta_{q+1} \sigma_{p+2} & \frac{\lambda_1 + \lambda_3}{2} \theta_{q+1} \sigma_{p+1} \\
\frac{\lambda_1 + \lambda_2}{2} \theta_{q+1} \sigma_{p+2} & \lambda_2 \theta_q \sigma_{p+2} & \frac{\lambda_2 + \lambda_3}{2} \theta_q \sigma_{p+1} \\
\frac{\lambda_1 + \lambda_3}{2} \theta_{q+1} \sigma_{p+1} & \frac{\lambda_2 + \lambda_3}{2} \theta_q \sigma_{p+1} & \lambda_3 \theta_q \sigma_p
\end{array} \right|$

$= \lambda_1 \lambda_2 \lambda_3 \theta_{q+1}^2 \theta_q \sigma_{p+2} \sigma_{p+1}^2 [\theta^2 - A_{12}^2] (\sigma^2 - A_{23}^2) - (A_{13} - A_{12} A_{23})^2],$
for $q = 0, 1, \ldots, p$ and $p = 0, 1, \ldots, n - 2$. 

\[\]
Using (3.2), we get $\Delta_3 > 0$. Consequently we have (3.18).

Substitution of the expressions of the partial derivatives given by Lemma 1 in the second integral yields

$$J = \int_\Omega \left[ \sum_{p=0}^{n-1} \sum_{q=0}^{n-p} C_{n-1}^p C_{p+1}^q \frac{1}{z_3^{n-1-p}} \times \right.$$

$$\left. \times (\theta_{q+1}\sigma_{p+1} F_1 + \theta_q\sigma_{p+1} F_2 + \theta_q\sigma_p F_3) \right] \, dx.$$ 

Using the expressions (2.7a), we get

$$\theta_{q+1}\sigma_{p+1} F_1 + \theta_q\sigma_{p+1} F_2 + \theta_q\sigma_p F_3 =$$

$$= \left( -\theta_{q+1}\sigma_{p+1} a_{21} + a_{21}\theta_q\sigma_{p+1} - a_{32}\theta_q\sigma_p \right) f + \left( \theta_{q+1}\sigma_{p+1} \mu_1 + \mu_2\theta_q\sigma_{p+1} \right) g +$$

$$+ \left( -\theta_{q+1}\sigma_{p+1} a_{23} + a_{23}\theta_q\sigma_{p+1} + a_{12}\theta_q\sigma_p \right) h =$$

$$= \left( a_{23}(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}) + a_{12}\theta_q\sigma_p \right) \left( \frac{a_{21}(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}) - a_{32}\theta_q\sigma_p}{a_{23}(\theta_q\sigma_{p+1} - \theta_{q+1}\sigma_{p+1}) + a_{12}\theta_q\sigma_p} \right) \times$$

$$\times \left( \frac{a_{21}}{a_{23}} \frac{\sigma_{p+1}}{\sigma_p} \left( \frac{\theta_q}{\theta_{q+1}} - 1 \right) - a_{32} \frac{\theta_q}{\sigma_p} + \frac{\mu_1}{\sigma_p} \frac{\sigma_{p+1}}{\sigma_p} \frac{\theta_q}{\theta_{q+1}} - a_{12} \frac{\theta_q}{\sigma_p} + \frac{\theta_q}{\theta_{q+1}} \right)$$

Since $\frac{\theta_q}{\theta_{q+1}}$ and $\frac{\sigma_{p+1}}{\sigma_p}$ are sufficiently large if we choose $\theta$ and $\sigma$ sufficiently large, using the condition (1.7) and the relation (2.6a) successively we get, for an appropriate constant $C_3$,

$$J \leq C_3 \int_\Omega \left[ \sum_{p=0}^{n-1} \sum_{q=0}^{n-p} (z_1 + z_2 + z_3 + 1) C_{n-1}^p C_{p+1}^q \frac{1}{z_3^{n-1-p}} \right] \, dx.$$ 

To prove that the functional $L$ is uniformly bounded on the interval $[0, T]$, we first write

$$\sum_{p=0}^{n-1} \sum_{q=0}^{n-p} (z_1 + z_2 + z_3 + 1) C_{n-1}^p C_{p+1}^q \frac{1}{z_3^{n-1-p}} =$$

$$= R_n(z_1, z_2, z_3) + S_n(z_1, z_2, z_3),$$

where $R_n(z_1, z_2, z_3)$ and $S_n(z_1, z_2, z_3)$ are two homogeneous polynomials of degrees $n$ and $n - 1$, respectively. First, since the polynomials $H_n$ and $R_n$ are of degree $n$, there exists a positive constant $C_4$ such that

$$\int_\Omega R_n(z_1, z_2, z_3) \, dx \leq C_4 \int_\Omega H_n(z_1, z_2, z_3) \, dx.$$
Applying Hölder’s inequality to the integral \( \int_{\Omega} S_{n-1} (z_1, z_2, z_3) \, dx \), one gets
\[
\int_{\Omega} S_{n-1} (z_1, z_2, z_3) \, dx \leq \left( \int_{\Omega} (S_{n-1} (z_1, z_2, z_3))^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}}.
\]
Since for all \( z_1 \geq 0 \) and \( z_2, z_3 > 0 \)
\[
\frac{(S_{n-1} (z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n (z_1, z_2, z_3)} = \frac{(S_{n-1} (\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n (\xi_1, \xi_2, 1)},
\]
where \( \xi_1 = \frac{z_1}{z_2}, \xi_2 = \frac{z_2}{z_3} \) and
\[
\lim_{\xi_1 \to +\infty, \xi_2 \to +\infty} \frac{(S_{n-1} (\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n (\xi_1, \xi_2, 1)} < +\infty,
\]
one asserts that there exists a positive constant \( C_5 \) such that
\[
\frac{(S_{n-1} (z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n (z_1, z_2, z_3)} \leq C_5 \quad \text{for all} \quad z_1, z_2, z_3 \geq 0.
\]
Hence the functional \( L \) satisfies the differential inequality
\[
L'(t) \leq C_6 L(t) + C_7 L^{\frac{n-1}{n}}(t),
\]
which for \( Z = L^\frac{1}{n} \) can be written as
\[
nZ' \leq C_6 Z + C_7.
\]
A simple integration gives a uniform bound of the functional \( L \) on the interval \([0,T] \). This completes the proof of Theorem 1. \( \square \)

**Corollary 1.** Suppose that the functions \( f(r_1, r_2, r_3) \), \( g(r_1, r_2, r_3) \) and \( h(r_1, r_2, r_3) \) are continuously differentiable on \( \Sigma \), point into \( \Sigma \) on \( \partial \Sigma \) and satisfy the condition (1.7). Then all uniformly bounded on \( \Omega \) solutions of (1.1)–(1.5) with the initial data in \( \Sigma \) are in \( L^\infty (0,T; L^p (\Omega)) \) for all \( p \geq 1 \).

**Proof.** The proof of this Corollary is an immediate consequence of Theorem 1, the trivial inequality
\[
\int_{\Omega} (z_1 + z_2 + z_3)^p \, dx \leq L(t) \quad \text{on} \quad [0,T^*],
\]
and (2.6a). \( \square \)

**Proposition 2.** Under the hypothesis of Corollary 1, if \( f(r_1, r_2, r_3), g(r_1, r_2, r_3) \) and \( h(r_1, r_2, r_3) \) are polynomially bounded, then all uniformly bounded on \( \Omega \) solutions of (1.1)–(1.4) with the initial data in \( \Sigma \) are global in time.
Proof. As has been mentioned above, it suffices to derive a uniform estimate of $\|F_1(z_1, z_2, z_3)\|_p$, $\|F_2(z_1, z_2, z_3)\|_p$ and $\|F_3(z_1, z_2, z_3)\|_p$ on $[0, T], T < T^*$ for some $p > \frac{N}{2}$. Since the reactions $f(u, v, w)$, $g(u, v, w)$ and $h(u, v, w)$ are polynomially bounded on $\Sigma$, by using relations (2.6a) and (2.7a) we get that so are $F_1(z_1, z_2, z_3)$, $F_2(z_1, z_2, z_3)$ and $F_3(z_1, z_2, z_3)$, and the proof becomes an immediate consequence of Corollary 1.

Acknowledgments

The authors would like to thank the anonymous referee for his/her valuable comments and suggestions that helped to improve the presentation of this paper.

References


(Received 22.01.2008; revised 23.09.2009)

Authors’ addresses:

Said Kouachi  
Department of Mathematics  
College of Science  
Qassim University  
P.O.Box 6644, Al-Gassim, Buraydah 51452  
Saudi Arabia  
E-mail: kouachi.said@caramil.com

Belgacem Rebiai  
Department of Mathematics  
University of Tebessa, 12002  
Algeria  
E-mail: brebiai@gmail.com